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DALE HOFFMAN

CONTEMPORARY CALCULUS

Contents

4
The Integral

Previous chapters dealt with **differential calculus**. They started with the "simple" geometrical idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, examined techniques for calculating derivatives, and applied these concepts and techniques to real-life situations. This chapter begins the development of **integral calculus** and starts with the "simple" geometric idea of **area**—an idea that will spawn its own combination of theory, techniques and applications.

One of the most important results in mathematics, the Fundamental Theorem of Calculus, appears in this chapter. It unifies differential and integral calculus into a single grand structure. Historically, this unification marked the beginning of modern mathematics, and it provided important tools for the growth and development of the sciences.

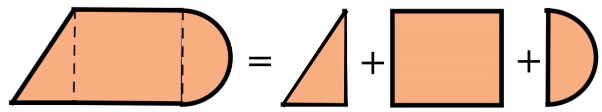
The chapter begins with a look at area, some geometric properties of areas, and some applications. First we will examine ways of approximating the areas of regions such as tree leaves bounded by curved edges and the areas of regions bounded by graphs of functions. Then we will develop ways to calculate the areas of some of these regions exactly. Finally, we will explore the rich variety of uses of "areas."

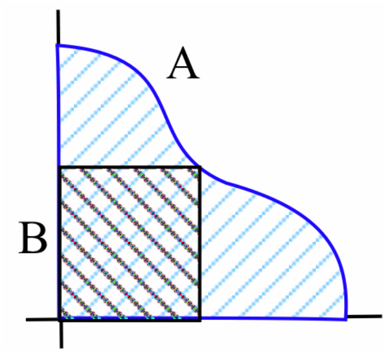
4.0 Area

The primary purpose of this introductory section is to help develop your intuition about areas and your ability to reason using geometric arguments about area. This type of reasoning will appear often in the rest of this book and is very helpful for applying the ideas of calculus.

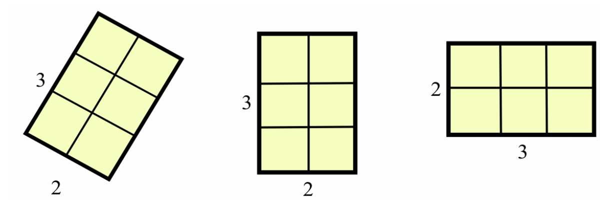
The basic shape we will use is the rectangle: the area of a rectangle is (base) $\cdot$ (height). If the units for each side of the rectangle are "meters," then the area will have units (meters) $\cdot$ (meters) $=$ "square meters" $= \text{m}^2$. The only other area formulas needed for this section are for triangles (area $= \frac{1}{2}b \cdot h$) and for circles (area $= \pi \cdot r^2$). In addition, we will use (and assume to be true) three other familiar properties of area:

- **Addition Property**: The total area of a region is the sum of the areas of the non-overlapping pieces that comprise the region:

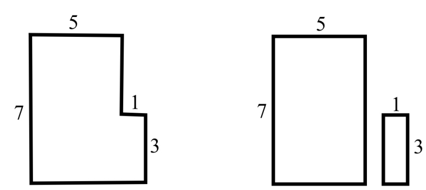




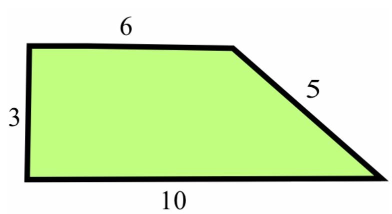
- **Inclusion Property**: If region B is inside region A (see margin), then the area of region B is less than or equal to the area of region A.
- **Location-Independence Property**: The area of a region does not depend on its location:



Example 1. Determine the area of the region shown below left.

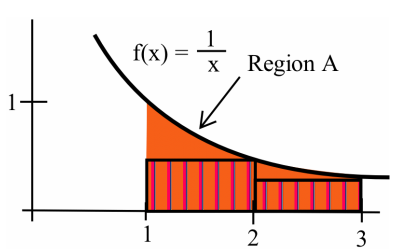


Solution. We can easily break the region into two rectangles (shown above right), with areas of 35 square inches and 3 square inches respectively, so the area of the original region is 38 square inches. ◀



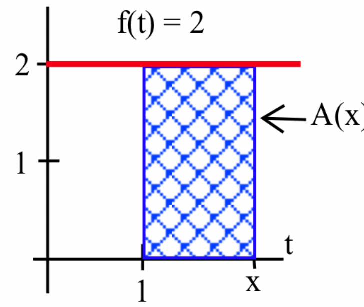
Practice 1. Determine the area of the trapezoidal region shown in the margin by cutting it in two ways: (a) into a rectangle and triangle and (b) into two triangles.

We can use our three area properties to deduce information about areas that are difficult to calculate exactly. Let A be the region bounded by the graph of $f(x) = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 3$. Because the two rectangles in the margin figure sit inside region A and do not overlap each other, the area of the rectangles, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, is less than the area of region A.

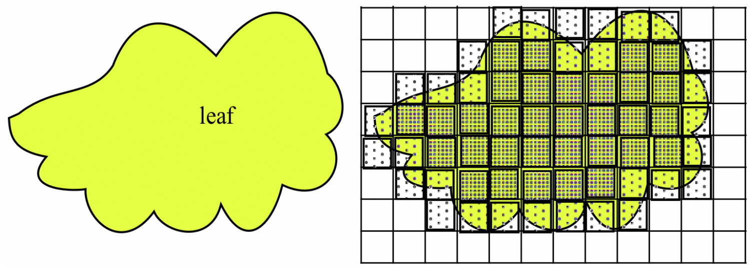


Practice 2. Build two rectangles, each with base 1 unit, with boundaries that extend outside the shaded region in the margin figure and use their areas to make another valid statement about the area of region A.

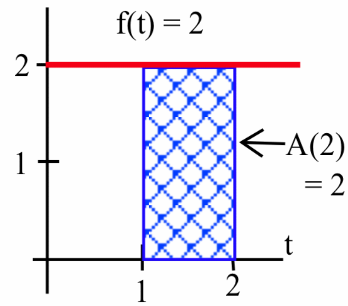
Practice 3. What can you say about the area of region A if we use "inside" and "outside" rectangles each with base $\frac{1}{2}$ unit?



Example 2. The figure below right includes 32 dark squares, each 1 centimeter on a side, and 31 lighter squares of the same size. We can be sure that the area of the leaf below left is smaller than what number?

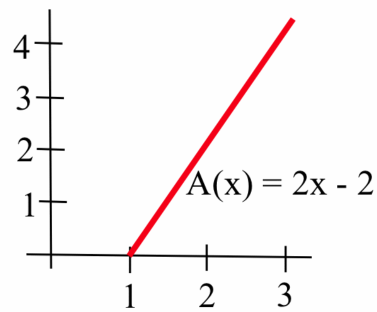


Solution. The area of the leaf is smaller than $32 + 31 = 63\,\text{cm}^2$. ◀



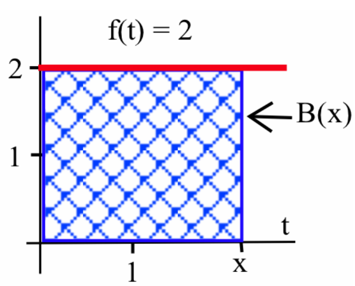
Practice 4. We can be sure that the area of the leaf is at least how large?

Functions can be defined in terms of areas. For the constant function $f(t) = 2$, define $A(x)$ to be the area of the rectangular region (top margin figure) bounded by the graph of f, the t-axis, and the vertical lines at $t = 1$ and $t = x$; we can easily see that $A(2) = 2$ (shaded region in the second margin figure). Similarly, $A(3) = 4$ and $A(4) = 6$. In general, $A(x) = (\text{base})(\text{height}) = (x - 1)(2) = 2x - 2$ for any $x \geq 1$. From the graph of $y = A(x)$ (in the third margin figure) we can see that $A'(x) = 2$ for every value of $x > 1$.



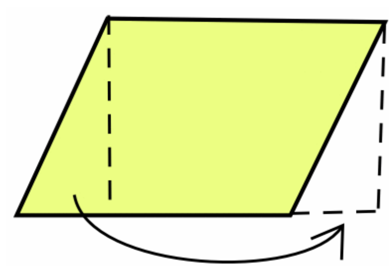
(The fact that $A'(x) = f(x)$ in the preceding discussion is not a coincidence, as we shall soon learn.)

Practice 5. For $f(t) = 2$, define $B(x)$ to be the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = 0$ and $t = x$ (see below left). Fill in the table below with the requested values of B. How are the graphs of $y = A(x)$ and $y = B(x)$ related?

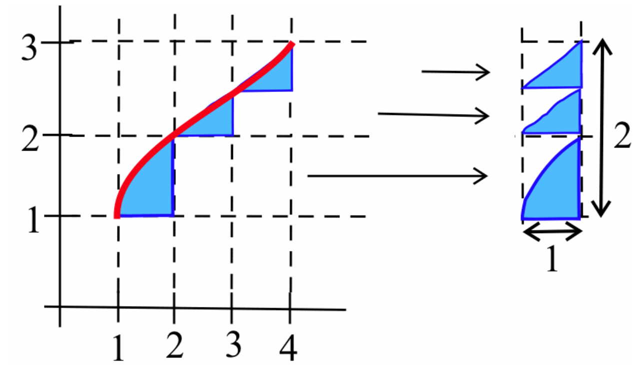


x	$B(x)$
0	
0.5	
1	
2	

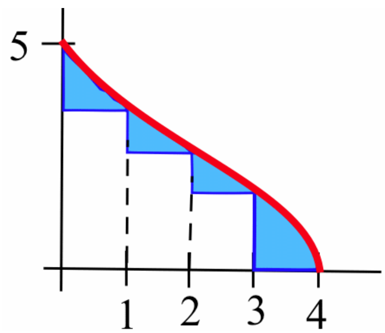
Sometimes it is useful to move regions around. The area of a parallelogram is obvious if we move the triangular region from one side of the parallelogram to fill the region on the other side, resulting in with a rectangle (see margin).



At first glance, it is difficult to estimate the total area of the shaded regions shown below left:



but if we slide all of them into a single column (above right), then becomes easy to determine that the shaded area is less than the area of the enclosing rectangle = (base)(height) = (1)(2) = 2.

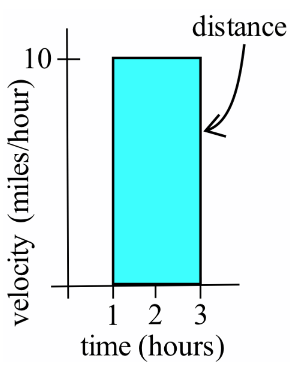


Practice 6. The total area of the shaded regions in the margin figure is less than what number?

Some Applications of "Area"

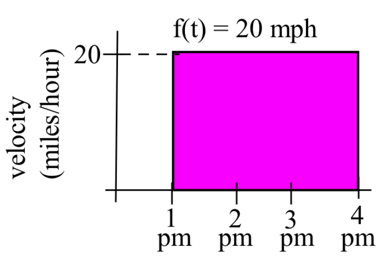
One reason "areas" are so useful is that they can represent quantities other than sizes of simple geometric shapes. For example, if the units of the base of a rectangle are "hours" and the units of the height are "$\frac{\text{miles}}{\text{hour}}$," then the units of the "area" of the rectangle are:

$$(\text{hours}) \cdot \left(\frac{\text{miles}}{\text{hour}}\right) = \text{miles}$$



a measure of distance. Similarly, if the base units are "pounds" and the height units are "feet," then the "area" units are "foot-pounds," a measure of work.

In the bottom margin figure, $f(t)$ is the velocity of a car in "miles per hour," and t is the time in "hours." So the shaded "area" will be $(\text{base}) \cdot (\text{height}) = (3\,\text{hours}) \cdot \left(20\,\frac{\text{miles}}{\text{hour}}\right) = 60$ miles, the distance traveled by the car in the 3 hours from 1:00 p.m. until 4:00 p.m.

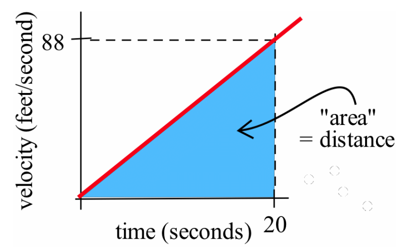


> **Distance as an "Area"**
> If $f(t)$ is the (positive) forward velocity of an object at time t, then the "area" between the graph of f and the t-axis and the vertical lines at times $t = a$ and $t = b$ will equal the distance that the object has moved forward between times a and b.

This "area as distance" concept can make some difficult distance problems much easier.

Example 3. A car starts from rest (velocity $= 0$) and steadily speeds up so that 20 seconds later its speed is 88 feet per second (60 miles per hour). How far did the car travel during those 20 seconds?

Solution. We could answer the question using the techniques of Chapter 3 (try this). But if "steadily" means that the velocity increases linearly, then it is easier to use the margin figure and the concept of "area as distance."



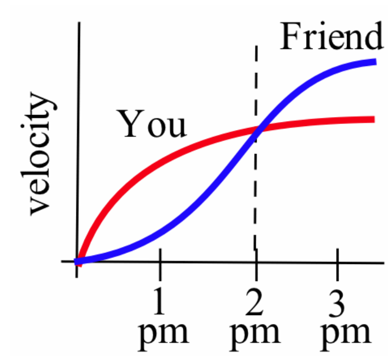
The "area" of the triangular region represents the distance traveled:

$$\text{distance} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(20\,\text{sec})\left(88\,\frac{\text{ft}}{\text{sec}}\right) = 880\,\text{ft}$$

The car travels a total of 880 feet during those 20 seconds. ◀

Practice 7. A train initially traveling at 45 miles per hour (66 feet per second) takes 60 seconds to decelerate to a complete stop. If the train slowed down at a steady rate (the velocity decreased linearly), how many feet did the train travel before coming to a stop?

Practice 8. You and a friend start off at noon and walk in the same direction along the same path at the rates shown in the margin figure.

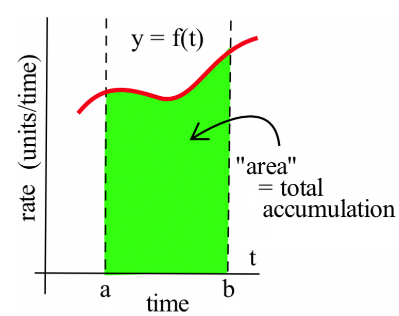


(a) Who is walking faster at 2:00 p.m.? Who is ahead at 2:00 p.m.?

(b) Who is walking faster at 3:00 p.m.? Who is ahead at 3:00 p.m.?

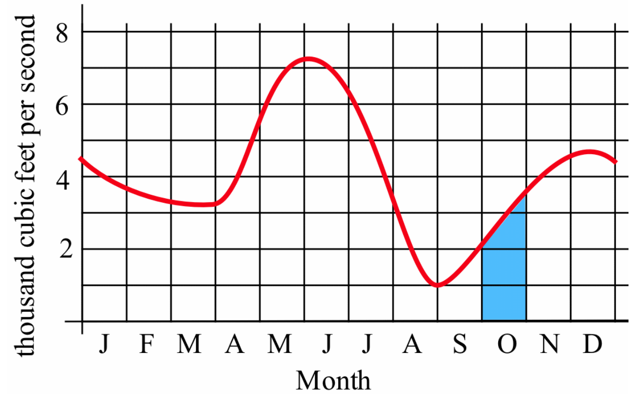
(c) When will you and your friend be together? (Answer in words.)

In the preceding Example and Practice problems, a function represented a rate of travel (in miles per hour, for instance) and the area represented the total distance traveled. For functions representing other rates, such as the production of a factory (bicycles per day) or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute) or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.



> **"Area" as a Total Accumulation**
> If $f(t)$ represents a positive rate (in units per time interval) at time t, then the "area" between the graph of f and the t-axis and the vertical lines at times $t = a$ and $t = b$ will be the total amount of {something} that accumulates between times a and b (see margin).

For example, the figure at the top of the next page shows the flow rate (in cubic feet per second) of water in the Skykomish River near the town of Gold Bar, Washington. The area of the shaded region represents the total volume (cubic feet) of water flowing past the town during the month of October:

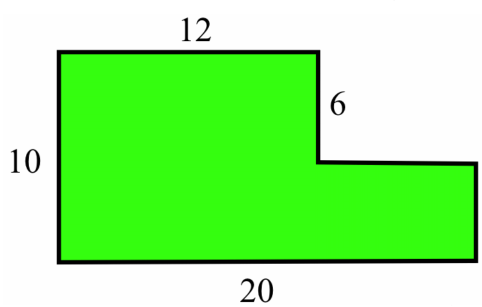


$$\begin{aligned}
\text{total water} &= \text{"area"} = \text{area of rectangle} + \text{area of triangle} \\
&\approx \left(2000\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) + \frac{1}{2}\left(1500\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) \\
&= \left(2750\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) = \left(2750\,\frac{\text{ft}^3}{\text{sec}}\right)(2592000\,\text{sec}) \\
&\approx 7.128 \times 10^9\,\text{ft}^3
\end{aligned}$$

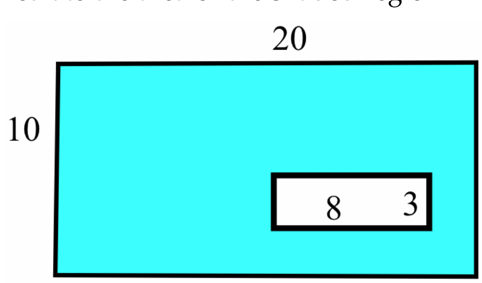
For comparison, the flow over Niagara Falls is about $2.12 \times 10^5\,\frac{\text{ft}^3}{\text{sec}}$.

4.0 Problems

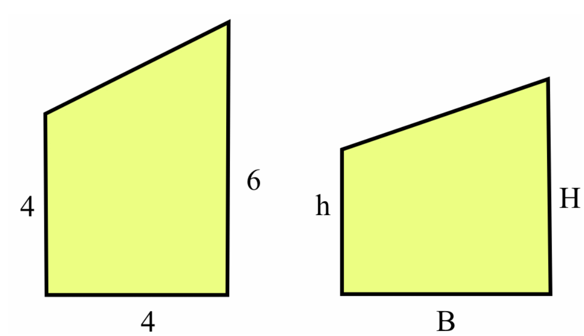
1. (a) Calculate the area of the shaded region:



(b) Calculate the area of the shaded region:

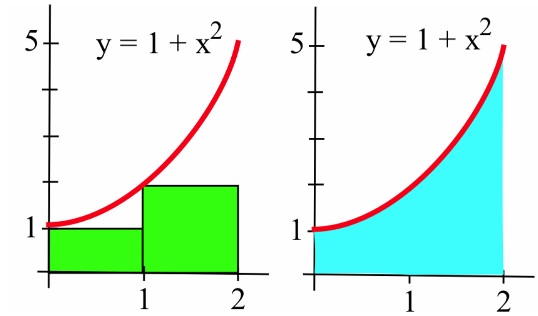


2. Calculate the area of the trapezoidal region in the figure below left by breaking it into a triangle and a rectangle.



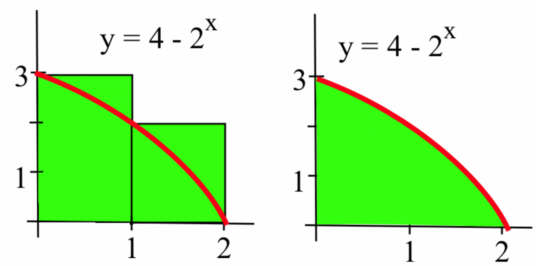
3. Break the region shown above right into a triangle and rectangle and verify that the total area of the trapezoid is $b \cdot \left(\frac{h+H}{2}\right)$.

