7
Transcendental Functions

An **algebraic number** is any number that can be a solution to a polynomial equation with integer coefficients. Any rational number is algebraic: for example, $\frac{5}{7}$ is a solution of $7x - 5 = 0$. Any square root or cube root is an algebraic number: $\sqrt{3}$ is a solution of $x^2 - 3 = 0$ and $-\sqrt[3]{5}$ is a solution of $x^3 + 5 = 0$. In fact, any number that can be expressed as the sum, difference, product, quotient or rational power of integers is an algebraic number, including something as horrible-looking as:

$$\sqrt[5]{\frac{\sqrt[4]{93} - \sqrt[3]{436 + \sqrt{2}}}{17 + \sqrt{37}}}$$

Any real number that is *not* an algebraic number is called a **transcendental number**. We have already met (and used extensively) two very important transcendental numbers: π and e.

It turns out that *proving* that these numbers are transcendental is rather difficult, although by the end of Chapter 8 we will have most of the tools necessary to prove that e is transcendental.

Functions defined using sums, differences, products, quotients or rational powers of rational coefficients and a real-valued variable x are called **algebraic functions**. Any non-algebraic functions of a real variable are called **transcendental functions**. Examples of transcendental functions with which you should be very familiar are $\sin(x)$, $\cos(x)$, $\tan(x)$ and the other trigonometric functions, as well as e^x.

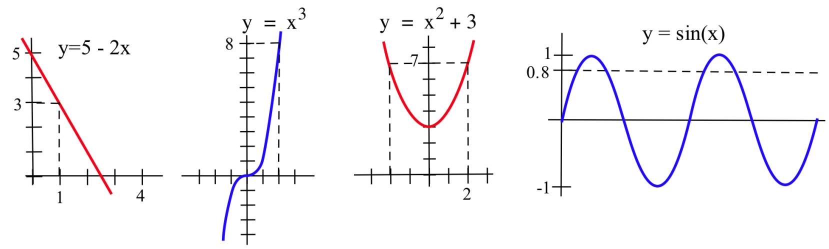
The geometric definitions of the trigonometric functions involve π, while the exponential function obviously involves e.

The inverse functions of these transcendental functions are also transcendental functions: $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$ and the other inverse trigonometric functions, as well as $\ln(x)$.

Because many important transcendental functions are defined as inverse functions, this chapter begins with a review of inverse functions and a discussion of finding derivatives of inverse functions. It continues with a review of inverse trigonometric functions and a discussion of their derivatives (and the usefulness of these derivative patterns in finding certain antiderivatives), then ties up some (very important) loose ends related to the definitions and properties of e^x and $\ln(x)$. The chapter concludes by introducing some new transcendental functions: the hyperbolic functions and their inverses, which play important roles in calculus and in applications.

7.1 *One-to-One Functions*

You've seen that some equations have only one solution (for example, $5-2x=3$ and $x^3=8$), while some have two solutions ($x^2+3=7$) and some even have an infinite number of solutions ($\sin(x)=0.8$). The graphs of $y=5-2x$, $y=x^3$, $y=x^2+3$ and $y=\sin(x)$ and the solutions of the equations mentioned above appear below:



Functions f for which equations of the form $f(x)=k$ have at most one solution for each value of k (that is, each outcome k comes from only one input x) arise often in applications and possess a number of useful mathematical properties. This brief section focuses on those functions and examines some of their properties.

Example 1. How many solutions does each equation have?

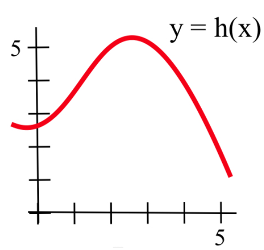
(a) $f(x)=0$ for $f(x)=x(x-4)$

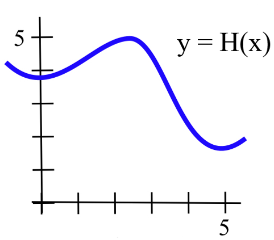
(b) $g(x)=3$ for g given in the margin table

(c) $h(x)=4$ for h given by the graph in the margin

(d) $f(x)=k$ for $f(x)=e^x$.

x	0	1	2	3	4	5
$g(x)$	5	7	3	5	0	7





Solution. (a) Two: $x(4-x)=0 \Rightarrow x=0$ or $x=4$. (b) One: $g(x)=3$ only if $x=2$. (c) Two: $h(x)=4$ if $x\approx 1.2$ or if $x\approx 4$. (d) If $k>0$, it has one solution: $x=\ln(k)$. If $k\le 0$, it has no solutions. ◀

Practice 1. How many solutions does each equation have?

(a) $f(x)=4$ for $f(x)=x(4-x)$

(b) $g(x)=7$ for g given by the margin table

(c) $H(x)=3$ for H given by the graph in margin

(d) $f(x)=5$ for $f(x)=\ln(x)$

Horizontal Line Test

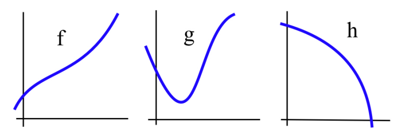
If not, review Section 0.3.

