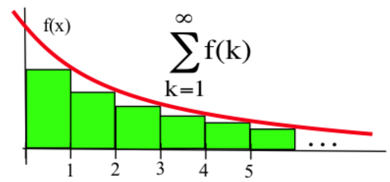
9.6 *Integral Test and P-Test*

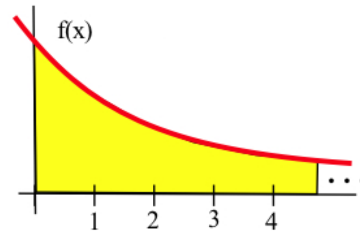
This section presents two methods for determining whether certain series converge or diverge. The first of these, the Integral Test, says that a given series converges if and only if a related improper integral converges. This lets us trade a question about the convergence of a series for a question about the convergence of an improper integral.

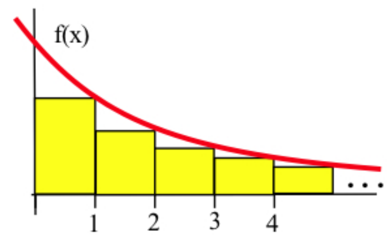
The second of these convergence tests, the P-Test, says that you can determine the convergence of a series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ by knowing the value of p. Both of these tests apply only to certain classes of series whose terms are positive and (unfortunately) the tests only tell us whether a series converges or diverges — they do *not* tell us the actual sum of the series. The Integral Test is the more fundamental and general of the two, and we will use it to prove the P-Test. The P-Test, however, is easier to apply and is likely to be the test you use more often.

Integral Test



We can represent a series geometrically as a sum of areas of rectangles with a width of one unit and a heights corresponding to the values a_k of the terms of the series (see margin). This area interpretation of series leads to a natural connection between series and integrals — and between the convergence of a series and the convergence of a corresponding improper integral.

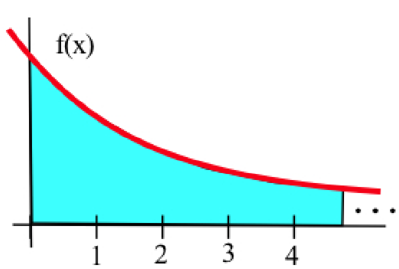


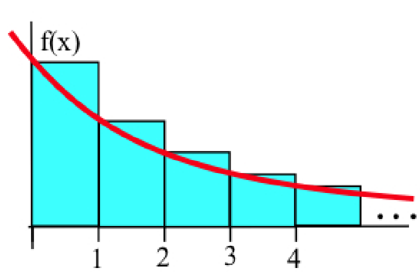


Example 1. If you can paint the shaded region in margin figure using 3 gallons of paint, how much paint do you need to paint all of the shaded rectangles in the lower margin figure?

Solution. We don't have enough information to determine the exact amount of paint needed for the rectangles, but the sum of the rectangular areas is smaller than the area in under the graph in the original margin figure, so we *can* say that we need less than 3 gallons of paint are needed for the rectangles. (We just can't say how much less.) ◀

Practice 1. If the area of the shaded region below left is infinite, what can you say about the total area of the rectangular regions below right?

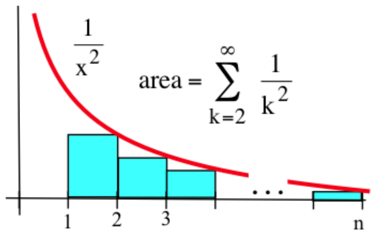


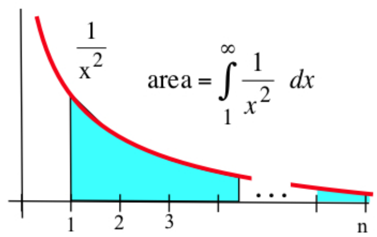


Example 2. Determine which is larger:

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \quad \text{or} \quad \int_{2}^{\infty} \frac{1}{x^2}\,dx$$

and use the result to show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges.