4. (a) Calculate the sum of the rectangular areas in the region shown below left.



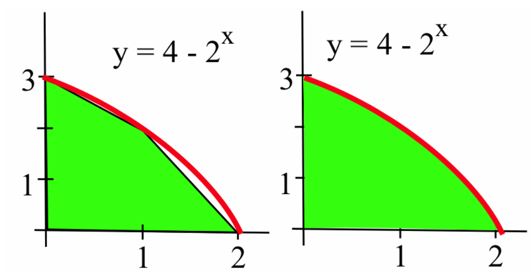
(b) What can you say about the area of the shaded region shown above right?

5. (a) Calculate the sum of the areas of the rectangles shown below left.



(b) What can you say about the area of the shaded region shown above right?

6. (a) Calculate the sum of the areas of the trapezoids shown below left.

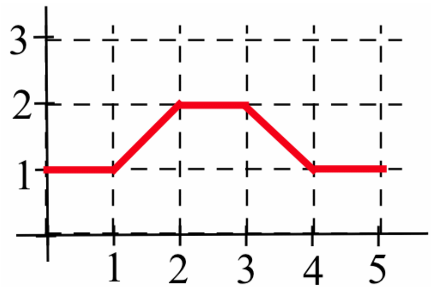


(b) What can you say about the area of the shaded region shown above right?

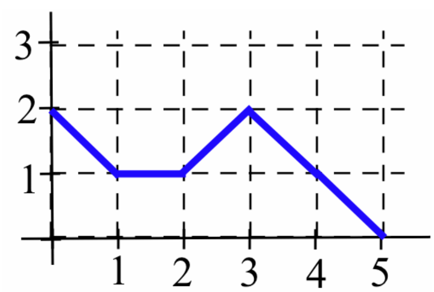
7. Consider the region bounded by the graph of $y = 2 + x^3$, the positive x-axis, the positive y-axis and the line $x = 2$. Use two well-placed rectangles to estimate the area of this region.

8. Consider the region bounded by the graph of $y = 9 - 3^x$, the positive x-axis and the positive y-axis. Use two well-placed trapezoids to estimate the area of this region.

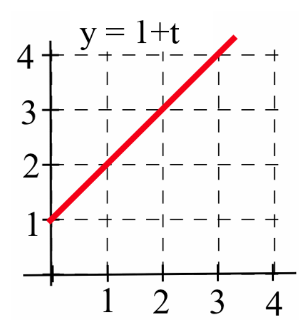
9. Let $A(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $A(x)$ for $x = 1, 2, 3, 4$ and 5.



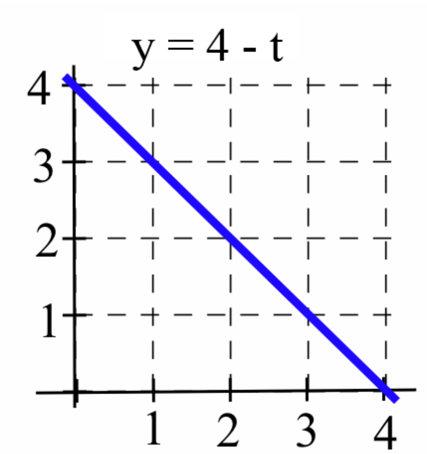
10. Let $B(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $B(x)$ for $x = 1, 2, 3, 4$ and 5.



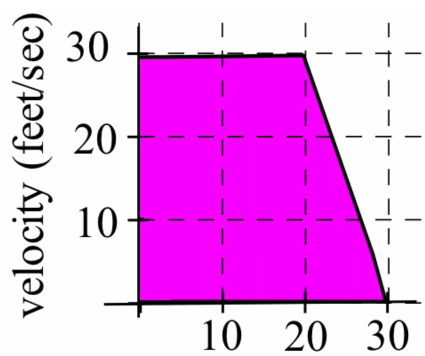
11. Let $C(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $C(x)$ for $x = 1, 2$ and 3, and use that information to deduce a formula for $C(x)$.



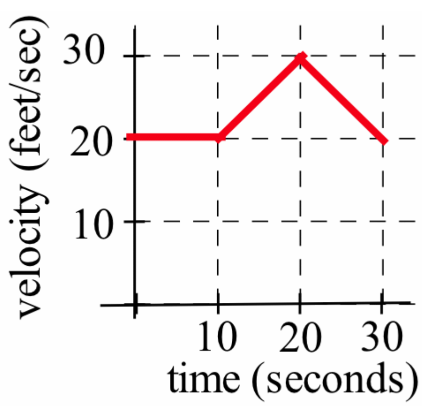
12. Let $A(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $A(x)$ for $x = 1$, 2 and 3, and find a formula for $A(x)$.



13. The figure below shows the velocity of a car during a 30-second time frame. How far did the car travel between $t = 0$ to $t = 30$ seconds?



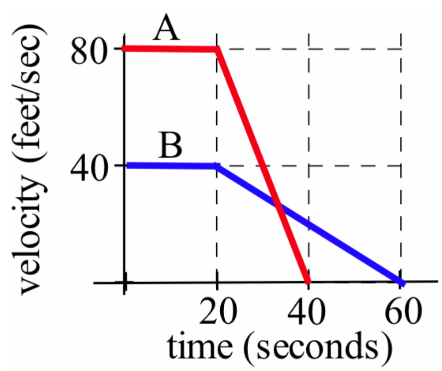
14. The figure below shows the velocity of a car during a 30-second time frame. How far did the car travel between $t = 0$ to $t = 30$ seconds?



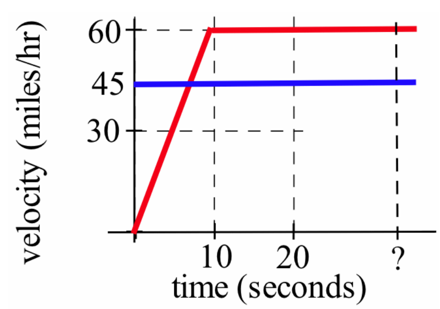
15. The figure below shows the velocities of two cars. From the time the brakes were applied:

(a) how long did it take each car to stop?

(b) which car traveled farther before stopping?



16. A speeder traveling 45 miles per hour (in a 25-mph zone) passes a stopped police car, which immediately takes off after the speeder. If the police car speeds up steadily to 60 mph over a 10-second interval and then travels at a constant 60 mph, how long—and how far—will it be before the police car catches the speeder, who continued traveling at 45 mph? (See figure below.)

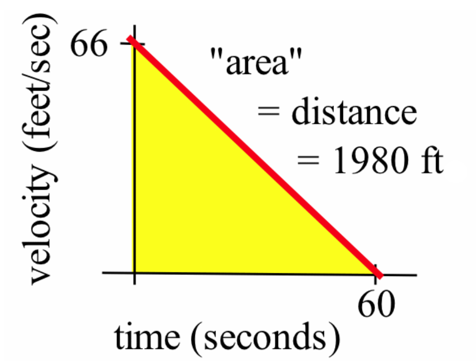


17. Fill in the table with the units for "area" of a rectangle with the given base and height units.

base	**height**	**"area"**
miles per second	seconds	
hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

4.0 Practice Answers

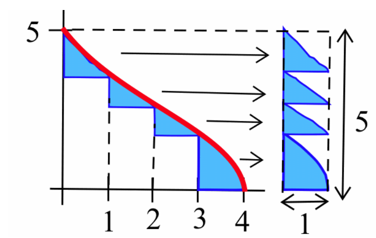
1. (a) $3(6) + \frac{1}{2}(4)(3) = 24$ (b) $\frac{1}{2}(3)(10) + \frac{1}{2}(6)(3) = 24$
2. outside rectangular area $= (1)(1) + (1)\left(\frac{1}{2}\right) = 1.5$
3. Using rectangles with base $= \frac{1}{2}$:
$$\text{inside area} = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3}\right) = \frac{57}{60} \approx 0.95$$
$$\text{outside area} = \frac{1}{2}\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) = \frac{72}{60} = 1.2$$
so the area of the region is between 0.95 and 1.2.
4. The leaf's area is larger than the area of the dark rectangles, $32\,\text{cm}^2$.
5. $y = B(x) = 2x$ is a line with slope 2, so it is parallel to the line $y = A(x) = 2x - 2$; see margin for table.
6. Area $<$ area of the rectangle enclosing the shifted regions $= 5$; see margin figure.
7. Draw a graph of the velocity function:



and then use the concept of "area as distance":
$$\begin{aligned}\text{distance} &= \text{area of shaded region}\\ &= \frac{1}{2}(\text{base})(\text{height})\\ &= \frac{1}{2}(60\,\text{sec})\left(66\,\frac{\text{ft}}{\text{sec}}\right) = 1980\,\text{feet}\end{aligned}$$

8. (a) At 2:00 p.m. both are walking at the same velocity. You are ahead.
   (b) At 3:00 p.m. your friend is walking faster than you, but you are still ahead. (The "area" under your velocity curve is larger than the "area" under your friend's.)
   (c) You and your friend will be together on the trail when the "areas" (distances) under the two velocity graphs are equal.

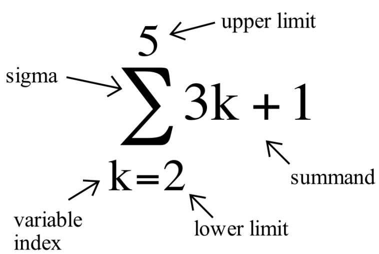
x	$B(x)$
0	0
0.5	1
1	2
2	4



4.1 Sigma Notation and Riemann Sums

One strategy for calculating the area of a region is to cut the region into simple shapes, calculate the area of each simple shape, and then add these smaller areas together to get the area of the whole region. When you use this approach with many sub-regions, it will be useful to have a notation for adding many values together: the sigma (Σ) notation.

summation	**sigma notation**	**how to read it**
$1^2+2^2+3^2+4^2+5^2$	$\sum\limits_{k=1}^{5} k^2$	the sum of k squared, from k equals 1 to k equals 5
$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$	$\sum\limits_{k=3}^{7} \frac{1}{k}$	the sum of 1 divided by k, from k equals 3 to k equals 7
$2^0+2^1+2^2+2^3+2^4+2^5$	$\sum\limits_{j=0}^{5} 2^j$	the sum of 2 to the j-th power, from j equals 0 to j equals 5
$a_2+a_3+a_4+a_5+a_6+a_7$	$\sum\limits_{n=2}^{7} a_n$	the sum of a sub n, from n equals 2 to n equals 7



The variable (typically i, j, k, m or n) used in the summation is called the **counter** or **index** variable. The function to the right of the sigma is called the **summand**, while the numbers below and above the sigma are called the **lower** and **upper limits** of the summation.

Practice 1. Write the summation denoted by each of the following:

$$\text{(a)} \sum_{k=1}^{5} k^3 \qquad \text{(b)} \sum_{j=2}^{7} (-1)^j \frac{1}{j} \qquad \text{(c)} \sum_{m=0}^{4} (2m+1)$$

In practice, we frequently use sigma notation together with the standard function notation:

$$\sum_{k=1}^{3} f(k+2) = f(1+2)+f(2+2)+f(3+2)$$
$$= f(3)+f(4)+f(5)$$
$$\sum_{j=1}^{4} f(x_j) = f(x_1)+f(x_2)+f(x_3)+f(x_4)$$

x	$f(x)$	$g(x)$	$h(x)$
1	2	4	3
2	3	1	3
3	1	−2	3
4	0	3	3
5	3	5	3

Example 1. Use the table to evaluate $\sum\limits_{k=2}^{5} 2\cdot f(k)$ and $\sum\limits_{j=3}^{5} [5+f(j-2)]$.

Solution. Writing out the sum and using the table values:

$$\sum_{k=2}^{5} 2\cdot f(k) = 2\cdot f(2)+2\cdot f(3)+2\cdot f(4)+2\cdot f(5)$$
$$= 2\cdot 3+2\cdot 1+2\cdot 0+2\cdot 3 = 14$$

while:

$$\begin{aligned}\sum_{j=3}^{5}[5+f(j-2)] &= [5+f(3-2)]+[5+f(4-2)]+[5+f(5-2)]\\ &= [5+f(1)]+[5+f(2)]+[5+f(3)]\\ &= [5+2]+[5+3]+[5+1]\end{aligned}$$

which adds up to 21. ◀

Practice 2. Use the values in the preceding margin table to evaluate:

$$\text{(a)} \sum_{k=2}^{5} g(k) \qquad \text{(b)} \sum_{j=1}^{4} h(j) \qquad \text{(c)} \sum_{k=3}^{5}[g(k)+f(k-1)]$$

Example 2. For $f(x)=x^2+1$, evaluate $\displaystyle\sum_{k=0}^{3} f(k)$.

Solution. Writing out the sum and using the function values:

$$\begin{aligned}\sum_{k=0}^{3} f(k) &= f(0)+f(1)+f(2)+f(3)\\ &= \left(0^2+1\right)+\left(1^2+1\right)+\left(2^2+1\right)+\left(3^2+1\right)\\ &= 1+2+5+10\end{aligned}$$

which adds up to 18. ◀

Practice 3. For $g(x)=\dfrac{1}{x}$, evaluate $\displaystyle\sum_{k=2}^{4} g(k)$ and $\displaystyle\sum_{k=1}^{3} g(k+1)$.

The summand need not contain the index variable explicitly: you can write a sum from $k=2$ to $k=4$ of the constant function $f(k)=5$ as $\displaystyle\sum_{k=2}^{4} f(k)$ or $\displaystyle\sum_{k=2}^{4} 5 = 5+5+5 = 3\cdot 5 = 15$. Similarly:

$$\sum_{k=3}^{7} 2 = 2+2+2+2+2 = 5\cdot 2 = 10$$

Because the sigma notation is simply a notation for addition, it possesses all of the familiar properties of addition.

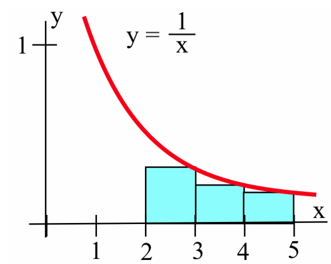
Summation Properties:

Sum of Constants:	$\displaystyle\sum_{k=1}^{n} C = C+C+C+\cdots+C = n\cdot C$
Addition:	$\displaystyle\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
Subtraction:	$\displaystyle\sum_{k=1}^{n}(a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
Constant Multiple:	$\displaystyle\sum_{k=1}^{n} C\cdot a_k = C\cdot\sum_{k=1}^{n} a_k$

Problems 16 and 17 illustrate that similar patterns for sums of products and quotients are *not* valid.

Sums of Areas of Rectangles

In Section 4.2, we will approximate areas under curves by building rectangles as high as the curve, calculating the area of each rectangle, and then adding the rectangular areas together.

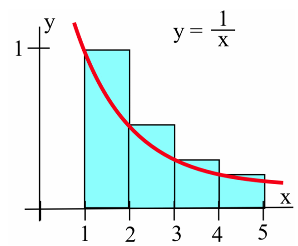


Example 3. Evaluate the sum of the rectangular areas in the margin figure, then write the sum using sigma notation.

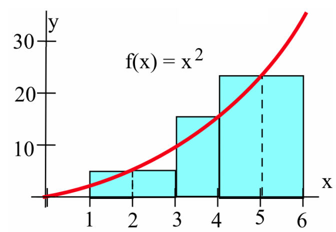
Solution. The sum of the rectangular areas is equal to the sum of (base) · (height) for each rectangle:

$$(1)\left(\frac{1}{3}\right)+(1)\left(\frac{1}{4}\right)+(1)\left(\frac{1}{5}\right)=\frac{47}{60}$$

which we can rewrite as $\sum_{k=3}^{5}\frac{1}{k}$ using sigma notation. ◀



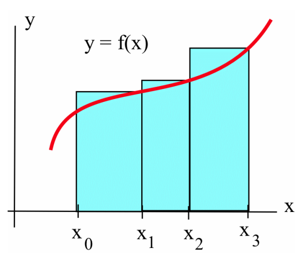
Practice 4. Evaluate the sum of the rectangular areas in the margin figure, then write the sum using sigma notation.



The bases of these rectangles need not be equal. For the rectangular areas associated with $f(x)=x^2$ in the margin figure:

rectangle	base	height	area
1	$3-1=2$	$f(2)=4$	$2\cdot 4=8$
2	$4-3=1$	$f(4)=16$	$1\cdot 16=16$
3	$6-4=2$	$f(5)=25$	$2\cdot 25=50$

so the sum of the rectangular areas is $8+16+50=74$.

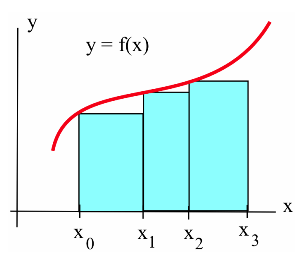


Example 4. Write the sum of the areas of the rectangles in the margin figure using sigma notation.

Solution. The area of each rectangle is (base) · (height):

rectangle	base	height	area
1	x_1-x_0	$f(x_1)$	$(x_1-x_0)\cdot f(x_1)$
2	x_2-x_1	$f(x_2)$	$(x_2-x_1)\cdot f(x_2)$
3	x_3-x_2	$f(x_3)$	$(x_3-x_2)\cdot f(x_3)$

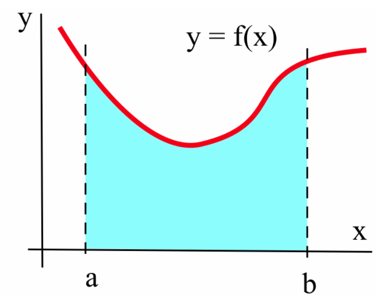
The area of the k-th rectangle is $(x_k-x_{k-1})\cdot f(x_k)$, so we can express the total area of the three rectangles as $\sum_{k=1}^{3}(x_k-x_{k-1})\cdot f(x_k)$. ◀

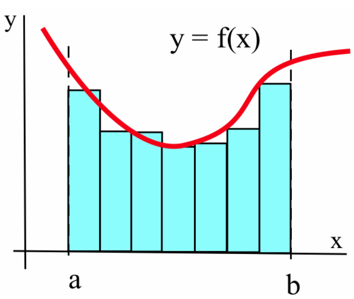


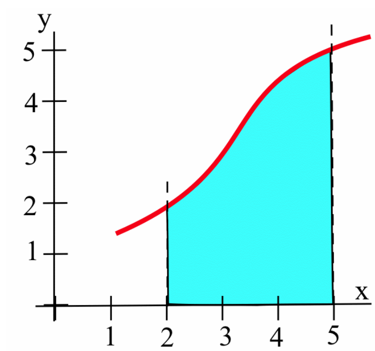
Practice 5. Write the sum of the areas of the shaded rectangles in the margin figure using sigma notation.

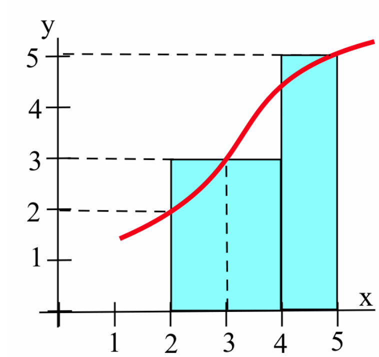
Area Under a Curve: Riemann Sums

Suppose we want to calculate the area between the graph of a positive function f and the x-axis on the interval $[a, b]$ (see below left).