You should be familiar with the Vertical Line Test, a graphical tool you can use to help determine whether or not a curve in the xy-plane is the graph of a function. A similar geometrical test leads to the definition of a **one-to-one function** and provides a tool for helping to determine when a function is one-to-one.

> **Horizontal Line Test (Definition of One-to-One)**:
> A function is **one-to-one** if each horizontal line intersects the graph of the function at most once.

Equivalently, a function $y = f(x)$ is one-to-one if two *distinct* x-values always produce two *distinct* y-values: that is, $a \neq b \Rightarrow f(a) \neq f(b)$. This immediately tells us that every strictly increasing function is one-to-one, and that every strictly decreasing function is one-to-one. (Why?)

For any function, if we know an input value we can calculate the output, but an output may arise from any of several different inputs. With a one-to-one function, each output comes from only one input.



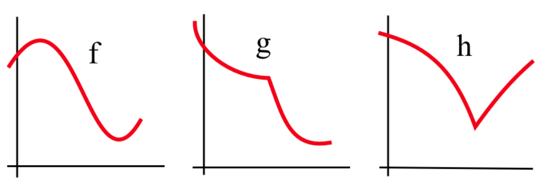
x	$f(x)$	$g(x)$	$h(x)$
0	5	7	2
1	2	3	-1
2	3	0	5
3	5	1	4
4	0	6	3
5	1	3	0

Example 2. (a) Which functions in the first margin figure are one–to-one? (b) Which functions in the first margin table are one-to-one?

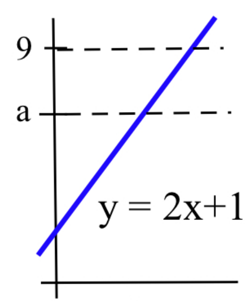
Solution. (a) In the figure, f and h are one-to-one; g fails the Horizontal Line Test, so g is not one-to-one. (b) In the table, h is one-to-one, while f and g are not one-to-one because $f(0) = f(3)$ and $g(1) = g(5)$. ◀

Practice 2. (a) Which functions graphed below are one-to-one? (b) Which functions in the second margin table are one-to-one?

x	$f(x)$	$g(x)$	$h(x)$
0	4	2	-2
1	2	3	5
2	-2	0	1
3	5	4	14
4	3	6	3
5	1	7	1



Example 3. Let $f(x) = 2x + 1$ (see margin). Find the values of x so that (a) $f(x) = 9$ and (b) $f(x) = a$ and then (c) solve $f(y) = x$ for y.



Solution. (a) $9 = f(x) = 2x + 1 \Rightarrow 8 = 2x \Rightarrow x = \frac{8}{2} = 4$ (b) $a = 2x + 1 \Rightarrow 2x = a - 1 \Rightarrow x = \frac{a-1}{2}$ (c) $x = f(y) = 2y + 1 \Rightarrow 2y = x - 1$ so $y = \dfrac{x-1}{2}$. Notice that this new function reverses the operations of $f(x)$, applied in reverse order: $f(x)$ multiplies x by 2, then adds 1; the new function subtracts 1, then divides by 2. ◀

Practice 3. Let $g(x) = 3x - 5$. Find the values of x so that (a) $g(x) = 7$ and (b) $g(x) = b$ and then (c) solve $g(y) = x$ for y.

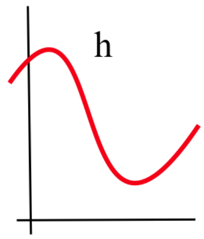
Practice 4. Show that exponential growth, for example $f(x) = e^{3x}$, and exponential decay, for example $g(x) = e^{-2x}$, are both one-to-one.

7.1 Problems

In Problems 1–4, explain why each given function is (or is not) one-to-one.

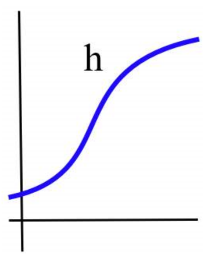
1. $f(x) = 3x - 5$, $y = 3 - x$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	3
1	4
2	5
3	2
4	4



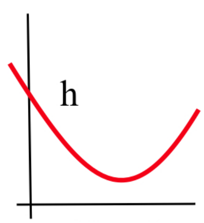
2. $f(x) = \frac{x}{4}$, $y = x^2 + 3$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	3
1	2
2	0
3	-2
4	1



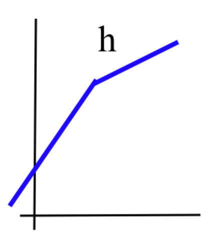
3. $f(x) = \sin(x)$, $y = e^x - 2$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	-1
1	5
2	3
3	1
4	0



4. $f(x) = 17$, $y = x^3 - 1$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	2
1	5
2	4
3	1
4	2



5. Is the relation between people and Social Security numbers a function? A one-to-one function?

6. Is the relation between people and phone numbers a function? If so, is it one-to-one?

7. What would it mean if the scores on a calculus test were one-to-one?

8. The relation given below represents "y is married to x." (a) Is this relation a function? (b) Is it one-to-one? (c) Is P breaking the law? (d) Is A breaking the law?