Solution. The margin figure illustrates that the area of the rectangles:

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots$$

is less than the area under the graph of the function $f(x) = \dfrac{1}{x^2}$ for $1 \le x \le n$ so that:

$$s_n = \sum_{k=2}^{n} \frac{1}{k^2} < \int_1^n \frac{1}{x^2}\,dx \le \int_1^{\infty} \frac{1}{x^2}\,dx$$

where the second inequality holds because $f(x) = \dfrac{1}{x^2} > 0$. Evaluating the improper integral:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{M\to\infty} \int_1^M x^{-2}\,dx = \lim_{M\to\infty} \left[-\frac{1}{x}\right]_1^M = \lim_{M\to\infty} \left[-\frac{1}{M} + 1\right] = 1$$

tells us that, for any integer n with $n \ge 2$, $s_n = \displaystyle\sum_{k=2}^{n} \frac{1}{k^2} \le 1$, so that the sequence of partial sums $\{s_n\}$ is bounded above. That sequence is also monotonically increasing because:

$$s_{n+1} = s_n + \frac{1}{(n+1)^2} > s_n$$

for any $n \ge 2$. The Monotone Convergence Theorem from Section 9.2 therefore tells us that $\{s_n\}$ converges, so $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges. ◀

Although we now know that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, the argument in Example 2 tells us nothing about the *sum* of the series (other than it is less than 1). It turns out the actual sum of the series is:

$$\sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1 \approx 0.644934$$

but this is not easy to prove.

The reasoning of Example 2 extends to a more general result comparing the convergence of infinte series to the convergence of certain improper integrals.

Integral Test

If f is a continuous, positive, decreasing function on $[1, \infty)$ and $a_k = f(k)$ for $k \ge 1$, then:

- $\displaystyle\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$ converges
- $\displaystyle\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$ diverges

Proof. Assume that f is a continuous, positive, decreasing function on $[1,\infty)$ and that $a_k = f(k)$ for $k \geq 1$.

To prove the first implication, assume that $\int_1^\infty f(x)\,dx$ converges so that:

The proof of the first statement mirrors the argument used in Example 2.

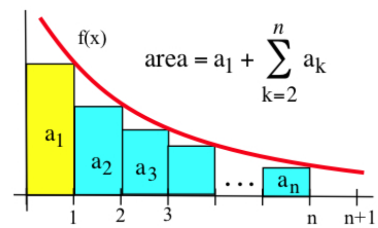
$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx = L$$

for some finite number L. Because $a_k = f(k)$ for $k \geq 1$ and because $f(x)$ is decreasing, we know that for $1 \leq x \leq 2$:

$$a_2 = f(2) \leq f(x) \;\Rightarrow \int_1^2 a_2\,dx \leq \int_1^2 f(x)\,dx \;\Rightarrow a_2 \leq \int_1^2 f(x)\,dx$$

Using similar reasoning for $k \geq 2$ we can conclude that:

$$a_k \leq \int_{k-1}^k f(x)\,dx \;\Rightarrow \sum_{k=2}^n a_k \leq \sum_{k=2}^n \int_{k-1}^k f(x)\,dx = \int_1^n f(x)\,dx$$

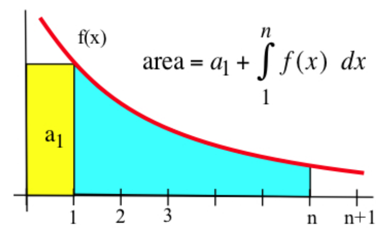


Adding the finite number a_1 to each side of this inequality yields:

$$s_n = \sum_{k=1}^n a_k = a_1 + \sum_{k=2}^n a_k \leq a_1 + \int_1^n f(x)\,dx$$

(see margin). Because $f(x) > 0$, $\int_1^n f(x)\,dx$ increases as n increases, so:

$$s_n \leq a_1 + \int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx = L$$



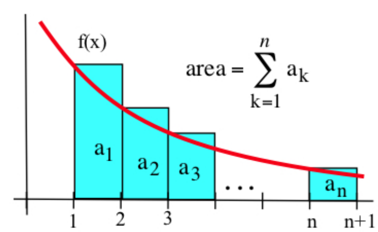
This tells us that the sequence $\{s_n\}$ is bounded above. Because $f(x) > 0$ we know that $a_{n+1} = f(n+1) > 0$ for any $n \geq 1$, so:

$$s_{n+1} = a_{n+1} + s_n > s_n$$

which means that $\{s_n\}$ is an increasing sequence. The Monotone Convergence Theorem then tells us that $\{s_n\}$ is a convergent sequence, so $\sum_{k=1}^\infty a_k$ must converge.