One method to approximate the area involves building several rectangles with bases on the x-axis spanning the interval $[a, b]$ and with sides that reach up to the graph of f (above right). We then compute the areas of the rectangles and add them up to get a number called a **Riemann sum** of f on $[a, b]$. The area of the region formed by the rectangles provides an approximation of the area we want to compute.

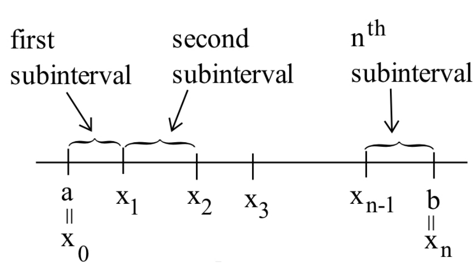
Example 5. Approximate the area shown in the margin between the graph of f and the x-axis spanning the interval $[2, 5]$ by summing the areas of the rectangles shown in the lower margin figure.

Solution. The total area is $(2)(3) + (1)(5) = 11$ square units. ◀

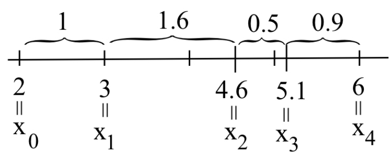
In order to effectively describe this process, some new vocabulary is helpful: a **partition** of an interval and the **mesh** of a partition.

A **partition** $\mathcal{P}$ of a closed interval $[a, b]$ into n subintervals consists of a set of $n + 1$ points $\{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ listed in increasing order, so that $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$. (A partition is merely a collection of points on the horizontal axis, unrelated to the function f in any way.)

The points of the partition $\mathcal{P}$ divide $[a, b]$ into n subintervals:



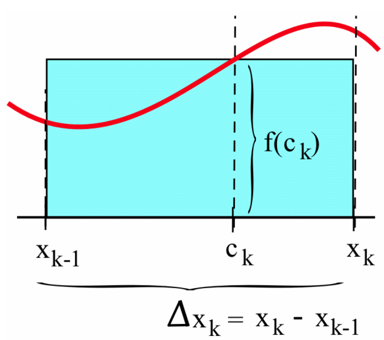
These intervals are $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\ldots$, $[x_{n-1}, x_n]$ with lengths $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, $\Delta x_3 = x_3 - x_2$, $\ldots$, $\Delta x_n = x_n - x_{n-1}$. The points x_k of the partition $\mathcal{P}$ mark the locations of the vertical lines for the sides of the rectangles, and the bases of the rectangles have



lengths Δx_k for $k = 1, 2, 3, \ldots, n$. The **mesh** or **norm** of a partition $\mathcal{P}$ is the length of the longest of the subintervals $[x_{k-1}, x_k]$ or, equivalently, the maximum value of Δx_k for $k = 1, 2, 3, \ldots, n$.

For example, the set $\mathcal{P} = \{2, 3, 4.6, 5.1, 6\}$ is a partition of the interval $[2, 6]$ (see margin) that divides the interval $[2, 6]$ into four subintervals with lengths $\Delta x_1 = 1$, $\Delta x_2 = 1.6$, $\Delta x_3 = 0.5$ and $\Delta x_4 = 0.9$, so the mesh of this partition is 1.6, the maximum of the lengths of the subintervals. (If the mesh of a partition is "small," then the length of each one of the subintervals is the same or smaller.)

Practice 6. $\mathcal{P} = \{3, 3.8, 4.8, 5.3, 6.5, 7, 8\}$ is a partition of what interval? How many subintervals does it create? What is the mesh of the partition? What are the values of x_2 and Δx_2?

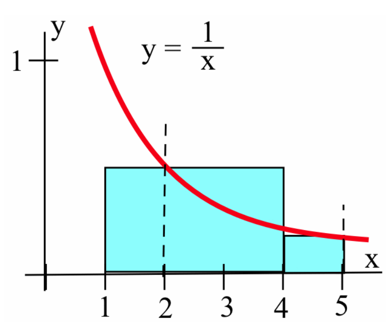


A function, a partition and a point chosen from each subinterval determine a **Riemann sum**. Suppose f is a positive function on the interval $[a, b]$ (so that $f(x) > 0$ when $a \leq x \leq b$), $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ is a partition of $[a, b]$, and c_k is an x-value chosen from the k-th subinterval $[x_{k-1}, x_k]$ (so $x_{k-1} \leq c_k \leq x_k$). Then the area of the k-th rectangle is:

$$f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$$

> **Definition**:
> A summation of the form $\displaystyle\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called a **Riemann sum** of f for the partition $\mathcal{P}$ and the chosen points $\{c_1, c_2, \ldots, c_n\}$.

This Riemann sum is the total of the areas of the rectangular regions and provides an approximation of the area between the graph of f and the x-axis on the interval $[a, b]$.



Example 6. Find the Riemann sum for $f(x) = \dfrac{1}{x}$ using the partition $\{1, 4, 5\}$ and the values $c_1 = 2$ and $c_2 = 5$ (see margin).

Solution. The two subintervals are $[1, 4]$ and $[4, 5]$, hence $\Delta x_1 = 3$ and $\Delta x_2 = 1$. So the Riemann sum for this partition is:

$$\sum_{k=1}^{2} f(c_k) \cdot \Delta x_k = f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2$$

$$= f(2) \cdot 3 + f(5) \cdot 1 = \frac{1}{2} \cdot 3 + \frac{1}{5} \cdot 1 = \frac{17}{10}$$

The value of the Riemann sum is 1.7. ◀

Practice 7. Calculate the Riemann sum for $f(x) = \dfrac{1}{x}$ on the partition $\{1, 4, 5\}$ using the chosen values $c_1 = 3$ and $c_2 = 4$.

Practice 8. What is the smallest value a Riemann sum for $f(x) = \frac{1}{x}$ can have using the partition $\{1,4,5\}$? (You will need to choose values for c_1 and c_2.) What is the largest value a Riemann sum can have for this function and partition?

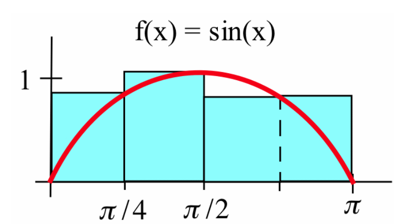
The table below shows the output of a computer program that calculated Riemann sums for the function $f(x) = \frac{1}{x}$ with various numbers of subintervals (denoted n) and different ways of choosing the points c_k in each subinterval.

n	mesh	c_k = left edge = x_{k-1}	c_k = "random" point	c_k = right edge = x_k
4	1.0	2.083333	1.473523	1.283333
8	0.5	1.828968	1.633204	1.428968
16	0.25	1.714406	1.577806	1.514406
40	0.10	1.650237	1.606364	1.570237
400	0.01	1.613446	1.609221	1.605446
4000	0.001	1.609838	1.609436	1.609038

When the mesh of the partition is small (and the number of subintervals, n, is large), it appears that all of the ways of choosing the c_k locations result in approximately the same value for the Riemann sum. For this decreasing function, using the left endpoint of the subinterval always resulted in a sum that was larger than the area approximated by the sum. Choosing the right endpoint resulted in a value smaller than that area. Why?

As the mesh gets smaller, all of the Riemann Sums seem to be approaching the same value, approximately 1.609. (As we shall soon see, these values are all approaching $\ln(5) \approx 1.609437912$.)

Example 7. Find the Riemann sum for the function $f(x) = \sin(x)$ on the interval $[0, \pi]$ using the partition $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\right\}$ and the chosen points $c_1 = \frac{\pi}{4}$, $c_2 = \frac{\pi}{2}$ and $c_3 = \frac{3\pi}{4}$.



Solution. The three subintervals (see margin) are $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$ so $\Delta x_1 = \frac{\pi}{4}$, $\Delta x_2 = \frac{\pi}{4}$ and $\Delta x_3 = \frac{\pi}{2}$. The Riemann sum for this partition is:

$$\sum_{k=1}^{3} f(c_k) \cdot \Delta x_k = \sin\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{4} + \sin\left(\frac{3\pi}{4}\right) \cdot \frac{\pi}{2}$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} + 1 \cdot \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{2} = \frac{(2+3\sqrt{2})\pi}{8}$$

or approximately 2.45148. ◀

Practice 9. Find the Riemann sum for the function and partition in the previous Example, but this time choose $c_1 = 0$, $c_2 = \frac{\pi}{2}$ and $c_3 = \frac{\pi}{2}$.

Two Special Riemann Sums: Lower and Upper Sums

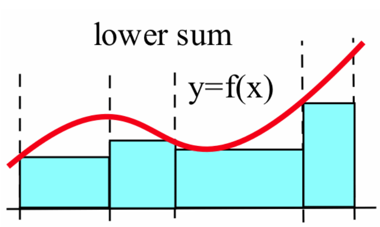
Two particular Riemann sums are of special interest because they represent the extreme possibilities for a given partition.

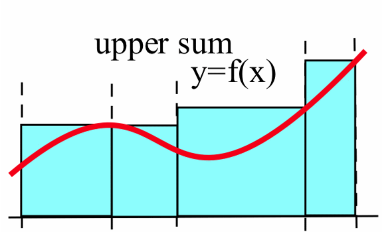
Definition:
If f is a positive, continuous function on $[a,b]$ and $\mathcal{P}$ is a partition of $[a,b]$, let m_k be the x-value in the k-th subinterval so that $f(m_k)$ is the minimum value of f on that interval, and let M_k be the x-value in the k-th subinterval so that $f(M_k)$ is the maximum value of f on that subinterval. Then:

$$\text{LS}_{\mathcal{P}} = \sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \text{ is the \textbf{lower sum} of } f \text{ for } \mathcal{P}$$

$$\text{US}_{\mathcal{P}} = \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k \text{ is the \textbf{upper sum} of } f \text{ for } \mathcal{P}$$

We need f to be continuous in order to assure that it attains its minimum and maximum values on any closed subinterval of the partition. If f is bounded—but not necessarily continuous—we can generalize this definition by replacing $f(m_k)$ with the **greatest lower bound** of all $f(x)$ on the interval and $f(M_k)$ with the **least upper bound** of all $f(x)$ on the interval.





Geometrically, a lower sum arises from building rectangles under the graph of f (see first margin figure) and *every* lower sum is less than or equal to the exact area A of the region bounded by the graph of f and the x-axis on the interval $[a,b]$: $\text{LS}_{\mathcal{P}} \leq A$ for every partition $\mathcal{P}$.

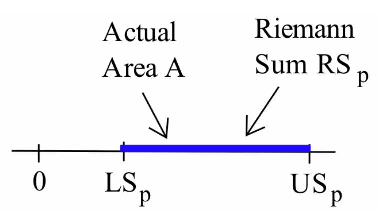
Likewise, an upper sum arises from building rectangles over the graph of f (see second margin figure) and *every* upper sum is greater than or equal to the exact area A of the region bounded by the graph of f and the x-axis on the interval $[a,b]$: $\text{US}_{\mathcal{P}} \geq A$ for every partition $\mathcal{P}$.

Together, the lower and upper sums provide bounds on the size of the exact area: $\text{LS}_{\mathcal{P}} \leq A \leq \text{US}_{\mathcal{P}}$.

For any c_k value in the k-th subinterval, $f(m_k) \leq f(c_k) \leq f(M_k)$, so, for *any* choice of the c_k values, the Riemann sum $\text{RS}_{\mathcal{P}} = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ satisfies the inequality:

$$\sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \quad \leq \quad \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \quad \leq \quad \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k$$

or, equivalently, $\text{LS}_{\mathcal{P}} \leq \text{RS}_{\mathcal{P}} \leq \text{US}_{\mathcal{P}}$. The lower and upper sums provide bounds on the size of *all* Riemann sums for a given partition.



The exact area A and every Riemann sum $\text{RS}_{\mathcal{P}}$ for partition $\mathcal{P}$ and any choice of points $\{c_k\}$ both lie between the lower sum and the upper sum for $\mathcal{P}$ (see margin). Therefore, if the lower and upper sums are close together, then the area and *any* Riemann sum for $\mathcal{P}$ (regardless of how you choose the points c_k) must also be close together. If we know that the upper and lower sums for a partition $\mathcal{P}$ are within 0.001 units of each other, then we can be sure that every Riemann sum for partition $\mathcal{P}$ is within 0.001 units of the exact area A.

Unfortunately, finding minimums and maximums for each subinterval of a partition can be a time-consuming (and tedious) task, so it is usually not practical to determine lower and upper sums for "wiggly" functions. If f is monotonic, however, then it is *easy* to find the values for m_k and M_k, and sometimes we can even explicitly calculate the limits of the lower and upper sums.

For a **monotonic**, bounded function we can guarantee that a Riemann sum is within a certain distance of the exact value of the area it is approximating.

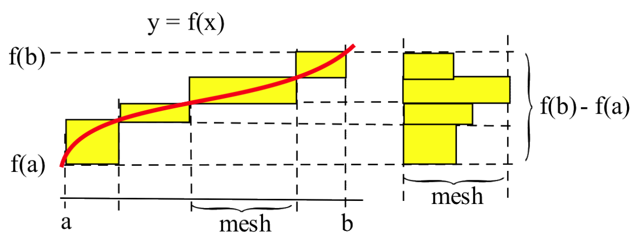
Recall from Section 3.3 that "monotonic" means "always increasing or always decreasing" on the interval in question.

Theorem:
If f is a positive, monotonic, bounded function on $[a, b]$
then for any partition $\mathcal{P}$ and any Riemann sum for f using $\mathcal{P}$,

$$|\mathrm{RS}_{\mathcal{P}} - A| \leq \mathrm{US}_{\mathcal{P}} - \mathrm{LS}_{\mathcal{P}} \leq |f(b) - f(a)| \cdot (\text{mesh of } \mathcal{P})$$

In words, this string of inequalities says that the distance between any Riemann sum and the area being approximated is no bigger than the difference between the upper and lower Riemann sums for the same partition, which in turn is no bigger than the distance between the values of the function at the endpoints of the interval times the mesh of the partition.

Proof. The Riemann sum and the exact area are both between the upper and lower sums, so the distance between the Riemann sum and the exact area is no bigger than the distance between the upper and lower sums. If f is monotonically increasing, we can slide the areas representing the difference of the upper and lower sums into a rectangle:



See Problem 56 for the monotonically decreasing case.

whose height equals $f(b) - f(a)$ and whose base equals the mesh of $\mathcal{P}$. So the total difference of the upper and lower sums is smaller than the area of that rectangle, $[f(b) - f(a)] \cdot (\text{mesh of } \mathcal{P})$. □

4.1 Problems

In Problems 1–6 , rewrite the sigma notation as a summation and perform the indicated addition.

1. $\displaystyle\sum_{k=2}^{4} k^2$
2. $\displaystyle\sum_{j=1}^{5} (1+j)$
3. $\displaystyle\sum_{n=1}^{3} (1+n)^2$
4. $\displaystyle\sum_{k=0}^{5} \sin(\pi k)$
5. $\displaystyle\sum_{j=0}^{5} \cos(\pi j)$
6. $\displaystyle\sum_{k=1}^{3} \frac{1}{k}$

In Problems 7–12, rewrite each summation using the sigma notation. Do not evaluate the sums.

7. $3+4+5+\cdots+93+94$

8. $4+6+8+\cdots+24$

9. $9+16+25+36+\cdots+144$

10. $\frac{3}{4}+\frac{3}{9}+\frac{3}{16}+\cdots+\frac{3}{100}$

11. $1\cdot 2^1+2\cdot 2^2+3\cdot 2^3+\cdots+7\cdot 2^7$

12. $3+6+9+\cdots+30$

In Problems 13–15, use this table:

k	a_k	b_k
1	1	2
2	2	2
3	3	2

to verify the equality for these values of a_k and b_k.

13. $\sum_{k=1}^{3}(a_k+b_k)=\sum_{k=1}^{3}a_k+\sum_{k=1}^{3}b_k$

14. $\sum_{k=1}^{3}(a_k-b_k)=\sum_{k=1}^{3}a_k-\sum_{k=1}^{3}b_k$

15. $\sum_{k=1}^{3}5a_k=5\cdot\sum_{k=1}^{3}a_k$

For Problems 16–18, use the values of a_k and b_k in the table above to verify the inequality.

16. $\sum_{k=1}^{3}a_k\cdot b_k\neq\left(\sum_{k=1}^{3}a_k\right)\left(\sum_{k=1}^{3}b_k\right)$

17. $\sum_{k=1}^{3}a_k^2\neq\left(\sum_{k=1}^{3}a_k\right)^2$

18. $\sum_{k=1}^{3}\frac{a_k}{b_k}\neq\frac{\sum_{k=1}^{3}a_k}{\sum_{k=1}^{3}b_k}$

For 19–30, $f(x)=x^2$, $g(x)=3x$ and $h(x)=\frac{2}{x}$. Evaluate each sum.

19. $\sum_{k=0}^{3}f(k)$

20. $\sum_{k=0}^{3}f(2k)$

21. $\sum_{j=0}^{3}2\cdot f(j)$

22. $\sum_{i=0}^{3}f(1+i)$

23. $\sum_{m=1}^{3}g(m)$

24. $\sum_{k=1}^{3}g(f(k))$

25. $\sum_{j=1}^{3}g^2(j)$

26. $\sum_{k=1}^{3}k\cdot g(k)$

27. $\sum_{k=2}^{4}h(k)$

28. $\sum_{i=1}^{4}h(3i)$

29. $\sum_{n=1}^{3}f(n)\cdot h(n)$

30. $\sum_{k=1}^{7}g(k)\cdot h(k)$

In 31–36, write out each summation and simplify the result. These are examples of "telescoping sums."

31. $\sum_{k=1}^{7}\left[k^2-(k-1)^2\right]$

32. $\sum_{k=1}^{6}\left[k^3-(k-1)^3\right]$

33. $\sum_{k=1}^{5}\left[\frac{1}{k}-\frac{1}{k+1}\right]$

34. $\sum_{k=0}^{4}\left[(k+1)^3-k^3\right]$

35. $\sum_{k=0}^{8}\left[\sqrt{k+1}-\sqrt{k}\right]$

36. $\sum_{k=1}^{5}\left[x_k-x_{k-1}\right]$

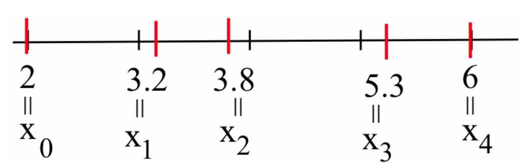
In 37–43, (a) list the subintervals determined by the partition $\mathcal{P}$, (b) find the values of Δx_k, (c) find the mesh of $\mathcal{P}$ and (d) calculate $\sum_{k=1}^{n}\Delta x_k$.

37. $\mathcal{P}=\{2,3,4.5,6,7\}$

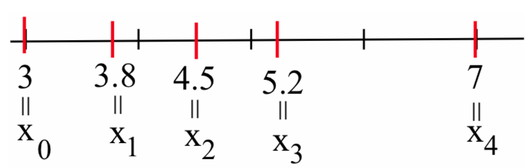
38. $\mathcal{P}=\{3,3.6,4,4.2,5,5.5,6\}$

39. $\mathcal{P}=\{-3,-1,0,1.5,2\}$

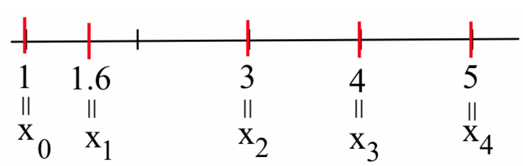
40. $\mathcal{P}$ as shown below:



41. $\mathcal{P}$ as shown below:



42. $\mathcal{P}$ as shown below:



43. For $\Delta x_k = x_k - x_{k-1}$, verify that:

$$\sum_{k=1}^{n} \Delta x_k = \text{length of the interval } [a, b]$$

For 44–48, sketch a graph of f, draw vertical lines at each point of the partition, evaluate each $f(c_k)$ and sketch the corresponding rectangle, and calculate and add up the areas of those rectangles.