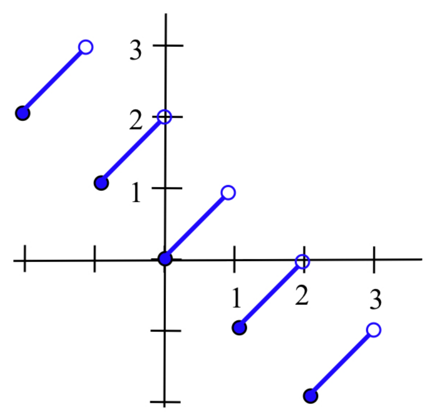
x	A	B	C	D
y	P	Q	P	R

9. In how many places can a one-to-one function touch the x-axis?

10. Can a continuous one-to-one function have the values given below? Explain.

x	1	3	5
$f(x)$	2	7	3

11. The graph of $f(x) = x - 2 \cdot \lfloor x \rfloor$ for $-2 \le x \le 3$ appears below.



(a) Is f a one-to-one function?
(b) Is f an increasing function?
(c) Is f a decreasing function?

12. Is every linear function $L(x) = ax + b$ one-to-one? If not, which linear functions *are* one-to-one?

13. Show that $f(x) = \ln(x)$ is one-to-one for $x > 0$.

14. Show that $g(x) = e^x$ is one-to-one.

15. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	c	f	e	b	a

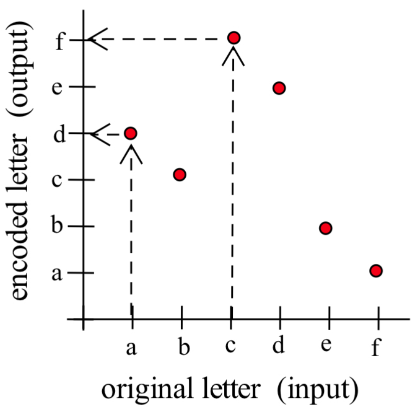
(a) Is the encoding rule a function?

(b) Is the encoding rule one-to-one?

(c) Encode the word "bad."

(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).

(e) A graph of the encoding rule appears below. Create a graph of the decoding rule.



(f) Compare the encoding and decoding graphs.

16. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
b	d	b	b	a	c

(a) Is the encoding rule a function?

(b) Is the encoding rule one-to-one?

(c) Encode the word "bad."

(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).

(e) Create a graph of the encoding rule.

(f) Create a graph of the decoding rule.

(g) Compare the encoding and decoding graphs.

17. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	f	e	a	c	b

(a) Is the encoding rule a function?

(b) Is the encoding rule one-to-one?

(c) Encode the word "bad."

(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).

(e) Create a graph of the encoding rule.

(f) Create a graph of the decoding rule.

(g) Compare the encoding and decoding graphs.

(h) What happens if you encode a word, then encode the encoded word? For example, encode (encode ("bad")) = ?

18. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
e	a	f	c	b	d

(a) Is the encoding rule a function?

(b) Is the encoding rule one-to-one?

(c) Encode the word "bad."

(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).

(e) Create a graph of the encoding rule.

(f) Create a graph of the decoding rule.

(g) Compare the encoding and decoding graphs.

(h) What happens if you apply this encoding rule three times in succession? For example, encode (encode (encode ("bad"))) = ?

7.1 Practice Answers

1. (a) One: solve $x(4-x) = 4$ to get $x = 2$.
   (b) Two: $x = 1$ and $x = 5$.
   (c) One: $x \approx 3.5$.
   (d) One: solve $5 = \ln(x)$ to get $x = e^5 \approx 148.4$.

2. (a) Only g is one-to-one; f and h fail the Horizontal Line Test.
   (b) Both f and g are one-to-one; h is not, because $h(2) = h(5)$.

3. (a) $3x - 5 = 7 \Rightarrow 3x = 12 \Rightarrow x = 4$
   (b) $3x - 5 = a \Rightarrow 3x = a + 5 \Rightarrow x = \dfrac{a+5}{3}$
   (c) $f(x) = 3x - 5 \Rightarrow f(y) = 3y - 5$ so $f(y) = x \Rightarrow 3y - 5 = x \Rightarrow 3y = x + 5 \Rightarrow y = \dfrac{x+5}{3}$

4. If $f(x) = e^{kx}$ where $k > 0$ then $f'(x) = k \cdot e^{kx} > 0$ so $f(x)$ is strictly increasing, hence one-to-one. If $g(x) = e^{rx}$ where $r < 0$ then $g'(x) = r \cdot e^{rx} < 0$ so $g(x)$ is strictly decreasing, hence one-to-one.