Now assume that $\int_1^\infty f(x)\,dx$ diverges. Because $a_k = f(k)$ for $k \geq 1$ and because $f(x)$ is decreasing, we know that for $1 \leq x \leq 2$:

$$a_1 = f(1) \geq f(x) \;\Rightarrow \int_1^2 a_1\,dx \geq \int_1^2 f(x)\,dx \;\Rightarrow a_1 \geq \int_1^2 f(x)\,dx$$

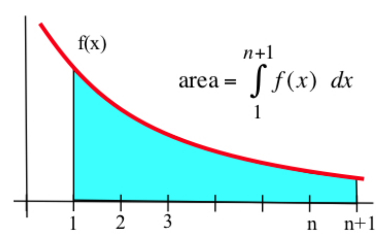


Using similar reasoning for $k \geq 1$ (see margin) we can conclude that:

$$a_k \geq \int_k^{k+1} f(x)\,dx \;\Rightarrow s_n = \sum_{k=1}^n a_k \geq \sum_{k=1}^n \int_k^{k+1} f(x)\,dx = \int_1^{n+1} f(x)\,dx$$

Because $\int_1^\infty f(x)\,dx$ diverges, we know that:

$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = \int_1^\infty f(x)\,dx = \infty$$



so $\{s_n\}$ is a divergent sequence and $\sum_{k=1}^\infty a_k$ must diverge. □

Altering the first few terms — or even the first million terms — of an infinite series does not affect whether that series converges or diverges (although changing these terms likely will affect the sum of the series). With this in mind, we can relax the hypotheses of the integral test.

Integral Test Corollary

If f is a continuous, positive, decreasing function on $[N, \infty)$ and $a_k = f(k)$ for $k \geq N$, then

- $\displaystyle\int_N^\infty f(x)\,dx$ converges $\Rightarrow \displaystyle\sum_{k=1}^\infty a_k$ converges
- $\displaystyle\int_N^\infty f(x)\,dx$ diverges $\Rightarrow \displaystyle\sum_{k=1}^\infty a_k$ diverges

Example 3. Use the Integral Test to determine whether the infinite series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^3}$ and $\displaystyle\sum_{k=2}^\infty \frac{1}{k \cdot \ln(k)}$ converge or diverge.

Solution. For the first series, let $f(x) = \dfrac{1}{x^3}$ so that $a_k = \dfrac{1}{k^3} = f(k)$ for $k \geq 1$. When $x \geq 1$, $f(x) = x^{-3} > 0$ and $f'(x) = -3x^{-2} < 0$, so $f(x)$ is continuous, positive and decreasing on $[1, \infty)$, as required by the Integral Test. Furthermore:

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{M\to\infty} \int_1^M x^{-3}\,dx = \lim_{M\to\infty} \left[-\frac{1}{2x^2}\right]_1^M = \lim_{M\to\infty} \left[-\frac{1}{2M^2} + \frac{1}{2}\right]$$

Although the Integral Test tells us that the series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^3}$ converges, it does *not* tell us the sum of that series.

This limit is $\dfrac{1}{2}$, so $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ converges, hence $\displaystyle\sum_{k=1}^\infty \frac{1}{k^3}$ also converges.

Next, let $g(x) = \dfrac{1}{x \cdot \ln(x)}$ so that $a_k = \dfrac{1}{k \cdot \ln(k)} = g(k)$ for $k \geq 2$. When $x \geq 2$, $g(x) > 0$ and:

$$g'(x) = \frac{[x \cdot \ln(x)] \cdot 0 - 1 \cdot \left[x \cdot \frac{1}{x} + \ln(x) \cdot 1\right]}{[x \cdot \ln(x)]^2} = \frac{-x - \ln(x)}{x^2 \cdot [\ln(x)]^2} < 0$$

We use the Integral Test Corollary because a_1 is undefined for $a_k = \dfrac{1}{k \cdot \ln(k)}$.