44. $f(x) = x + 1$, $\mathcal{P} = \{1, 2, 3, 4\}$
  (a) $c_1 = 1$, $c_2 = 3$, $c_3 = 3$
  (b) $c_1 = 2$, $c_2 = 2$, $c_3 = 4$

45. $f(x) = 4 - x^2$, $\mathcal{P} = \{0, 1, 1.5, 2\}$
  (a) $c_1 = 0$, $c_2 = 1$, $c_3 = 2$
  (b) $c_1 = 1$, $c_2 = 1.5$, $c_3 = 1.5$

46. $f(x) = \sqrt{x}$, $\mathcal{P} = \{0, 2, 5, 10\}$
  (a) $c_1 = 1$, $c_2 = 4$, $c_3 = 9$
  (b) $c_1 = 0$, $c_2 = 3$, $c_3 = 7$

47. $f(x) = \sin(x)$, $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$
  (a) $c_1 = 0$, $c_2 = \dfrac{\pi}{4}$, $c_3 = \dfrac{\pi}{2}$
  (b) $c_1 = \dfrac{\pi}{4}$, $c_2 = \dfrac{\pi}{2}$, $c_3 = \pi$

48. $f(x) = 2^x$, $\mathcal{P} = \{0, 1, 3\}$
  (a) $c_1 = 0$, $c_2 = 2$
  (b) $c_1 = 1$, $c_2 = 3$

For 49–52, sketch the function and find the smallest possible value and the largest possible value for a Riemann sum for the given function and partition.

49. $f(x) = 1 + x^2$
  (a) $\mathcal{P} = \{1, 2, 4, 5\}$
  (b) $\mathcal{P} = \{1, 2, 3, 4, 5\}$
  (c) $\mathcal{P} = \{1, 1.5, 2, 3, 4, 5\}$

50. $f(x) = 7 - 2x$
  (a) $\mathcal{P} = \{0, 2, 3\}$
  (b) $\mathcal{P} = \{0, 1, 2, 3\}$
  (c) $\mathcal{P} = \{0, .5, 1, 1.5, 2, 3\}$

51. $f(x) = \sin(x)$
  (a) $\mathcal{P} = \{0, \frac{\pi}{2}, \pi\}$
  (b) $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$
  (c) $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\}$

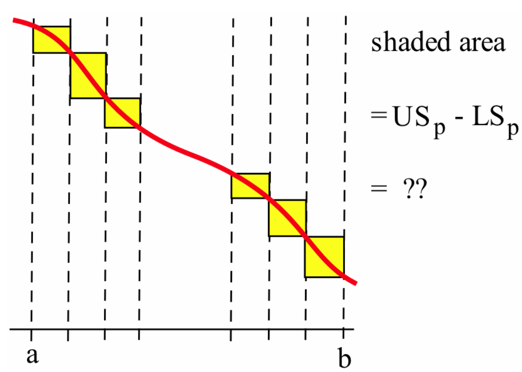
52. $f(x) = x^2 - 2x + 3$
  (a) $\mathcal{P} = \{0, 2, 3\}$
  (b) $\mathcal{P} = \{0, 1, 2, 3\}$
  (c) $\mathcal{P} = \{0, 0.5, 1, 2, 2.5, 3\}$

53. Suppose $\text{LS}_{\mathcal{P}} = 7.362$ and $\text{US}_{\mathcal{P}} = 7.402$ for a positive function f and a partition $\mathcal{P}$ of $[1, 5]$.
  (a) You can be certain that every Riemann sum for the partition $\mathcal{P}$ is within what distance of the exact value of the area between the graph of f and the x-axis on the interval $[1, 5]$?
  (b) What if $\text{LS}_{\mathcal{P}} = 7.372$ and $\text{US}_{\mathcal{P}} = 7.390$?

54. Suppose you divide the interval $[1, 4]$ into 100 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^2$ by randomly selecting a point c_k in each subinterval.
  (a) You can be certain that the value of the Riemann sum is within what distance of the exact value of the area between the graph of f and the x-axis on interval $[1, 4]$?
  (b) What if you use 200 equally wide subintervals?

55. If you divide $[2, 4]$ into 50 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^3$ by randomly selecting a point c_k in each subinterval, then you can be certain that the Riemann sum is within what distance of the exact value of the area between f and the x-axis on the interval $[2, 4]$?

56. If f is monotonic decreasing on $[a, b]$ and you divide $[a, b]$ into n equally wide subintervals:



then you can be certain that the Riemann sum is within what distance of the exact value of the area between f and the x-axis on the interval $[a, b]$?

Summing Powers of Consecutive Integers

The formulas below are included here for your reference. They will not be used in the following sections, except for a handful of exercises in Section 4.2.

Formulas for some commonly encountered summations can be useful for explicitly evaluating certain special Riemann sums.

The summation formula for the first n positive integers is relatively well known, has several easy but clever proofs, and even has an interesting story behind it.

$$1+2+3+\cdots+(n-1)+n=\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

Proof. Let $S=1+2+3+\cdots+(n-2)+(n-1)+n$, which we can also write as $S=n+(n-1)+(n-2)+\cdots+3+2+1$. Adding these two representations of S together:

$$\begin{array}{rccccccccccccc} S= & 1 & + & 2 & + & 3 & + & \cdots & + & (n-2) & + & (n-1) & + & n \\ +\,S= & n & + & (n-1) & + & (n-2) & + & \cdots & + & 3 & + & 2 & + & 1 \\ \hline 2S= & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) & + & (n+1) \end{array}$$

So $2S=n\cdot(n+1)\Rightarrow S=\dfrac{n(n+1)}{2}$, the desired formula. $\square$

Karl Friedrich Gauss (1777–1855), a German mathematician sometimes called the "prince of mathematics."

Gauss supposedly discovered this formula for himself at the age of five when his teacher, planning to keep the class busy for a while, asked the students to add up the integers from 1 to 100. Gauss thought a few minutes, wrote his answer on his slate, and turned it in, then sat smugly while his classmates struggled with the problem.

57. Find the sum of the first 100 positive integers in two ways: (a) using Gauss' formula, and (b) using Gauss' method (from the proof).

58. Find the sum of the first 10 odd integers. (Each odd integer can be written in the form $2k-1$ for $k=1, 2, 3, \ldots$.)

59. Find the sum of the integers from 10 to 20.

Formulas for other integer powers of the first n integers are also known:

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{1}{2}n^2+\frac{1}{2}n=\frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n=\frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2=\left(\frac{n(n+1)}{2}\right)^2 \\ \sum_{k=1}^{n} k^4 &= \frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned}$$

In Problems 60–62, use the properties of summation and the formulas for powers given above to evaluate each sum.

60. $\sum_{k=1}^{10}\left(3+2k+k^2\right)$ 61. $\sum_{k=1}^{10} k\cdot\left(k^2+1\right)$ 62. $\sum_{k=1}^{10} k^2\cdot(k-3)$

4.1 Practice Answers

1. (a) $\sum_{k=1}^{5} k^3 = 1+8+27+64+125$

   (b) $\sum_{j=2}^{7}(-1)^j\cdot\frac{1}{j} = \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}$

   (c) $\sum_{m=0}^{4}(2m+1) = 1+3+5+7+9$

2. (a) $\sum_{k=2}^{5} g(k) = g(2)+g(3)+g(4)+g(5) = 1+(-2)+3+5 = 7$

   (b) $\sum_{j=1}^{4} h(j) = h(1)+h(2)+h(3)+h(4) = 3+3+3+3 = 12$

   (c) $$\begin{aligned}\sum_{k=3}^{5}(g(k)+f(k-1)) &= (g(3)+f(2))+(g(4)+f(3))+(g(5)+f(4))\\ &= (-2+3)+(3+1)+(5+0) = 10\end{aligned}$$

3. $\sum_{k=2}^{4} g(k) = g(2)+g(3)+g(4) = \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{13}{12}$

   $\sum_{k=1}^{3} g(k+1) = g(2)+g(3)+g(4) = \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{13}{12}$

4. Rectangular areas $= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{25}{12} = \sum_{j=1}^{4}\frac{1}{j}$

5. $f(x_0)\cdot(x_1-x_0)+f(x_1)\cdot(x_2-x_1)+f(x_2)\cdot(x_3-x_2) = \sum_{j=1}^{3} f\left(x_{j-1}\right)\cdot\left(x_j-x_{j-1}\right)$

   or $\sum_{k=0}^{2} f(x_k)\cdot(x_{k+1}-x_k)$

6. Interval is $[3,8]$; six subintervals; mesh $=1.2$; $x_2 = 4.8$; $\Delta x_2 = x_2 - x_1 = 4.8-3.8 = 1$.

7. $\text{RS} = (3)\left(\frac{1}{3}\right)+(1)\left(\frac{1}{4}\right) = 1.25$

8. smallest $\text{RS} = (3)\left(\frac{1}{4}\right)+(1)\left(\frac{1}{5}\right) = 0.95$

   largest $\text{RS} = (3)(1)+(1)\left(\frac{1}{4}\right) = 3.25$

9. $\text{RS} = (0)\left(\frac{\pi}{4}\right)+(1)\left(\frac{\pi}{4}\right)+(1)\left(\frac{\pi}{2}\right) \approx 2.356$

4.2 The Definite Integral

Each particular Riemann sum depends on several things: the function f, the interval $[a,b]$, the partition $\mathcal{P}$ of that interval, and the chosen values c_k from each subinterval of that partition. Fortunately—for most of the functions needed for applications—as the approximating rectangles get thinner (and as the meshes of the partitions $\mathcal{P}$ approach 0 and the number of subintervals n in those partitions approaches ∞) the values of the Riemann sums approach the same number, independent of the particular partitions P and the chosen points c_k in the subintervals of those partitions.

This limit of the Riemann sums will become the next big topic in calculus: the **definite integral**. Integrals arise throughout the rest of this book and in applications in almost every field that uses mathematics.

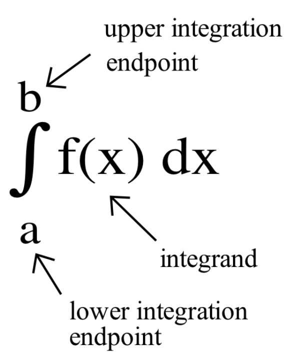
Definition of the Definite Integral:

If $\displaystyle\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k\right)$ equals a finite number I, where each $\mathcal{P}$ is a partition of the interval $[a,b]$, then we say a bounded function f is **integrable** on the interval $[a,b]$ and call the number I the **definite integral** of f on $[a,b]$ and write it as $\displaystyle\int_a^b f(x)\,dx$.

Here we use the notation $\|\mathcal{P}\|$ to mean "the mesh of $\mathcal{P}$" and we assume $b > a$ so that $[a,b]$ is not a single point.

The dx is a differential (see Section 3.6), the limit of the discrete quantity Δx in the Riemann sum.

We read the symbol $\displaystyle\int_a^b f(x)\,dx$ as "the integral from a to b of 'eff' of x 'dee' x" or "the integral from a to b of $f(x)$ with respect to x." Furthermore, we call $f(x)$ the **integrand**, a the **lower endpoint of integration** and b the **upper endpoint** of integration. (We will sometimes also call a and b the **upper and lower limits** of integration.)



Example 1. Describe the area between the graph of $f(x) = \dfrac{1}{x}$, the x-axis, and the vertical lines at $x = 1$ and $x = 5$ as a limit of Riemann sums and as a definite integral.

Solution. Here $f(x) = \dfrac{1}{x}$, $a = 1$ and $b = 5$, so:

$$\text{area} = \lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}\frac{1}{c_k}\cdot\Delta x_k\right) = \int_1^5 \frac{1}{x}\,dx$$

which, according to estimations made in Section 4.1, is approximately equal to 1.609. ◀

You may have noticed that we did not precisely define what $\displaystyle\lim_{\|\mathcal{P}\|\to 0}$ means or how to compute this limit. Providing a definition turns out to be more complicated than the limits we have encountered so far, and in practice we will rarely need to compute such a limit, so a formal definition is left to more advanced textbooks.

Practice 1. Describe the area between the graph of $f(x) = \sin(x)$, the x-axis, and the vertical lines at $x = 0$ and $x = \pi$ as a limit of Riemann sums and as a definite integral.

Example 2. Using the concept of area, determine the values of:

(a) $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (1 + c_k) \cdot \Delta x_k \right)$ on the interval $[1, 3]$

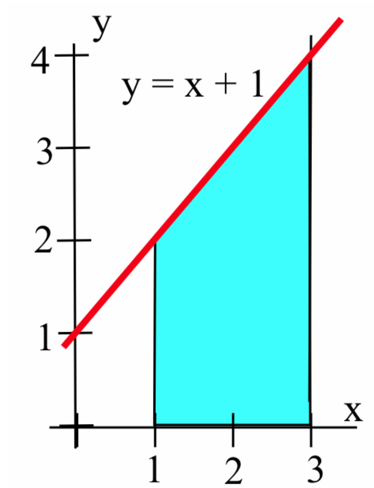
(b) $\int_0^4 (5 - x)\; dx$

(c) $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$

Solution. (a) The limit of the Riemann sums represents the area between the graph of $f(x) = 1 + x$, the x-axis, and the vertical lines at $x = 1$ and $x = 3$ (see margin); this area equals 6 square units.

(b) The definite integral represents the area between $f(x) = 5 - x$, the x-axis and the vertical lines at $x = 0$ and $x = 4$, which is a trapezoid with area 12 square units.

(c) The definite integral represents the area of the upper half of the circle $x^2 + y^2 = 1$, which has radius 1 and center at $(0, 0)$. The area of this semicircle is $\frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi \cdot 1^2 = \frac{\pi}{2}$. ◀



Practice 2. Using the concept of area, determine the values of:

(a) $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (2c_k) \cdot \Delta x_k \right)$ on the interval $[1, 3]$ (b) $\int_3^8 4\, dx$

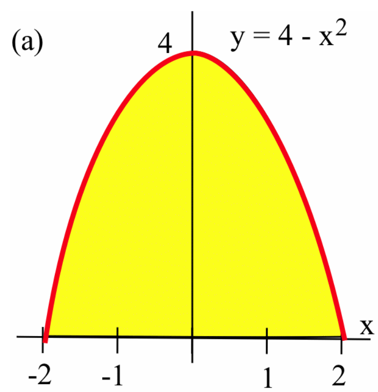
Example 3. Represent each limit of Riemann sums as a definite integral.

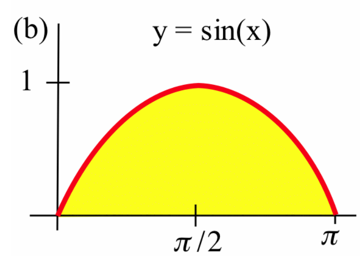
(a) $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (3 + c_k)\, \Delta x_k \right)$ on $[1, 4]$ (b) $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} \sqrt{c_k}\, \Delta x_k \right)$ on $[0, 9]$

Solution. (a) $\int_1^4 (3 + x)\; dx$ (b) $\int_0^9 \sqrt{x}\, dx$ ◀

Example 4. Represent each shaded area in the margin figure as a definite integral. (Do not attempt to evaluate the definite integral, just translate the picture into symbols.)

Solution. (a) $\int_{-2}^{2} \left(4 - x^2\right)\; dx$ (b) $\int_0^{\pi} \sin(x)\, dx$ ◀



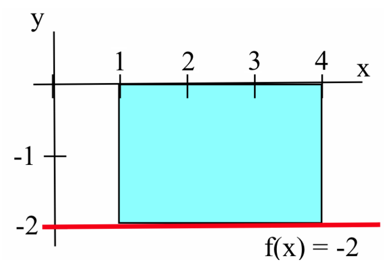


The value of a definite integral $\int_a^b f(x)\, dx$ depends only on the function f being integrated and on the interval $[a, b]$. Replacing the variable x that appears in $\int_a^b f(x)\, dx$, sometimes called a "dummy variable," does not change the value of the integral. The following definite integrals each represent the integral of the function f on the interval $[a, b]$, and they are all equal:

$$\int_a^b f(x)\, dx \quad = \quad \int_a^b f(t)\, dt \quad = \quad \int_a^b f(u)\, du \quad = \quad \int_a^b f(w)\, dw$$

Definite Integrals of Negative Functions

A definite integral is a limit of Riemann sums, and you can form Riemann sums using any integrand function f, positive or negative (or both), continuous or discontinuous. The definite integral of an integrable function will still have a geometric meaning even if the function is sometimes (or always) negative, and definite integrals of negative functions also have meaningful interpretations in applications.



Example 5. Find the definite integral of $f(x) = -2$ on $[1,4]$.

Solution. Writing a Riemann sum for $f(x) = -2$ on the interval $[1,4]$:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} (-2) \cdot \Delta x_k = -2 \cdot \sum_{k=1}^{n} \Delta x_k = -2\,(3) = -6$$

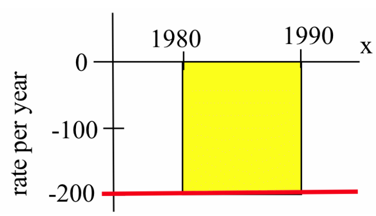
for every partition $\mathcal{P}$ and every choice of values for c_k so:

$$\int_1^4 -2\,dx = \lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} (-2) \cdot \Delta x_k \right) = \lim_{\|\mathcal{P}\|\to 0} -6 = -6$$

The **area** of the region in in the margin figure is 6 units, but because the region is below the x-axis, the value of the **integral** is -6. ◀

If the graph of $f(x)$ is below the x-axis for $a \le x \le b$ (f is negative) then $\int_a^b f(x)\,dx$ is -1 times the area of the region below the x-axis and above the graph of $f(x)$ between $x = a$ and $x = b$.

If $f(t)$ represents the rate of population change (people per year) for a town, then negative values of f for a given time interval would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would represent the change in the population—a decrease—during that time interval.



Example 6. In 1980 there were 12,000 ducks nesting around a lake. The **rate** of population change is shown in the margin. Write a definite integral to represent the **total change** in the duck population between 1980 and 1990, then estimate the population in 1990.

Solution. The total change in population is given by $\int_{1980}^{1990} f(t)\,dt$ and this definite integral is equal to -1 times the area of the rectangle in the margin figure:

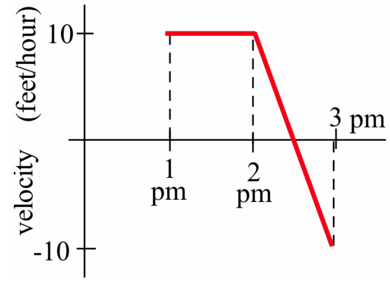
$$-200\,\frac{\text{ducks}}{\text{year}} \cdot 10\,\text{years} = -2000\,\text{ducks}$$

so:

$$\begin{aligned}[\text{1990 population}] &= [\text{1980 population}] + [\text{change from 1980 to 1990}] \\ &= 12000 + (-2000) = 10000\end{aligned}$$

Approximately 10,000 ducks were nesting around the lake in 1990. ◀

If $f(t)$ represents the velocity of a car (in miles per hour) moving in the positive direction along a straight line at time t, then negative values of f indicate that the car was travelling in the negative direction (that is, backwards). The definite integral of f (the integral will be a negative number) represents the change in position of the car during the time interval: how far the car travelled in the negative direction.

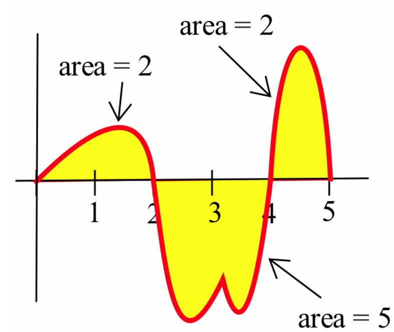


Practice 3. A bug starts at the location $x = 12$ on the x-axis at 1:00 p.m. and walks along the axis with the velocity shown in the margin figure. How far does the bug travel between 1:00 p.m. and 3:00 p.m.? Where is the bug at 3:00 p.m.?

Frequently an integrand function will be positive some of the time and negative some of the time. If f represents a rate of population increase, then the integral of the positive parts of f will give the increase in population and the integral of the negative parts of f will give the decrease in population. Altogether, the integral of f over the entire time interval will give the **total (net) change** in the population.

Geometrically, we can now interpret a definite integral as a difference of areas of the region(s) between the graph of f and the horizontal axis:

$$\int_a^b f(x)\,dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$



Example 7. Use the margin figure to calculate $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$ and $\int_0^5 f(x)\,dx$.

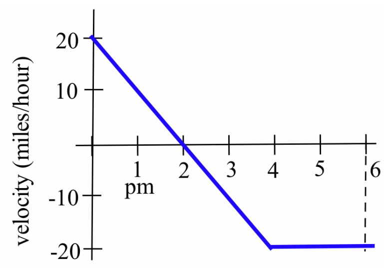
Solution. Using the areas indicated in the figure, $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = -5$ and $\int_4^5 f(x)\,dx = 2$, while

$$\int_0^5 f(x)\,dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$
$$= [2+2] - [5] = -1$$

where we added the areas of the regions above the x-axis and subtracted the area of the region below the x-axis. ◀

Practice 4. Use geometric reasoning to evaluate $\int_0^{2\pi} \sin(x)\,dx$.

If f represents a velocity, then integrals on the intervals where f is positive measure distances moved in the forward direction and integrals on the intervals where f is negative measure distances moved in the backward direction. The integral over the whole time interval gives the **total (net) change** in position: the distance moved forward minus the distance moved backward.



Practice 5. A car travels west with the velocity shown in the margin.

(a) How far does the car travel between noon and 6:00 p.m.?

(b) At 6:00 p.m., where is the car relative to its position at noon?

Units for the Definite Integral

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if x measures time in "seconds" and $f(x)$ gives a velocity with units "feet per second," then Δx has the units "seconds" and $f(x) \cdot \Delta x$ has units:

$$\left(\frac{\text{feet}}{\text{second}}\right)(\text{seconds}) = \text{feet}$$

which is a measure of distance. Because each Riemann sum $\sum f(x) \cdot \Delta x$ is a sum of "feet" and the definite integral is a limit of these Riemann sums, the definite integral has the same units, "feet."

If the units of $f(x)$ are "square feet" and the units of x are "feet," then $\int_a^b f(x)\,dx$ is a number with the units $(\text{feet}^2) \cdot (\text{feet}) = \text{feet}^3$, or cubic feet, a measure of volume. If $f(x)$ represents a force in pounds and x is a distance in feet, then $\int_a^b f(x)\,dx$ is a number with the units foot-pounds, a measure of work.

In general, the units for $\int_a^b f(x)\,dx$ are (units for $f(x)$) $\cdot$ (units for x). A quick check of the units when computing a definite integral can help avoid errors when solving an applied problem.

4.2 Problems

In Problems 1–4 , rewrite each limit of Riemann sums as a definite integral.

1. $\lim\limits_{\|\mathcal{P}\| \to 0} \left(\sum\limits_{k=1}^{n} (2 + 3c_k) \cdot \Delta x_k \right)$ on $[0, 4]$

2. $\lim\limits_{\|\mathcal{P}\| \to 0} \left(\sum\limits_{k=1}^{n} (c_k)^3 \cdot \Delta x_k \right)$ on $[0, 11]$

3. $\lim\limits_{\|\mathcal{P}\| \to 0} \left(\sum\limits_{k=1}^{n} \cos(5c_k) \cdot \Delta x_k \right)$ on $[2, 5]$

4. $\lim\limits_{\|\mathcal{P}\| \to 0} \left(\sum\limits_{k=1}^{n} \sqrt{c_k} \cdot \Delta x_k \right)$ on $[1, 4]$

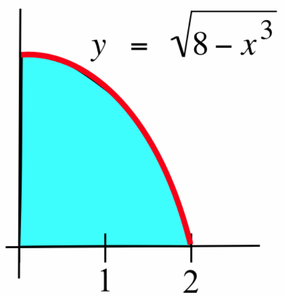
In Problems 5–10, represent the area of each bounded region as a definite integral. (Do not attempt to evaluate the integral, just translate the area into an integral.)

5. The region bounded by $y = x^3$, the x-axis, and the lines $x = 1$ and $x = 5$.

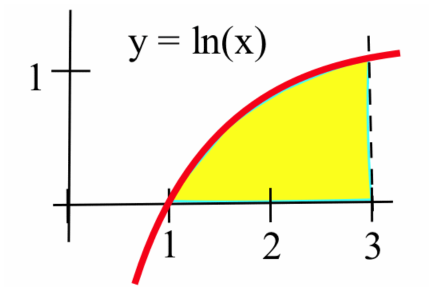
6. The region bounded by $y = \sqrt{x}$, the x-axis and the line $x = 9$.

7. The region bounded by $y = x \cdot \sin(x)$, the x-axis, and the lines $x = \frac{1}{2}$ and $x = 2$.

8. The shaded region shown below:



9. The shaded region shown below:



10. The shaded region shown above for $2 \le x \le 3$.

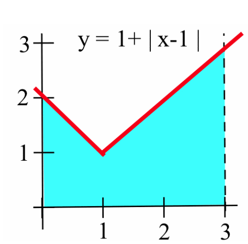
In Problems 11–15, represent the area of each bounded region as a definite integral, then use geometry to determine the value of that definite integral.

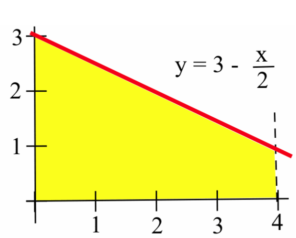
11. The region bounded by $y = 2x$, the x-axis, and the lines $x = 1$ and $x = 3$.

12. The region bounded by $y = 4 - 2x$, the x-axis and the y-axis.

13. The region bounded by $y = |x|$, the x-axis and the line $x = -1$.

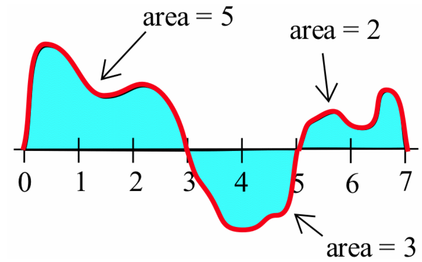
14. The shaded region shown below left.





15. The shaded region shown above right.

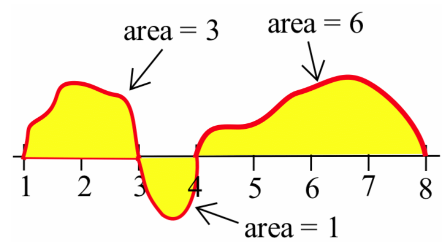
16. Evaluate each integral using the figure below showing the graph of f and various areas.



(a) $\int_0^5 f(x)\,dx$ (b) $\int_3^5 f(x)\,dx$ (c) $\int_5^7 f(x)\,dx$

(d) $\int_0^5 |f(x)|\,dx$ (e) $\int_3^7 f(x)\,dx$

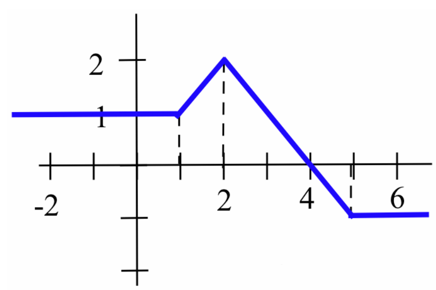
17. Evaluate each integral using the figure below showing the graph of g and various areas.



(a) $\int_1^3 g(x)\,dx$ (b) $\int_3^4 g(x)\,dx$ (c) $\int_4^8 g(x)\,dx$

(d) $\int_1^8 g(x)\,dx$ (e) $\int_3^8 |g(x)|\,dx$

18. Evaluate each integral using the figure below showing the graph of h.

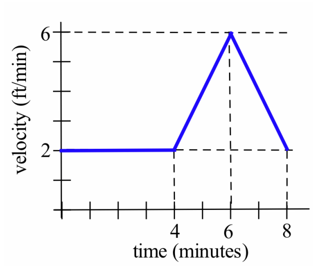


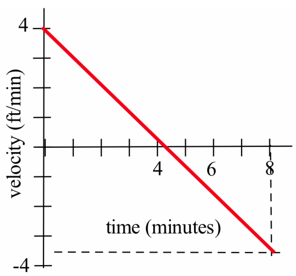
(a) $\int_{-1}^1 h(x)\,dx$ (b) $\int_4^6 h(x)\,dx$ (c) $\int_{-2}^6 h(x)\,dx$

(d) $\int_{-2}^4 h(x)\,dx$ (e) $\int_{-2}^4 |h(x)|\,dx$ (f) $\int_{-2}^6 |h(x)|\,dx$

For Problems 19–20, the figure shows your velocity (in feet per minute) along a straight path. (a) Sketch a graph of your location. (b) How many feet did you walk in 8 minutes? (c) Where, relative to your starting location, are you after 8 minutes?

19. See figure below left.





20. See figure above right.

Problems 21–27 give the units for x and $f(x)$. Specify the units of the definite integral $\int_a^b f(x)\,dx$.

21. x is time in "seconds"; $f(x)$ is velocity in "meters per second"
22. x is time in "hours"; $f(x)$ is a flow rate in "gallons per hour"
23. x a position in "feet"; $f(x)$ area in "square feet"
24. x is a time in "days"; $f(x)$ is a temperature in "degrees Celsius"
25. x a height in "meters"; $f(x)$ force in "grams"
26. x is a position in "inches"; $f(x)$ is a density in "pounds per inch"
27. x is a time in "seconds"; $f(x)$ is an acceleration in "feet per second per second" $\left(\frac{\text{ft}}{\text{sec}^2}\right)$

The remaining problems use the summation formulas given at the end of Section 4.1, as demonstrated in the following Example.

Example 8. For $f(x) = x^2$, divide the interval $[0,2]$ into n equally wide subintervals, evaluate the lower sum, and compute the limit of that lower sum as $n \to \infty$.

Solution. The width of the interval is $b - a = 2 - 0 = 2$ so each of the n subintervals should have width $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$. The endpoints of the k-th interval in the partition are therefore $(k-1)\cdot\frac{2}{n}$ and $k\cdot\frac{2}{n}$ for $k = 1, 2, \ldots, n$.

Because $f(x) = x^2$ is increasing on $[0,2]$ the minimum value of the function on each subinterval occurs at the left endpoint of the subinterval, hence we need to choose $c_k = (k-1)\cdot\frac{2}{n}$. So:

$$\begin{aligned}
\text{LS} &= \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n}\left((k-1)\cdot\frac{2}{n}\right)^2\cdot\frac{2}{n} = \frac{8}{n^3}\cdot\sum_{k=1}^{n}(k-1)^2 \\
&= \frac{8}{n^3}\cdot\sum_{k=1}^{n}\left(k^2 - 2k + 1\right) = \frac{8}{n^3}\left[\sum_{k=1}^{n} k^2 \;-\; 2\sum_{k=1}^{n} k \;+\; \sum_{k=1}^{n} 1\right] \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\cdot\frac{n(n+1)}{2} + n\right] = \frac{8}{n^3}\left[\frac{2n^3 - 3n^2 + n}{6}\right] \\
&= \frac{8}{6}\left[2 - \frac{3}{n} + \frac{1}{n^2}\right]
\end{aligned}$$

As $n \to \infty$, $\text{LS} \to \frac{8}{6}(2) = \frac{8}{3}$ so we can be certain that $\int_0^2 x^2\,dx \geq \frac{8}{3}$. ◀

Practice 6. Redo Example 6 but find the upper Riemann sum for n equally wide partition intervals and show that the limit of these upper sums, as $n \to \infty$, is $\frac{8}{3}$.

From the previous Example and Practice problem, we know that

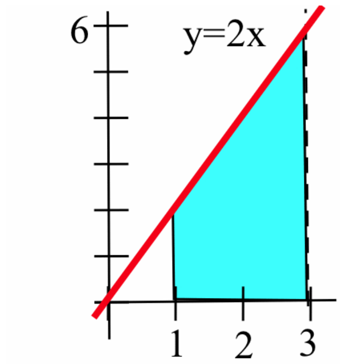
$$\frac{8}{3} \le \int_0^2 x^2\,dx \le \frac{8}{3}$$

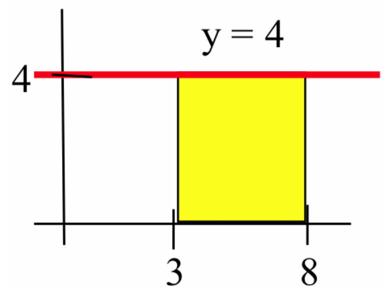
so we can conclude that $\int_0^2 x^2 = \frac{8}{3}$. We will discover a much easier method for evaluating this integral in Section 4.4.

28. For $f(x) = 3 + x$, partition the interval $[0, 2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.
    (a) Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n \to \infty$.
    (b) Compute the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

29. For $f(x) = x^3$, partition the interval $[0, 2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.
    (a) Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n \to \infty$.
    (b) Compute the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

30. For $f(x) = \sqrt{x}$, partition the interval $[0, 9]$ into n subintervals by taking $x_k = \frac{9}{n^2} \cdot k^2$ for $k = 1, 2, \ldots, n$.
    (a) Choose $c_k = x_k$ for each subinterval and compute the upper sum for this function and partition, then calculate the limit of that upper sum as $n \to \infty$.
    (b) Compute the lower sum for this function and partition and find the limit of the lower sum as $n \to \infty$.





4.2 Practice Answers

1. area $= \lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} \sin(c_k) \cdot \Delta x_k \right) = \int_0^{\pi} \sin(x)\,dx$

2. $\lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} (2c_k) \cdot \Delta x_k \right) =$ area of trapezoid in margin $= 8$

   $\int_3^8 4\,dx =$ area of rectangle in margin $= 20$

3. (a) 12.5 feet forward and 2.5 feet backward = 15 feet total

   (b) The bug ends up 10 feet forward of its starting position at $x = 12$, so the bug's final location is at $x = 22$.

4. Between $x = 0$ and $x = 2\pi$, the graph of $y = \sin(x)$ has the same area above the x-axis as below the x-axis so the two areas cancel and the definite integral is 0: $\int_0^{2\pi} \sin(x)\,dx = 0$.

5. (a) 20 miles west (from noon to 2:00 p.m.) plus 60 miles east (from 2:00 p.m. to 6:00 p.m.) yields a total travel distance of 80 miles. (At 4:00 p.m. the driver is back at the starting position after driving 40 miles: 20 miles west and then 20 miles east.)

   (b) The car is 40 miles east of the starting location. (East is the "negative" of west.)

6. $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $M_k = \frac{2}{n} \cdot k$ so $f(M_k) = \left(\frac{2}{n} \cdot k\right)^2 = \frac{4}{n^2} \cdot k^2$. Then:

$$\text{US} = \sum_{k=1}^{n} f(M_k) \cdot \Delta x = \sum_{k=1}^{n} \frac{4}{n^2} \cdot k^2 \cdot \frac{2}{n}$$
$$= \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \frac{8}{n^3} \left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right] = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

so the limit of these upper sums as $n \to \infty$ is $\frac{8}{3}$.

4.3 Properties of the Definite Integral

We have defined definite integrals as limits of Riemann sums, which can often be interpreted as "areas" of geometric regions. These two powerful concepts of the definite integral can help us understand integrals and use them in a variety of applications.

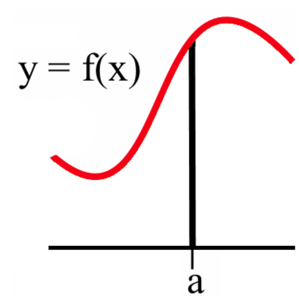
This section continues to emphasize this dual view of definite integrals and presents several properties of definite integrals. We will justify these properties using the properties of summations and the definition of a definite integral as a Riemann sum, but they also have natural interpretations as properties of areas of regions.

We will then use these properties to help understand functions that are defined by integrals, and later to help calculate the values of definite integrals.

Properties of the Definite Integral

As you read each statement about definite integrals, draw a sketch or examine the accompanying figure to interpret the property as a statement about areas.

$$\boxed{\int_a^a f(x)\,dx = 0}$$



This property says that the definite integral of a function over an interval consisting of a single point must be 0. Geometrically, we can see that the area under the graph of a a function above a single point should be 0 because the "width" of a point is 0. In terms of Riemann sums, we can't partition a single point, so instead we must *define* the value of any definite integral over a non-existent "interval" to be 0.

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

In words, this property says that if we reverse the limits of integration, we must multiply the value of the definite integral by -1.

Geometrically, if $a < b$ then the x-values in the first integral are moving "backwards" from $x = b$ to $x = a$, so it might seem reasonable that we should get a negative answer.

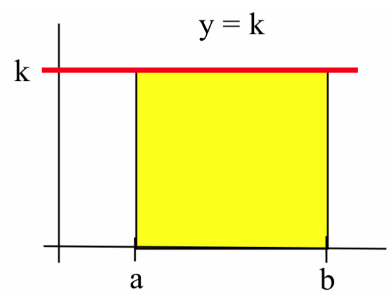
In terms of Riemann sums, if we move from right to left, each Δx_k in any partition $\mathcal{P}$ will be negative:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} f(c_k) \cdot (-|\Delta x_k|) = -1 \cdot \sum_{k=1}^{n} f(c_k) \cdot |\Delta x_k|$$

resulting in -1 times the Riemann sum we would use for $\int_a^b f(x)\,dx$.

Our definition of a Riemann sum only allows each Δx_k to be positive, however, so we can simply treat this integral property as another definition.

$$\boxed{\int_a^b k\,dx = k\,(b-a) \qquad (k \text{ is any constant})}$$



Thinking geometrically, if $k > 0$ (see margin), then $\int_a^b k\,dx$ represents the area of a rectangle with base $b-a$ and height k, so:

$$\int_a^b k\,dx = (\text{height}) \cdot (\text{base}) = k \cdot (b-a)$$

Alternatively, for any $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ that partitions the interval $[a,b]$, and every choice of points c_j from the subintervals of that partition, the Riemann sum is:

$$\sum_{j=1}^{n} f\left(c_j\right) \cdot \Delta x_j = \sum_{j=1}^{n} k \cdot \Delta x_j = k \sum_{j=1}^{n} \Delta x_j = k \cdot (b-a)$$

Here we use the fact that the sum of the lengths of the subinterval of any partition of the interval $[a,b]$ is equal to the width of $[a,b]$, which is $b-a$.