so $g(x)$ is continuous, positive and decreasing on $[2, \infty)$, as required. Furthermore, using the substitution $u = \ln(x) \Rightarrow du = \dfrac{1}{x}\,dx$:

$$\int \frac{1}{x \cdot \ln(x)}\,dx = \int \frac{1}{u}\,du = \ln(|u|) + C = \ln(|\ln(x)|) + C$$

so that:

$$\int_2^\infty \frac{1}{x \cdot \ln(x)}\,dx = \lim_{M\to\infty} \Big[\ln(|\ln(x)|)\Big]_2^M = \lim_{M\to\infty} \Big[\ln(\ln(M)) - \ln(\ln(2))\Big]$$

which diverges to ∞, so $\displaystyle\sum_{k=2}^\infty \frac{1}{k \cdot \ln(k)}$ must also diverge. ◀

Practice 2. Determine whether $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}}$ and $\sum_{k=1}^{\infty} e^{-k}$ converge or diverge.

The P-Test

In the preceding Examples and Practice problems, you have observed that $\sum_{k=2}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^3}$ both converge, while $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=4}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ diverges. These series all have the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ for some power p. Rather than continuing to employ the Integral Test every time we encounter a series of this form, we can instead develop a test that applies to all such series.

P-Test: The "p-series" $\sum_{k=1}^{\infty} \frac{1}{k^p}$:

- converges if $p > 1$
- diverges if $p \leq 1$

Proof. If $p \leq 0$, $\lim_{k \to \infty} \frac{1}{k^p} \neq 0$, so the Test for Divergence shows $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges. If $p = 1$, $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series, which we already know diverges (from Section 9.4). If $p > 0$, let $f(x) = \frac{1}{x^p} = x^{-p}$ so that $f(k) = \frac{1}{k^p}$ for any integer $k \geq 1$. When $x \geq 1$, $f(x) > 0$ and:

$$f'(x) = -p \cdot x^{-p-1} = \frac{-p}{x^{p+1}} < 0$$

so $f(x)$ is continuous, positive and decreasing, as required by the Integral Test. Furthermore, for $p \neq 1$:

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{M \to \infty} \int_1^M x^{-p}\, dx = \lim_{M \to \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^M$$
$$= \lim_{M \to \infty} \left[\frac{M^{1-p}}{1-p} - \frac{1}{1-p} \right]$$

This limit converges when $p > 1$ and diverges when $0 < p < 1$, so $\sum_{k=1}^{\infty} \frac{1}{k^p}$ also converges when $p > 1$ and diverges when $p \leq 1$. □

Example 4. Do $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}}$ converge or diverge?

Solution. For $\sum_{k=1}^{\infty} \frac{1}{k^2}$, $p = 2 > 1$ so the series converges; for $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}}$, $p = \frac{1}{2} \leq 1$, so the series diverges. ◀

Like the Integral Test, the P-Test tells us whether or not a particular series converges, but it does *not* give the value of a convergent infinite series.

Estimating Sums

As mentioned several times so far, neither the Integral Test nor the P-Test can tell us the value of an infinite sum — these tests only tell us whether certain series converge or diverge. But some of the concepts we used to prove the Integral Test *can* help us estimate how close a partial sum comes to approximating the value of an infinite series.

If $f(x) > 0$ is continuous and decreasing on $[1, \infty)$ with $a_k = f(k)$:

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k \le a_1 + \int_1^n f(x)\,dx$$

For $a_k = \frac{1}{k^2}$ and $n = 1000$ we can say that:

Evaluating these improper integrals requires much less arithmetic than computing s_{1000}.

$$\int_1^{1001} \frac{1}{x^2}\,dx \le \sum_{k=1}^{1000} \frac{1}{k^2} \le \frac{1}{1^2} + \int_1^{1000} \frac{1}{x^2}\,dx$$

Evaluating these integrals reveals that:

$$0.999 \le \sum_{k=1}^{1000} \frac{1}{k^2} \le 1.999$$

Unfortunately, this is does not provide a very accurate estimate of s_{1000}. Letting $n \to \infty$ in the first inequality above:

$$\int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} f(x)\,dx$$

For $a_k = \frac{1}{k^2}$, this tells us that

This merely provides us with a "ballpark" estimate for $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$1 = \int_1^{\infty} \frac{1}{x^2}\,dx \le \sum_{k=1}^{\infty} \frac{1}{k^2} \le \frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$$