Because every Riemann sum equals $k \cdot (b-a)$, the limit of those sums, as $\|\mathcal{P}\| \to 0$, must also be $k \cdot (b-a)$.

$$\boxed{\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx \qquad (k \text{ is any constant})}$$

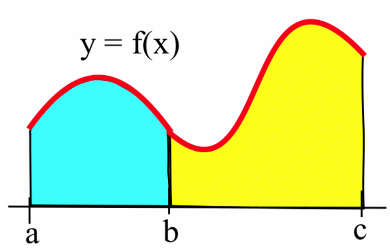
In words, this property says that multiplying an integrand by a constant k has the same result as multiplying the value of the definite integral by that constant.

Geometrically, multiplying a function by a positive constant k stretches the graph of $y = f(x)$ by a factor of k in the vertical direction, which should multiply the area of the region between that graph and the x-axis by the same factor.

Thinking in terms of Riemann sums:

$$\sum_{j=1}^{n} k \cdot f\left(c_j\right) \cdot \Delta x_j = k \cdot \sum_{j=1}^{n} f\left(c_j\right) \cdot \Delta x_j$$

so the limit of the sum on the left over all possible partitions $\mathcal{P}$, as $\|\mathcal{P}\| \to 0$, is $\int_a^b k \cdot f(x)\,dx$, while the corresponding limit of the sums on the right yields $k \cdot \int_a^b f(x)\,dx$.



$$\boxed{\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \;=\; \int_a^c f(x)\,dx}$$

This property is most easily understood (and believed) in terms of a picture (see margin). We can also justify this property using Riemann sums by restricting our partitions to include the point $x = b$ between $x = a$ and $x = c$ and then splitting that partition into two sub-partitions that partition $[a,b]$ and $[b,c]$, respectively.

This property remains true, however, even when $b \geq c$ or $b \leq a$.

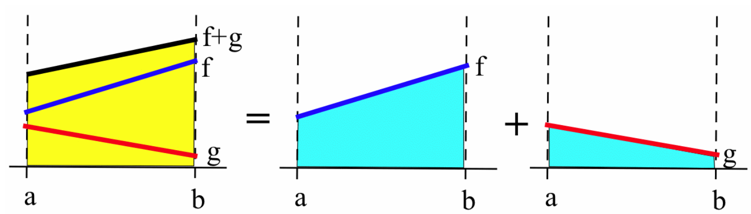
Properties of Definite Integrals of Combinations of Functions

The next two properties relate the values of integrals of sums and differences of functions to the sums and differences of integrals of the individual functions. You will find these properties very useful when computing integrals of functions that involve the sum or difference of several terms (such as a polynomial): you can integrate each term and then add or subtract the individual results to get the answer. Both properties have natural interpretations as statements about areas.

$$\boxed{\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx}$$

In words, this says "the integral of a sum is the sum of the integrals."

The following graph supplies a geometrical justification:



Using Riemann sums, we can write:

$$\sum_{j=1}^{n} \left[f\left(c_j\right)+g\left(c_j\right)\right]\cdot\Delta x_j = \sum_{j=1}^{n} f\left(c_j\right)\cdot\Delta x_j \;+\; \sum_{j=1}^{n} g\left(c_j\right)\cdot\Delta x_j$$

and then take the limit on each side as $\|\mathcal{P}\| \to 0$.

$$\boxed{\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx \;-\; \int_a^b g(x)\,dx}$$

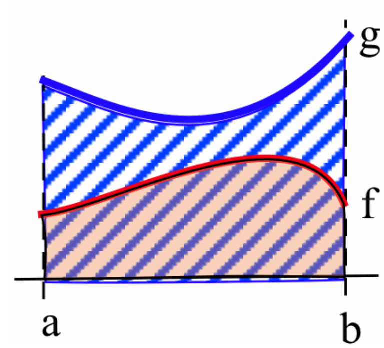
In words, this says "the integral of a difference is the difference of the integrals."

The justification for this difference property is quite similar to the justification of the sum property. (Or we can combine the sum property with the constant-multiple property, setting $k=-1$.)

Practice 1. Given that $\int_1^4 f(x)\,dx = 7$ and that $\int_1^4 g(x)\,dx = 3$, evaluate the definite integral $\int_1^4 [f(x)-g(x)]\,dx$.

$$\boxed{\text{If } f(x)\le g(x) \text{ for all } x \text{ in } [a,b]\text{, then } \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx}$$

Geometrically, the margin figure illustrates that if f and g are both positive and that $f(x)\le g(x)$ on the interval $[a,b]$, then the area of region between the graph of f and the x-axis is smaller than the area of region between the graph of g and the x-axis.



Similar sketches for the situations where f or g are sometimes or always negative illustrate that the property holds in other situations as well, but we can avoid all of those different cases using Riemann sums.

If we use the same partition $\mathcal{P}$ and chosen points c_j for Riemann sums for f and g, then $f(c_j) \leq g(c_j)$ for each j, so:

$$\sum_{j=1}^{n} f(c_j) \cdot \Delta x_j \leq \sum_{j=1}^{n} g(c_j) \cdot \Delta x_j$$

Taking the limit over all such partitions as the mesh of those partitions approaches 0, we get $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

If	$m \leq f(x) \leq M$ for all x in $[a,b]$
then	$m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M \cdot (b-a)$

You may have noticed that we haven't called the justifications of these properties "proofs," in part because we haven't precisely defined what $\lim\limits_{\|\mathcal{P}\|\to 0}$ means, but also because of some other technical details left to more advanced textbooks.

This property follows easily from the previous one. First let $g(x) = M$ so that $f(x) \leq M = g(x)$ for all x in $[a,b]$, hence

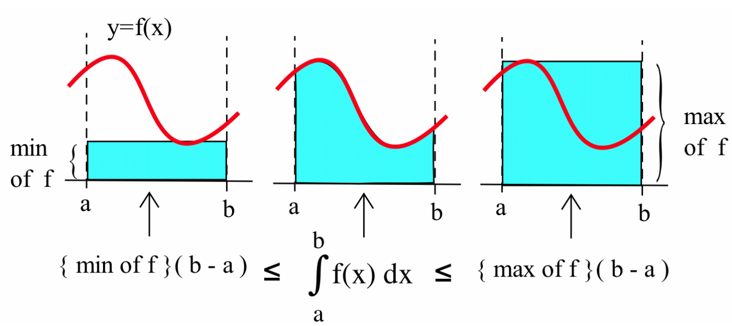
$$\int_a^b f(x)\,dx \leq \int_a^b M\,dx = M \cdot (b-a)$$

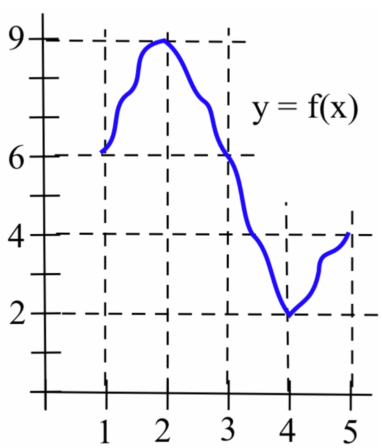
(using one of our previous properties). Likewise, taking $g(x) = m$ so that $f(x) \geq m = g(x)$ for all x in $[a,b]$:

$$\int_a^b f(x)\,dx \geq \int_a^b m\,dx = m \cdot (b-a)$$

Geometrically, this says that if we can "trap" the output values of a function on the interval $[a,b]$ between two upper and lower bounds, m and M, then the value of the definite integral must lie between the areas of the rectangles with heights m and M.

If f is continuous on the closed interval $[a,b]$, then we know that f takes on a minimum value on that interval (call it m) and a maximum value (call it M), in which case this property just uses the lower and upper Riemann sums for the simplest possible partition of $[a,b]$:





Example 1. Determine lower and upper bounds for the value of $\int_1^5 f(x)\,dx$ with $f(x)$ given graphically in the margin.

Solution. If $1 \leq x \leq 5$, then we can estimate (from the graph) that $2 \leq f(x) \leq 9$ so a lower bound for $\int_1^5 f(x)\,dx$ is

$$(b-a)\cdot(\text{minimum of } f \text{ on } [a,b]) = (4)(2) = 8$$

and an upper bound is:

$$(b-a)\cdot(\text{maximum of } f \text{ on } [a,b]) = (4)(9) = 36$$

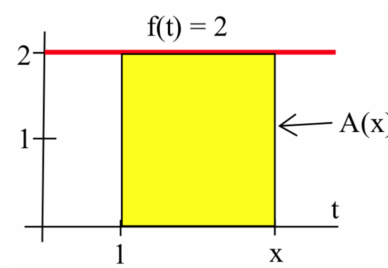
We can conclude that $8 \leq \int_1^5 f(x)\,dx \leq 36$. ◀

Knowing that the value of a definite integral is somewhere between 8 and 36 is not useful for finding its exact value, but the preceding estimation property is very easy to use and provides a "ballpark estimate" that will help you avoid reporting an unreasonable value.

Practice 2. Determine a lower bound and an upper bound for the value of $\int_3^5 f(x)\,dx$ with f as in the previous Example.

Functions Defined by Integrals

If one of the endpoints a or b of the interval $[a,b]$ changes, then the value of the integral $\int_a^b f(t)\,dt$ typically changes. A definite integral of the form $\int_a^x f(t)\,dt$ defines a function of x that possesses interesting and useful properties. The next examples illustrate one such property: the derivative of a function defined by an integral is closely related to the integrand, the function "inside" the integral.



Example 2. For the function $f(t) = 2$, define $A(x)$ to be the area of the region bounded by f, the t-axis, and vertical lines at $t = 1$ and $t = x$.

(a) Evaluate $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

(b) Find an algebraic formula for $A(x)$ valid for $x \geq 1$.

(c) Calculate $A'(x)$.

(d) Express $A(x)$ as a definite integral.

Solution. (a) Referring to the graph in the margin, we can see that $A(1) = 0$, $A(2) = 2$, $A(3) = 4$ and $A(4) = 6$. (b) Using the same area idea to compute a more general area:

$$A(x) = \text{area of a rectangle} = (\text{base})\,(\text{height}) = (x-1)(2) = 2x - 2$$

(c) $A'(x) = \dfrac{d}{dx}(2x-2) = 2$ (d) $A(x) = \int_1^x 2\,dt$ ◀

Practice 3. Answer the questions in the previous Example for $f(x) = 3$.

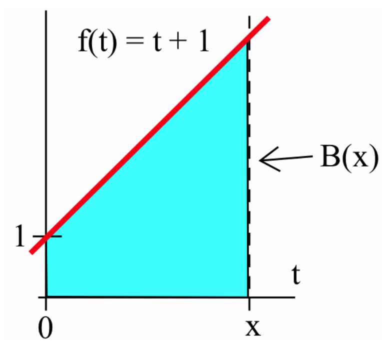
Example 3. For the function $f(t) = 1 + t$, define $B(x)$ to be the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = 0$ and $t = x$ (see margin).

(a) Evaluate $B(0)$, $B(1)$, $B(2)$ and $B(3)$.

(b) Find an algebraic formula for $B(x)$ valid for $x \geq 0$.

(c) Calculate $B'(x)$.

(d) Express $B(x)$ as a definite integral.



Solution. (a) From the graph, $B(0) = 0$, $B(1) = 1.5$, $B(2) = 4$ and $B(3) = 7.5$. (b) Using the same area concept:

$$\begin{aligned} B(x) &= \text{area of trapezoid} = (\text{base}) \cdot (\text{average height}) \\ &= (x) \cdot \left(\frac{1 + (1+x)}{2} \right) = x + \frac{1}{2}x^2 \end{aligned}$$

(c) $B'(x) = \dfrac{d}{dx}\left(x + \dfrac{1}{2}x^2 \right) = 1 + x$ (d) $B(x) = \displaystyle\int_0^x [1+t]\, dt$ ◀

Practice 4. Answer the questions in the previous Example for $f(t) = 2t$.

A curious "coincidence" appeared in each of these Examples and Practice problems: the derivative of the function defined by the integral was the same as the integrand, the function "inside" the integral. Stated another way, the function defined by the integral was an "antiderivative" of the function "inside" the integral. In Section 4.4 we will see that this "coincidence" is actually a property shared by all functions defined by an integral in this way. And it is such an important property that it is part of a result called the Fundamental Theorem of Calculus. Before we study the Fundamental Theorem of Calculus, however, we need to consider an "existence" question: Which functions can be integrated?

Which Functions Are Integrable?

This important question was finally answered in the 1850s by Bernhard Riemann, a name that should be familiar to you by now. Riemann proved that a function must be *badly* discontinuous in order to not be integrable.

Due to our inexact definition of the limit involved in the definition of the definite integral, we defer a proof of this theorem to more advanced textbooks.

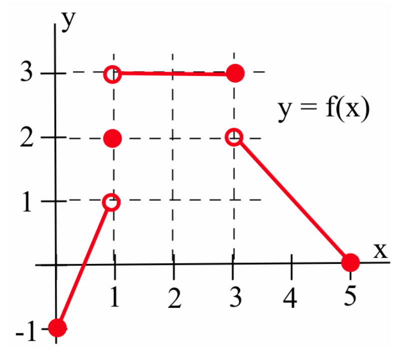
Theorem: Every continuous function is integrable.

This result says that if f is continuous on the interval $[a, b]$, then $\displaystyle\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ approaches the same finite number, $\displaystyle\int_a^b f(x)\, dx$, as $\|\mathcal{P}\| \to 0$, no matter how we choose the partitions $\mathcal{P}$.

In fact, we can generalize this result to functions that have a finite number of breaks or jumps, as long as the function is bounded:

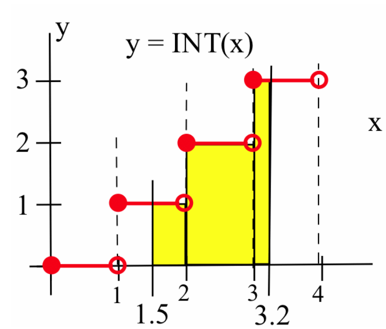
> **Theorem:**
> If f is defined on an interval $[a,b]$ and bounded
> ($|f(x)| \le M$ for some number M for all x in $[a,b]$)
> and continuous except at a finite number of points in $[a,b]$
> then f is integrable on $[a,b]$.

The function f graphed in the margin is always between -3 and 3 (in fact, always between -1 and 3), so it is bounded, and it is continuous except at $x=1$ and $x=3$. As long as the values of $f(1)$ and $f(3)$ are finite numbers, their actual values will not affect the value of the definite integral, and we can compute the value of the integral by computing the areas of the (triangular and rectangular) regions between the graph of f and the x-axis:

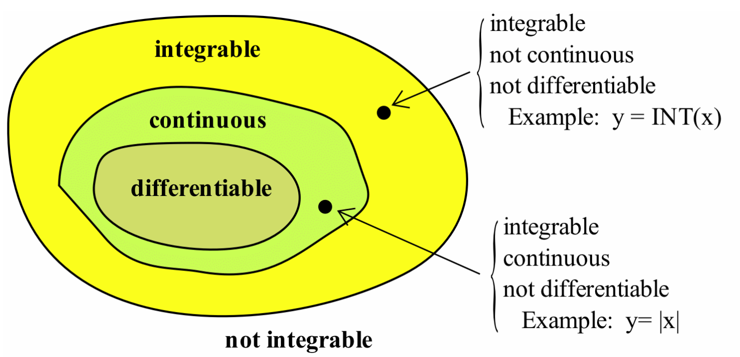


$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = 0+6+2=8$$

Practice 5. Evaluate $\int_{1.5}^{3.2} \lfloor x \rfloor\,dx$ (see margin).



The figure below depicts graphically the relationships between differentiable, continuous and integrable functions:



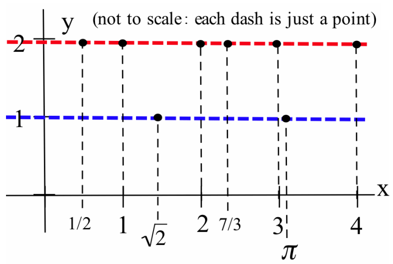
This says:

- Every differentiable function is continuous, but there are continuous functions that are not differentiable: a simple example of the latter is $f(x) = |x|$, which is continuous but not differentiable at $x=0$.
- Every continuous function is integrable, but there are integrable functions that are not continuous: a simple example of the latter situation is the function $f(x)$ graphed in the margin, which is integrable on $[0,5]$ but discontinuous at $x=2$ and $x=3$.
- Finally, as demonstrated by the next example, there are functions that are not integrable.

A Non-integrable Function

If f is continuous or piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$. Fortunately, nearly all of the functions we will use throughout the rest of this book are integrable, as are the functions you are likely to need for common applications.

There are functions, however, for which the limit of the Riemann sums does not exist and hence, by definition, are not integrable. Recall the "holey" function from Section 0.4:



> The function
>
> $$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$
>
> is **not** integrable on $[0, 3]$.

Proof. For any partition $\mathcal{P}$ of $[0, 3]$, suppose that you, a very rational person, always choose values of c_k that are rational numbers. (Any open interval on the real-number line contains rational numbers and irrational numbers, so for each subinterval of the partition $\mathcal{P}$ you can always choose c_k to be a rational number.)

Then $h(c_k) = 2$, so for your Riemann sum:

$$\text{YS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} 2 \cdot \Delta x_k = 2 \cdot \sum_{k=1}^{n} \Delta x_k = 2 \cdot (3 - 0) = 6$$

Suppose your friend, however, always selects values of c_k that are irrational numbers. Then $h(c_k) = 1$ for each c_k, so for your friend's Riemann sum:

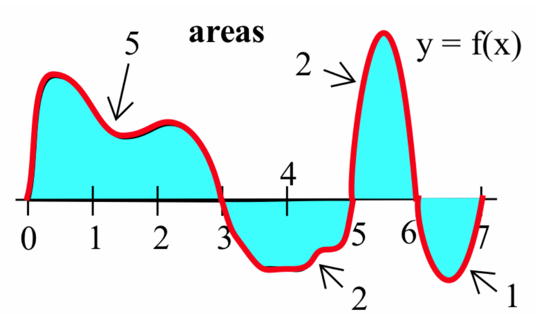
$$\text{FS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = 1 \cdot \sum_{k=1}^{n} \Delta x_k = 1 \cdot (3 - 0) = 3$$

So the limit of your Riemann sums, as the mesh of $\mathcal{P}$ approaches 0, will be 6, while the limit of your friend's sums will be 3. This means that $\lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} h(c_k) \cdot \Delta x_k \right)$ does not exist (because there is no single limiting value of the Riemann sums as $\|\mathcal{P}\| \to 0$) so $h(x)$ is not integrable on $[0, 3]$. $\square$

A similar argument shows that $h(x)$ is not integrable on *any* interval of the form $[a, b]$ (where $a < b$).

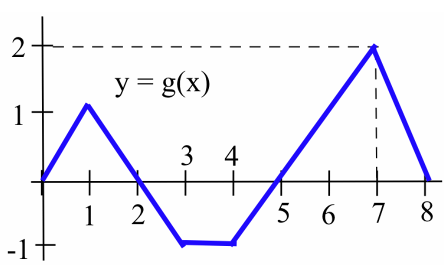
4.3 Problems

In Problems 1–20, refer to the graph of f given below to determine the value of each definite integral.



1. $\int_0^3 f(x)\,dx$
2. $\int_3^5 f(x)\,dx$
3. $\int_2^2 f(x)\,dx$
4. $\int_6^7 f(w)\,dw$
5. $\int_0^5 f(x)\,dx$
6. $\int_0^7 f(x)\,dx$
7. $\int_3^6 f(t)\,dt$
8. $\int_5^7 f(x)\,dx$
9. $\int_3^0 f(x)\,dx$
10. $\int_5^3 f(x)\,dx$
11. $\int_6^0 f(x)\,dx$
12. $\int_0^3 2\cdot f(x)\,dx$
13. $\int_4^4 f^2(s)\,ds$
14. $\int_0^3 [1+f(x)]\,dx$
15. $\int_0^3 [x+f(x)]\,dx$
16. $\int_3^5 [3+f(x)]\,dx$
17. $\int_0^5 [2+f(x)]\,dx$
18. $\int_3^5 |f(x)|\,dx$
19. $\int_0^5 |f(x)|\,dx$
20. $\int_7^3 [1+|f(x)|]\,dx$

Problems 21–30 refer to the graph of g given below. Use the graph to evaluate each integral.



21. $\int_0^2 g(x)\,dx$
22. $\int_1^3 g(t)\,dt$
23. $\int_0^5 g(x)\,dx$
24. $\int_4^2 g(x)\,dx$
25. $\int_0^8 g(s)\,ds$
26. $\int_1^4 |g(x)|\,dx$
27. $\int_0^3 2\cdot g(t)\,dt$
28. $\int_5^8 [1+g(x)]\,dx$
29. $\int_6^3 g(u)\,du$
30. $\int_0^8 [t+g(t)]\,dt$

For 31–34 , use the constant functions $f(x)=4$ and $g(x)=3$ on the interval $[0,2]$. Calculate the value of each integral and verify that the value obtained in part (a) is **not** equal to the value in part (b).

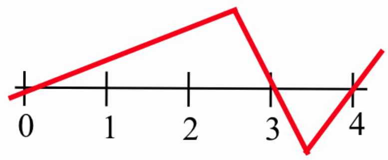
31. (a) $\int_0^2 f(x)\,dx\cdot\int_0^2 g(x)\,dx$ (b) $\int_0^2 f(x)\cdot g(x)\,dx$
32. (a) $\dfrac{\int_0^2 f(x)\,dx}{\int_0^2 g(x)\,dx}$ (b) $\int_0^2 \dfrac{f(x)}{g(x)}\,dx$
33. (a) $\int_0^2 [f(x)]^2\,dx$ (b) $\left(\int_0^2 f(x)\,dx\right)^2$
34. (a) $\int_0^2 \sqrt{f(x)}\,dx$ (b) $\sqrt{\int_0^2 f(x)\,dx}$

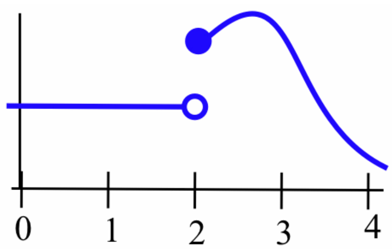
For 35–42, sketch a graph of the integrand function and use it to help evaluate the integral.

35. $\int_0^4 |x|\,dx$
36. $\int_0^4 [1+|t|]\,dt$
37. $\int_{-1}^2 |x|\,dx$
38. $\int_0^2 [|x|-1]\,dx$
39. $\int_1^3 \lfloor u\rfloor\,du$
40. $\int_1^{3.5} \lfloor x\rfloor\,dx$
41. $\int_1^3 [2+\lfloor t\rfloor]\,dt$
42. $\int_3^1 \lfloor x\rfloor\,dx$

For Problems 43–46, sketch (a) a graph of $y=A(x)=\int_0^x f(t)\,dt$ and (b) a graph of $y=A'(x)$.

43. $f(x)=x$
44. $f(x)=x-2$

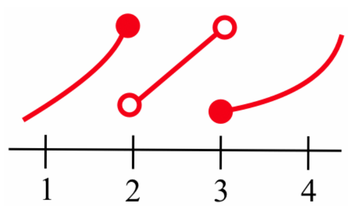
45. 

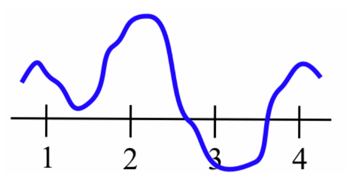
46. 

For 47–50, state whether or not each function is: (a) continuous on $[1,4]$ (b) differentiable on $[1,4]$ (c) integrable on $[1,4]$

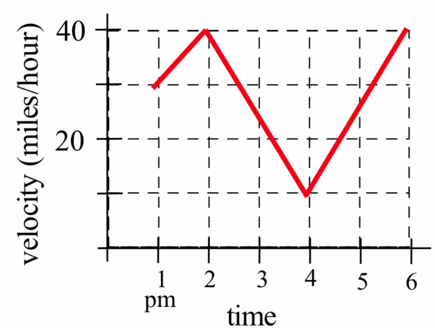
47. $f(x)$ from Problem 45.

48. $f(x)$ from Problem 46.

49. 

50. 

51. The figure below shows the velocity of a car. Write the total distance traveled by the car between 1:00 p.m. and 4:00 p.m. as a definite integral and estimate the value of that integral.



52. Write the total distance traveled by the car in the previous problem between 3:00 p.m. and 6:00 p.m. as a definite integral and estimate the value of that integral.

53. Define $g(x) = 7$ for $x \neq 2$ and $g(2) = 5$.

(a) Show that the Riemann sum for $g(x)$ for any partition $\mathcal{P}$ of the interval $[1,4]$ is equal to $5w + 7(3-w)$, where w is the width of the subinterval that includes $x = 2$.

(b) Compute the limit of these sums, as $\|\mathcal{P}\| \to 0$

(c) Compare the values of $\int_1^4 g(x)\,dx$ and $\int_1^4 7\,dx$.

(d) What can you conclude about how changing the value of an integrable function at a single point affects the value of its definite integral?

4.3 Practice Answers

1. $\displaystyle\int_1^4 [f(x) - g(x)]\,dx = 7 - 3 = 4$

2. $m = 2$ and $M = 6$ so $(2)(5-3) = 4 \leq \displaystyle\int_3^5 f(x)\,dx \leq 12 = (6)(5-3)$

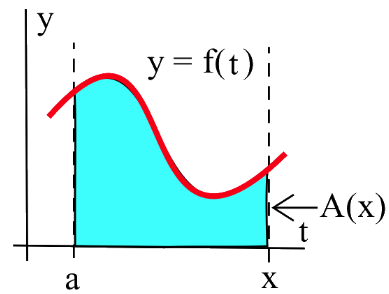
3. (a) $A(1) = 0$, $A(2) = 3$, $A(3) = 6$, $A(4) = 9$
(b) $A(x) = (x-1)(3) = 3x - 3$ (c) $A'(x) = 3$ (d) $A(x) = \displaystyle\int_1^x 3\,dt$

4. (a) $B(0) = 0$, $B(1) = 1$, $B(2) = 4$, $B(3) = 9$
(b) $B(x) = \frac{1}{2}(x)(2x) = x^2$ (c) $B'(x) = 2x$ (d) $B(x) = \displaystyle\int_0^x 2t\,dt$

5. $(0.5)(1) + (1)(2) + (0.2)(3) = 3.1$

4.4 *Areas, Integrals and Antiderivatives*

This section explores properties of functions defined as areas and examines some connections among areas, integrals and antiderivatives. In order to focus on these connections and their geometric meaning, all of the functions in this section are nonnegative, but in the next section we will generalize (and prove) the results for all continuous functions. This section also introduces examples showing how you can use the relationships between areas, integrals and antiderivatives in various applications.



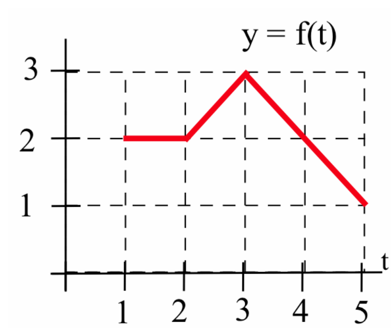
When f is a continuous, nonnegative function, the "area function" $A(x) = \int_a^x f(t)\,dt$ represents the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = a$ and $t = x$ (see margin figure), and the derivative of $A(x)$ represents the rate of change (growth) of $A(x)$ as the vertical line $t = x$ moves rightward. Examples 2 and 3 of Section 4.3 showed that for certain functions f, $A'(x) = f(x)$ so that $A(x)$ was an antiderivative of $f(x)$. The next theorem says the result is true for every continuous, nonnegative function f.

The Area Function Is an Antiderivative

If	f is a continuous, nonnegative function
	and $A(x) = \int_a^x f(t)\,dt$ for $x \geq a$
then	$\dfrac{d}{dx}\left(\int_a^x f(t)\,dt\right) = A'(x) = f(x)$
so	$A(x)$ is an antiderivative of $f(x)$.

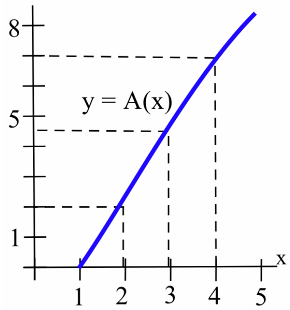
This result relating integrals and antiderivatives is a special case (for nonnegative functions f) of the first part of the Fundamental Theorem of Calculus (FTC^1), which we will prove in Section 4.5. This result is important for two reasons:

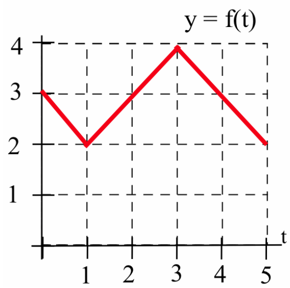
- It says that a large collection of functions have antiderivatives.
- It leads to an **easy** way to **exactly** evaluate definite integrals.



Example 1. Define $A(x) = \int_1^x f(t)\,dt$ for the function $f(t)$ shown in the margin. Estimate the values of $A(x)$ and $A'(x)$ for $x = 2, 3, 4$ and 5 and use these values to sketch a graph of $y = A(x)$.

Solution. Dividing the region into squares and triangles, it is easy to see that $A(2) = 2$, $A(3) = 4.5$, $A(4) = 7$ and $A(5) = 8.5$. Because $A'(x) = f(x)$, we know that $A'(2) = f(2) = 2$, $A'(3) = f(3) = 3$, $A'(4) = f(4) = 2$ and $A'(5) = f(5) = 1$. A graph of $y = A(x)$ appears in the margin at the top of the next page. ◀

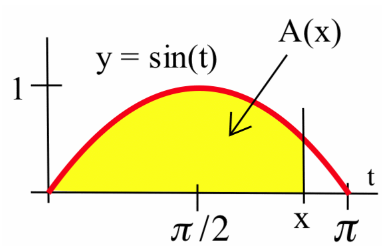




It is important to recognize that f is not differentiable at $x = 2$ or $x = 3$ but that the values of A change smoothly near $x = 2$ and $x = 3$, and the function A *is* differentiable at those points and at every other point between $x = 1$ and $x = 5$. Also note that $f'(4) = -1$ (f is clearly decreasing near $x = 4$) but that $A'(4) = f(4) = 2$ is positive (the area A is growing even though f is getting smaller).

Practice 1. Let $B(x)$ be the area bounded by the horizontal axis, vertical lines at $t = 0$ and $t = x$, and the graph of $f(t)$ shown in the margin. Estimate the values of $B(x)$ and $B'(x)$ for $x = 1, 2, 3, 4$ and 5.

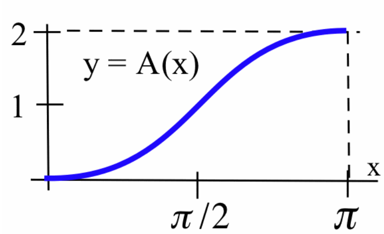
Example 2. Let $G(x) = \dfrac{d}{dx}\left(\int_0^x \sin(t)\,dt\right)$. Evaluate $G(x)$ for $x = \dfrac{\pi}{4}$, $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{4}$.

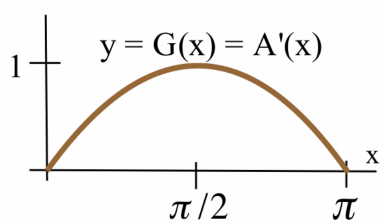


Solution. The middle margin figure shows $A(x) = \int_0^x \sin(t)\,dt$ graphically. By the theorem, $A'(x) = \sin(x)$, so:

$$G\left(\frac{\pi}{4}\right) = A'\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707$$
$$G\left(\frac{\pi}{2}\right) = A'\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$
$$G\left(\frac{3\pi}{4}\right) = A'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707$$

The penultimate margin figure shows a graph of $y = A(x)$ and the bottom margin figure shows the graph of $y = A'(x) = G(x)$. ◀





Using Antiderivatives to Evaluate $\int_a^b f(x)\,dx$

Now we combine the ideas of areas and antiderivatives to devise a technique for evaluating definite integrals that is exact—and often easy.

If $A(x) = \int_a^x f(t)\,dt$, then we know that $A(a) = \int_a^a f(t)\,dt = 0$, $A(b) = \int_a^b f(t)\,dt$ and that $A(x)$ is an antiderivative of f, so $A'(x) = f(x)$. We also know that if $F(x)$ is **any** antiderivative of f, then $F(x)$ and $A(x)$ have the same derivative so $F(x)$ and $A(x)$ are "parallel" functions and differ by a constant: $F(x) = A(x) + C$ for all x and some constant C. As a consequence:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a)$$
$$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt$$

This result says that, to evaluate a definite integral $A(b) = \int_a^b f(t)\,dt$, we can find **any** antiderivative F of f and simply evaluate $F(b) - F(a)$.

This result is a special case of the second part of the Fundamental Theorem of Calculus (FTC^2, stated and proved in Section 4.5), which you will use hundreds of times over the next several chapters.

> **Antiderivatives and Definite Integrals**
>
> If f is a continuous, nonnegative function and F is any antiderivative of f (so that $F'(x) = f(x)$) on $[a, b]$
>
> then $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$

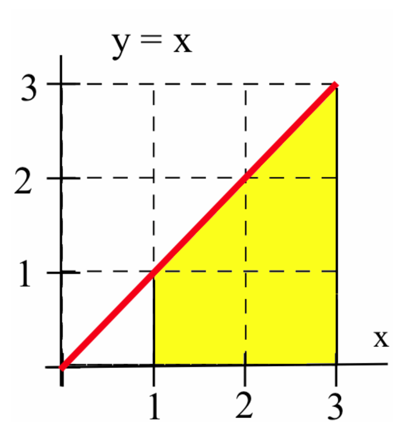
The problem of finding the exact value of a definite integral has been reduced to finding some (any) antiderivative F of the integrand and then evaluating $F(b) - F(a)$. Even finding one antiderivative can be difficult, so for now we will restrict our attention to functions that have "easy" antiderivatives. Later we will explore some methods for finding antiderivatives of more "difficult" functions.

Because an evaluation of the form $F(b) - F(a)$ will occur quite often, we represent it symbolically as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

Example 3. Evaluate $\displaystyle\int_1^3 x\,dx$ in two ways:

(a) by sketching a graph of $y = x$ and finding the area represented by the definite integral.

(b) by finding an antiderivative $F(x)$ of $f(x) = x$ and evaluating $F(3) - F(1)$.



Solution. (a) A graph of $y = x$ appears in the margin; the area of the trapezoidal region in question has area 4. (b) One antiderivative of x is $F(x) = \dfrac{1}{2}x^2$ (you should check for yourself that $\mathbf{D}\left(\dfrac{x^2}{2}\right) = x$), so:

$$F(x)\Big|_1^3 = F(3) - F(1) = \frac{1}{2}(3)^2 - \frac{1}{2}(1)^2 = \frac{9}{2} - \frac{1}{2} = 4$$

which agrees with the area from part (a).

If someone chose another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$ (you should check for yourself that $\mathbf{D}\left(\dfrac{x^2}{2} + 7\right) = x$), then:

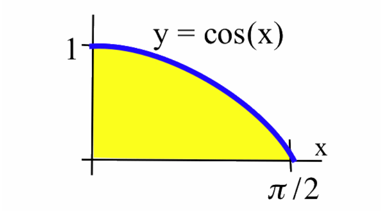
$$F(x)\Big|_1^3 = F(3) - F(1) = \left[\frac{1}{2}(3)^2 + 7\right] - \left[\frac{1}{2}(1)^2 + 7\right] = \frac{23}{2} - \frac{15}{2} = 4$$

No matter which antiderivative F we choose, $F(3) - F(1) = 4$. ◀

Practice 2. Evaluate $\displaystyle\int_1^3 (x-1)\,dx$ in the two ways specified in the previous Example.

This antiderivative method provides an extremely powerful way to evaluate some definite integrals, and we will use it often.

Example 4. Find the area of the region in the first quadrant bounded by the graph of $y = \cos(x)$, the horizontal axis, and the line $x = 0$.



Solution. The area we want (see margin) is $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$ so we need an antiderivative of $f(x) = \cos(x)$. $F(x) = \sin(x)$ is one such antiderivative (you should check that $\mathbf{D}(\sin(x)) = \cos(x)$), so

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \sin(x)\Big|_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$$

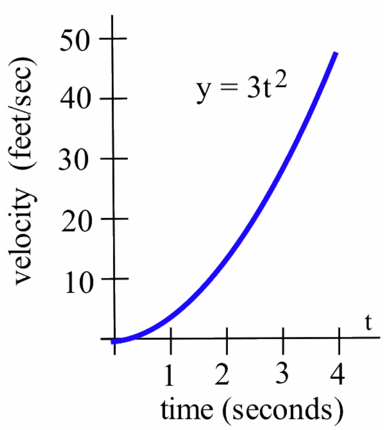
is the area of the region in question. ◀

Practice 3. Find the area of the region bounded by the graph of $y = 3x^2$, the horizontal axis and the vertical lines $x = 1$ and $x = 2$.

Integrals, Antiderivatives and Applications

The antiderivative method for evaluating definite integrals can also be used when we need to find a more general "area," so it is often useful for solving applied problems.

Example 5. A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet per second.



(a) How far will the robot travel during its first four seconds of movement?

(b) How far will the robot travel during its next four seconds of movement?

(c) How long will it take for the robot to move 729 feet from its starting place?

Solution. (a) The distance during the first four seconds will be the area under the graph of the velocity function (see margin figure) from $t = 0$ to $t = 4$, an area we can compute with the definite integral $\int_0^4 3t^2\,dt$. One antiderivative of $3t^2$ is t^3 so:

$$\int_0^4 3t^2\,dt = \left[t^3\right]_0^4 = 4^3 - 0^3 = 64$$

and we can conclude that the robot will be 64 feet away from its starting position after four seconds.

(b) Proceeding similarly:

$$\int_4^8 3t^2\,dt = \left[t^3\right]_4^8 = 8^3 - 4^3 = 512 - 64 = 448 \text{ feet}$$

(c) This question is different from the first two. Here we know the lower integration endpoint, $t = 0$, and the total distance, 729 feet, and need to find the upper integration endpoint (the time when the robot is 729 feet away from its starting position). Calling this upper endpoint T, we know that:

$$729 = \int_0^T 3t^2\,dt = \left[t^3\right]_0^T = T^3 - 0^3 = T^3$$

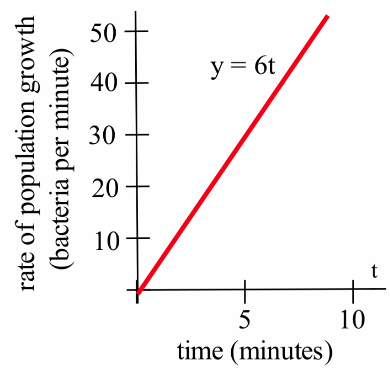
so $T = \sqrt[3]{729} = 9$. The robot is 729 feet away after 9 seconds. ◀

Practice 4. Refer to the robot from the previous Example.