If we start the approximation at $k = n+1$ instead of $k = 1$:

$$\int_{n+1}^{\infty} f(x)\,dx \le \sum_{k=n+1}^{\infty} a_k \le a_{n+1} + \int_{n+1}^{\infty} f(x)\,dx$$

Adding s_n to each expression in this inequality yields:

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \le s_{n+1} + \int_{n+1}^{\infty} f(x)\,dx$$

For $a_k = \frac{1}{k^2}$ and $n = 10$ we can compute:

$$s_{10} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{10^2} \approx 1.5497677$$

so that $s_{11} = s_{10} + \frac{1}{11^2} \approx 1.5580322$ and:

$$\int_{11}^{\infty} \frac{1}{x^2}\,dx = \lim_{M\to\infty} \left[-\frac{1}{x}\right]_{11}^{M} = \lim_{M\to\infty} \left[-\frac{1}{M} + \frac{1}{11}\right] = \frac{1}{11}$$

Therefore:

$$1.6407 \approx s_{10} + \int_{11}^{\infty} \frac{1}{x^2}\,dx \le \sum_{k=1}^{\infty} \frac{1}{k^2} \le s_{11} + \int_{11}^{\infty} \frac{1}{x^2}\,dx \approx 1.6489$$

Advanced mathematical techniques can be used to show that:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.6449$$

which is consistent with this estimate.

Practice 3. Find an upper and lower bound for $a_k = \frac{1}{k^3}$ using s_{10}.

9.6 Problems

In Problems 1–16, use the Integral Test to determine whether the series converges or diverges. (Be sure to verify the hypotheses of the Integral Test hold.)

1. $\sum_{k=1}^{\infty} \frac{1}{2k+5}$
2. $\sum_{k=1}^{\infty} \frac{1}{(2k+5)^2}$
3. $\sum_{k=1}^{\infty} \frac{1}{(2k+5)^{\frac{3}{2}}}$
4. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k}$
5. $\sum_{k=2}^{\infty} \frac{1}{k \cdot [\ln(k)]^2}$
6. $\sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \sin\left(\frac{1}{k}\right)$
7. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$
8. $\sum_{k=1}^{\infty} \frac{1}{k^2+100}$
9. $\sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+3}\right]$
10. $\sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right]$
11. $\sum_{k=1}^{\infty} \frac{1}{k(k+5)}$
12. $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$
13. $\sum_{k=1}^{\infty} k \cdot e^{-k^2}$
14. $\sum_{k=1}^{\infty} k^2 \cdot e^{-k^3}$
15. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{6k+10}}$
16. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$

In Problems 17–28, use the P-Test to determine whether the series converges or diverges.

17. $\sum_{k=1}^{\infty} \frac{1}{k^4}$
18. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$
19. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[5]{k}}$
20. $\sum_{k=1}^{\infty} \frac{1}{k^5}$
21. $\sum_{k=1}^{\infty} \frac{1}{k}$
22. $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}}$
23. $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$
24. $\sum_{k=1}^{\infty} \frac{1}{k^e}$
25. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt[3]{k}}$
26. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^4}}$
27. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}}$
28. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[5]{k^4}}$

In Problems 29–34, use the inequality:

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k \le a_1 + \int_1^n f(x)\,dx$$

to estimate s_{10}, s_{100} and $s_{1000000}$ for the series.

29. $\sum_{k=1}^{\infty} \frac{1}{k^3}$
30. $\sum_{k=1}^{\infty} \frac{1}{k^4}$
31. $\sum_{k=1}^{\infty} \frac{1}{k}$
32. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$
33. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$
34. $\sum_{k=1}^{\infty} \frac{1}{k^2+100}$

In Problems 35–40, use the inequality:

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \le s_{n+1} + \int_{n+1}^{\infty} f(x)\,dx$$

with $n = 10$ and $n = 20$ to estimate the sum.

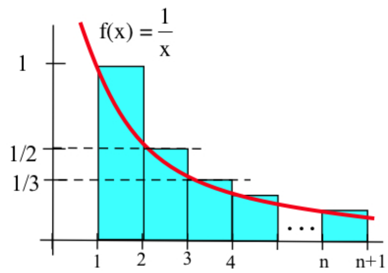
35. $\sum_{k=1}^{\infty} \frac{1}{k^4}$
36. $\sum_{k=1}^{\infty} \frac{1}{k}$
37. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$
38. $\sum_{k=1}^{\infty} \frac{1}{k^2+9}$
39. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$
40. $\sum_{k=1}^{\infty} \frac{1}{2^k}$
41. Show that $\int_2^{\infty} \frac{1}{x \cdot [\ln(x)]^q}\,dx$ converges for $q > 1$ and diverges for $q \le 1$, then use this result to state a "Q-Test" for the series $\sum_{k=2}^{\infty} \frac{1}{k \cdot [\ln(k)]^q}$.

In Problems 42–45, use the result for Problem 41 to determine whether the series converges or diverges.

42. $\sum_{k=2}^{\infty} \frac{1}{k \cdot \ln(k)}$
43. $\sum_{k=2}^{\infty} \frac{1}{k \cdot [\ln(k)]^3}$
44. $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln(k)}}$
45. $\sum_{k=2}^{\infty} \frac{1}{k \cdot \ln(k^3)}$

46. For $n \geq 1$, define $g_n = \left[\sum_{k=1}^{n} \frac{1}{k}\right] - \ln(n)$ so that:

$$g_1 = 1 - \ln(1) = 1$$
$$g_2 = 1 + \frac{1}{2} - \ln(2) \approx 0.806853$$
$$g_2 = 1 + \frac{1}{2} + \frac{1}{3} - \ln(3) \approx 0.734721$$



(a) Make several copies of the margin figure and shade the regions represented by g_2, g_3, g_4 and g_n.

(b) Provide a geometric argument that $g_n > 0$ for all $n \geq 1$.

(c) Provide a geometric argument that $\{g_n\}$ is monotonically decreasing; that is: $g_{n+1} < g_n$ for all $n \geq 1$.

(d) Conclude from your geometric results and the Monotone Convergence Theorem that $\{g_n\}$ converges.

The value to which $\{g_n\}$ converges, denoted by γ (the lowercase Greek letter gamma) is called **Euler's constant**. Although $\gamma \approx 0.5772157$ has been approximated to over 100,000 digits, no one (yet) knows whether or not γ is a rational number.

9.6 Practice Answers

1. The area of the rectangles exceeds the area under the curve, which is infinite, so the total area of the rectangles must be infinite.

2. For the first series, let $f(x) = x^{-\frac{1}{2}}$. For $x \geq 1$, $f(x)$ is continuous, $f(x) > 0$ and $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} < 0$ (so $f(x)$ is decreasing), while $f(k) = \frac{1}{\sqrt{k}}$ for all integers $k \geq 1$. Furthermore:

$$\int_1^{\infty} x^{-\frac{1}{2}}\, dx = \lim_{M\to\infty} \left[2\sqrt{x}\right]_1^M = \lim_{M\to\infty} \left[2\sqrt{M} - 2\right] = \infty$$

This is a p-series, with $p = \frac{1}{2} < 1$, hence it must diverge.

Because this improper integral diverges, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ also diverges.

For the second series, let $g(x) = e^{-x}$. For any x, $g(x)$ is continuous, $g(x) > 0$ and $g'(x) = -e^{-x} < 0$, so $g(x)$ is decreasing, while $g(k) = e^{-k}$ for all integers k. Furthermore:

$$\int_1^{\infty} e^{-x}\, dx = \lim_{M\to\infty} \left[-e^{-x}\right]_1^M = \lim_{M\to\infty} \left[-\frac{1}{e^M} + \frac{1}{e}\right] = \frac{1}{e}$$

This is actually a geometric series with ratio $e^{-1} < 1$, hence it converges to:

$$\frac{1}{1 - e^{-1}} - 1 = \frac{1}{e - 1} \approx 0.582$$

Because this improper integral converges, $\sum_{k=1}^{\infty} e^{-k}$ also converges.

3. $1.197532 + \frac{1}{242} \leq \sum_{k=1}^{\infty} \frac{1}{k^3} \leq 1.198283 + \frac{1}{242} \Rightarrow 1.20166 < \sum_{k=1}^{\infty} \frac{1}{k^3} < 1.20242$