(a) How far will the robot travel between $t = 1$ and $t = 5$ seconds?

(b) How long will it take for the robot to move 343 feet from its starting place?

Example 6. Suppose that t minutes after placing 1,000 bacteria on a Petri plate the rate of growth of the bacteria population is $6t$ bacteria per minute.



(a) How many new bacteria are added to the population during the first seven minutes?

(b) What is the total population after seven minutes?

(c) When will the total population reach 2,200 bacteria?

Solution. (a) The number of new bacteria is represented by the area under the rate-of-growth graph (see margin) and one antiderivative of $6t$ is $3t^2$ (check that $\mathbf{D}\left(3t^2\right) = 6t$) so:

$$\text{new bacteria} = \int_0^7 6t\,dt = \left[3t^2\right]_0^7 = 3(7)^2 - 3(0)^2 = 147$$

(b) [old population] + [new bacteria] $= 1000 + 147 = 1147$ bacteria.

(c) When the total population reaches 2,200 bacteria, then there are $2200 - 1000 = 1200$ new bacteria, hence we need to find the time T required for that many new bacteria to grow:

$$1200 = \int_0^T 6t\,dt = \left[3t^2\right]_0^T = 3(T)^2 - 3(0)^2 = 3T^2$$

so $T^2 = 400 \Rightarrow T = 20$. After 20 minutes, the total bacteria population will be $1000 + 1200 = 2200$. ◀

Practice 5. Refer to the bacteria population from the previous Example.

(a) How many new bacteria will be added to the population between $t = 4$ and $t = 8$ minutes?

(b) When will the total population reach 2,875 bacteria?

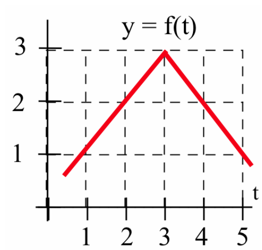
4.4 Problems

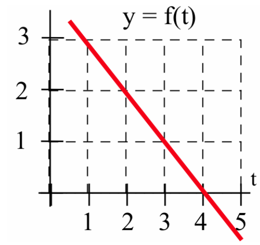
In Problems 1–8, $A(x) = \int_1^x f(t)\,dt$ with $f(t)$ given.

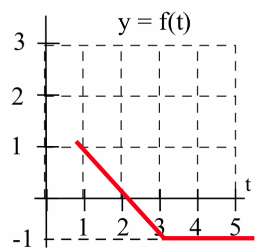
(a) Graph $y = A(x)$ for $1 \le x \le 5$.

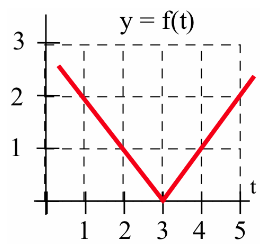
(b) Estimate the values of $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

(c) Estimate $A'(1)$, $A'(2)$, $A'(3)$ and $A'(4)$.

1. 

2. 

3. 

4. 

5. $f(t) = 2$

6. $f(t) = 1 + t$

7. $f(t) = 6 - t$

8. $f(t) = 1 + 2t$

In Problems 9–18, use the **Antiderivatives and Definite Integrals** Theorem to evaluate each integral.

9. (a) $\int_0^3 2x\,dx$ (b) $\int_1^3 2x\,dx$ (c) $\int_0^1 2x\,dx$

10. (a) $\int_0^2 4x^3\,dx$ (b) $\int_0^1 4x^3\,dx$ (c) $\int_1^2 4x^3\,dx$

11. (a) $\int_1^3 6x^2\,dx$ (b) $\int_1^2 6x^2\,dx$ (c) $\int_0^3 6x^2\,dx$

12. (a) $\int_{-2}^2 2x\,dx$ (b) $\int_{-2}^{-1} 2x\,dx$ (c) $\int_{-2}^0 2x\,dx$

13. (a) $\int_0^3 4x^3\,dx$ (b) $\int_1^3 4x^3\,dx$ (c) $\int_0^1 4x^3\,dx$

14. (a) $\int_0^5 4x^3\,dx$ (b) $\int_0^2 4x^3\,dx$ (c) $\int_2^5 4x^3\,dx$

15. (a) $\int_{-3}^3 3x^2\,dx$ (b) $\int_{-3}^0 3x^2\,dx$ (c) $\int_0^3 3x^2\,dx$

16. (a) $\int_0^3 5\,dx$ (b) $\int_0^2 5\,dx$ (c) $\int_2^3 5\,dx$

17. (a) $\int_0^2 3x^2\,dx$ (b) $\int_1^3 3x^2\,dx$ (c) $\int_3^1 3x^2\,dx$

18. (a) $\int_{-2}^2 \left[12 - 3x^2\right]\,dx$ (b) $\int_1^2 \left[12 - 3x^2\right]\,dx$

In 19–21, use the given velocity of a car (in feet per second) after t seconds to find:

(a) how far the car travels during the first 10 seconds.

(b) how many seconds it takes the car to travel half the distance in part (a).

19. $v(t) = 2t$

20. $v(t) = 3t^2$

21. $v(t) = 4t^3$

Problems 22–23 give the velocity of a car (in feet per second) after t seconds.

(a) How many seconds does it take for the car to come to a stop (velocity $= 0$)?

(b) How far does the car travel before coming to a stop?

(c) How many seconds does it take the car to travel half the distance in part (b)?

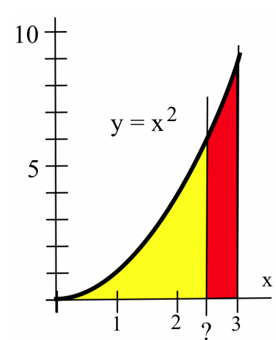
22. $v(t) = 20 - 2t$

23. $v(t) = 75 - 3t^2$

24. Find the exact area under half of one arch of the sine curve: $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$.

25. An artist you know wants to make a figure consisting of the region between the curve $y = x^2$ and the x-axis for $0 \le x \le 3$.

(a) Where should the artist divide the region with a vertical line (see figure below) so that each piece has the same area?



(b) Where should she divide the region with vertical lines to get three pieces with equal areas?

4.4 Practice Answers

1. $B(1) = 2.5$, $B(2) = 5$, $B(3) = 8.5$, $B(4) = 12$, $B(5) = 14.5$
$B(x) = \int_0^x f(t)\,dt \Rightarrow B'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$
(by the **Area Function Is an Antiderivative** Theorem), hence:
$B'(1) = f(1) = 2$, $B'(2) = f(2) = 3$, $B'(3) = 4$, $B'(4) = 3$ and $B'(5) = 2$.

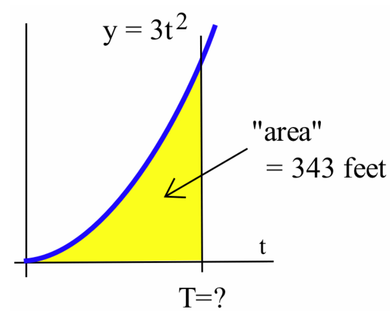
2. (a) $\int_1^3 (x-1)\,dx$ gives the area of the triangular region between the graph of $y = x - 1$ and the x-axis for $1 \le x \le 3$:

$$\text{area} = \frac{1}{2}\,(\text{base})\,(\text{height}) = \frac{1}{2}(2)(2) = 2$$

(b) $F(x) = \frac{1}{2}x^2 - x$ is an antiderivative of $f(x) = x - 1$ so:

$$\int_1^3 (x-1)\,dx = F(3) - F(1) = \left[\frac{1}{2}\cdot 3^3 - 3\right] - \left[\frac{1}{2}\cdot 1^3 - 1\right] = 2$$

3. Area $= \int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = 8 - 1 = 7$



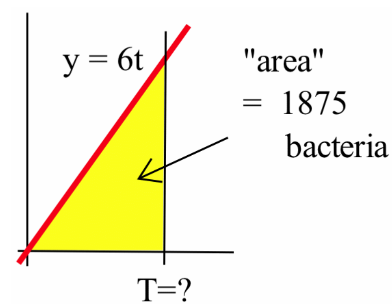
4. (a) distance $= \int_1^5 3t^2\,dt = t^3\Big|_1^5 = 125 - 1 = 124$ feet.

(b) We know the starting point is $x = 0$ and the total distance ("area" under the velocity curve) is 343 feet. We need to find the time T (see margin figure) so that 343 feet $= \int_0^T 3t^2\,dt$:

$$343 = \int_0^T 3t^2\,dt = t^3\Big|_0^T = T^3 - 0 = T^3$$

hence $T = \sqrt[3]{343} = 7$ seconds.

5. (a) new bacteria $= \int_4^8 6t\,dt = 3t^2\Big|_4^8 = 3\cdot 64 - 3\cdot 16 = 144$ bacteria.



(b) We know the total new population ("area" under the rate-of-change graph) is $2875 - 1000 = 1875$ so:

$$1875 = \int_0^T 6t\,dt = 3t^2\Big|_0^T = 3T^2 - 0 = 3T^2 \Rightarrow T^2 = 625$$

hence $T = \sqrt{625} = 25$ minutes.

4.5 *The Fundamental Theorem of Calculus*

This section contains the most important and most frequently used theorem of calculus, **THE** Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz during the late 1600s, it establishes a connection between derivatives and integrals, provides a way to easily calculate many definite integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is the most important brick in that beautiful structure.

Prior sections have emphasized the meaning of the definite integral, defined it, and began to explore some of its applications and properties. In this section, the emphasis shifts to the Fundamental Theorem of Calculus. You will use this theorem often in later sections.

The Fundamental Theorem has two parts. They resemble results in the previous section but apply to more general situations. The first part (FTC^1) says that every continuous function has an antiderivative and shows how to differentiate a function defined as an integral. The second part (FTC^2) shows how to evaluate the definite integral of any function if we know a formula for an antiderivative of that function.

Part 1: Antiderivatives

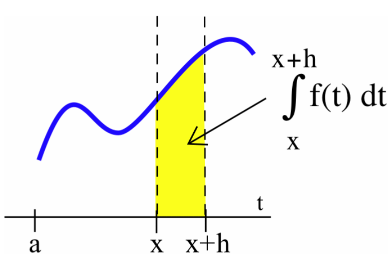
Every continuous function has an antiderivative, even functions with "corners," such as the absolute value function $f(x) = |x|$, that fail to be differentiable at one or more points.

The Fundamental Theorem of Calculus Part 1 (FTC^1)

If f is continuous and $A(x) = \int_a^x f(t)\,dt$

then $A'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

so $A(x)$ is an antiderivative of $f(x)$.



Proof. For a continuous function f, let $A(x) = \int_a^x f(t)\,dt$. By the definition of derivative,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Using one of the integral properties from Section 4.3, we know that:

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

$$\Rightarrow \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

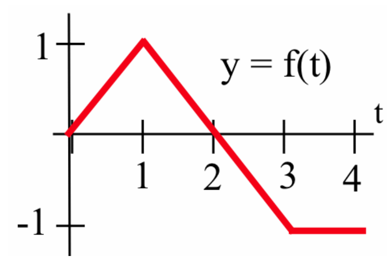
Assume for the moment that $h > 0$. Because f is continuous on $[x, x+h]$ we know that f attains a maximum and minimum on that interval, so there are values m_h and M_h with $x < m_h < x+h$ and $x < M_h < x+h$ so that $f(m_h) \leq f(t) \leq f(M_h)$ when $x \leq t \leq x+h$. Hence:

$$\int_x^{x+h} f(m_h)\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} f(M_h)\,dt$$
$$\Rightarrow \quad f(m_h)\cdot h \leq \int_x^{x+h} f(t)\,dt \leq f(M_h)\cdot h$$
$$\Rightarrow \quad f(m_h) \leq \frac{\int_x^{x+h} f(t)\,dt}{h} \leq f(M_h)$$

Because $x < m_h < x+h$, we know $\lim_{h\to 0^+} m_h = x$; consequently—because $f(t)$ is continuous—we also know that $\lim_{h\to 0^+} f(m_h) = f(x)$. Likewise, $\lim_{h\to 0^+} f(M_h) = f(x)$, so the Squeezing Theorem tells us that:

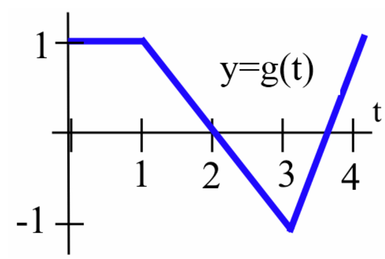
$$\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$

Repeating this argument for $h < 0$ is relatively straightforward. □



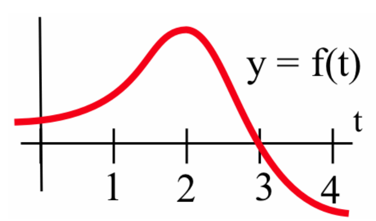
Example 1. Define $A(x) = \int_0^x f(t)\,dt$ for f in the margin figure. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Solution. $A(1) = \int_0^1 f(t)\,dt = \frac{1}{2}$, $A(2) = \int_0^2 f(t)\,dt = 1$, $A(3) = \int_0^3 f(t)\,dt = \frac{1}{2}$ and $A(4) = \int_0^4 f(t)\,dt = -\frac{1}{2}$. Because f is continuous, FTC[1] tells us that $A'(x) = f(x)$, so $A'(1) = f(1) = 1$, $A'(2) = f(2) = 0$, $A'(3) = f(3) = -1$ and $A'(4) = f(4) = -1$. ◀



Practice 1. Define $A(x) = \int_0^x g(t)\,dt$ for g in the margin figure. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Example 2. Define $A(x) = \int_0^x f(t)\,dt$ for f in the margin figure. For which value of x is $A(x)$ maximum? For which x is the rate of change of A maximum?



Solution. Because A is differentiable, the only critical points are where $A'(x) = 0$ or at endpoints. $A'(x) = f(x) = 0$ at $x = 3$, and A has a maximum at $x = 3$. Notice that the values of $A(x)$ increase as x goes from 0 to 3 and then the values of A decrease. The rate of change of $A(x)$ is $A'(x) = f(x)$, and $f(x)$ appears to have a maximum at $x = 2$, so the rate of change of $A(x)$ is maximum when $x = 2$. Near $x = 2$, a slight increase in the value of x yields the maximum increase in the value of $A(x)$. ◀

Part 2: Evaluating Definite Integrals

If we know a formula for an antiderivative of a function, then we can compute any definite integral of that function.

> **The Fundamental Theorem of Calculus Part 2 (FTC^2)**
>
> If $f(x)$ is continuous
> and $F(x)$ is any antiderivative of f (so that $F'(x) = f(x)$)
>
> then $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$

Proof. Define $A(x) = \displaystyle\int_a^x f(t)\,dt$. If F is an antiderivative of f, then $F'(x) = f(x)$ and by FTC^1 we know that $A'(x) = f(x)$ so $F'(x) = A'(x)$, hence $F(x)$ and $A(x)$ differ by a constant: $A(x) - F(x) = C$ for all x and some constant C. At $x = a$, we have $C = A(a) - F(a) = 0 - F(a) = -F(a)$ so $C = -F(a)$ and the equation $A(x) - F(x) = C$ becomes $A(x) - F(x) = -F(a)$. Then $A(x) = F(x) - F(a)$ for all x, so setting $x = b$ yields $A(b) = F(b) - F(a)$, hence $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, the formula we wanted. □

We can evaluate the definite integral of a continuous function f by finding an antiderivative of f (*any* antiderivative of f will work) and then doing some arithmetic with this antiderivative. FTC^2 does not tell us *how* to find an antiderivative of f, and it does not tell us how to find the definite integral of a discontinuous function. It is possible to evaluate definite integrals of some discontinuous functions (as we saw in Section 4.3) but not by using FTC^2 directly.

Example 3. Evaluate $\displaystyle\int_0^2 \left(x^2 - 1\right)\,dx$.

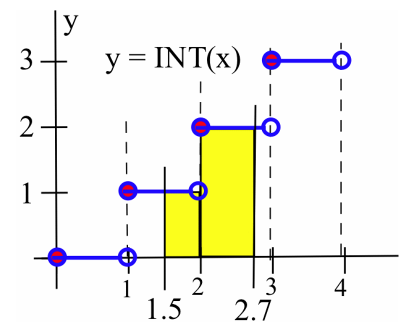
Solution. $F(x) = \frac{1}{3}x^3 - x$ is an antiderivative of $f(x) = x^2 - 1$ (you should check that $\mathbf{D}\left(\frac{1}{3}x^3 - x\right) = x^2 - 1$), so:

$$\int_0^2 \left(x^2 - 1\right)\,dx = \left[\frac{1}{3}x^3 - x\right]_0^2 = \left[\frac{1}{3}\cdot 2^3 - 2\right] - \left[\frac{1}{3}\cdot 0^3 - 0\right] = \frac{2}{3}$$

If your friend had picked a different antiderivative of $x^2 - 1$, say $G(x) = \frac{1}{3}x^3 - x + 4$, then her calculations would be slightly different :

$$\begin{aligned}\int_0^2 \left(x^2 - 1\right)\,dx &= \left[\frac{1}{3}x^3 - x + 4\right]_0^2 \\ &= \left[\frac{1}{3}\cdot 2^3 - 2 + 4\right] - \left[\frac{1}{3}\cdot 0^3 - 0 + 4\right] = \frac{2}{3} + 4 - 4 = \frac{2}{3}\end{aligned}$$

but the result would be the same. ◀



Practice 2. Evaluate $\int_1^3 \left(3x^2 - 1\right)\, dx$.

Example 4. Evaluate $\int_{1.5}^{2.7} \lfloor x \rfloor\, dx$ (where $\lfloor x \rfloor = \text{INT}(x)$ is the largest integer less than or equal to x, as in the margin figure).

Solution. $f(x) = \lfloor x \rfloor$ is not continuous at $x = 2$ in the interval $[1.5, 2.7]$, so we cannot employ the Fundamental Theorem of Calculus directly. We can, however, use our understanding of the geometric meaning of a definite integral to compute:

$$\begin{aligned}
\int_{1.5}^{2.7} \lfloor x \rfloor\, dx &= (\text{area below } y = \lfloor x \rfloor \text{ for } 1.5 \le x \le 2) + (\text{area below } y = \lfloor x \rfloor \text{ for } 2 \le x \le 2.7) \\
&= (\text{first base})\,(\text{first height}) + (\text{second base})\,(\text{second height}) \\
&= (0.5)(1) + (0.7)(2) = 1.9
\end{aligned}$$

We could also split the integral into two pieces:

$$\begin{aligned}
\int_{1.5}^{2.7} \lfloor x \rfloor\, dx &= \int_{1.5}^{2.0} \lfloor x \rfloor\, dx + \int_{2.0}^{2.7} \lfloor x \rfloor\, dx \\
&= \int_{1.5}^{2.0} 1\, dx + \int_{2.0}^{2.7} 2\, dx = \Big[x\Big]_{1.5}^{2.0} + \Big[x\Big]_{2.0}^{2.7} \\
&= [2.0 - 1.5] + [2(2.7) - 2(2.0)] = 0.5 + 1.4 = 1.9
\end{aligned}$$

using the fact that $\lfloor x \rfloor = 1$ for $1.5 \le x < 2.0$ and the fact that $\lfloor x \rfloor = 2$ for $2.0 \le x \le 2.7$. (We also need to redefine the first integrand to equal 1 at its right endpoint and the second integrand to equal 2 at its right endpoint so that each integrand is continuous on a closed interval). ◀

Problem 53 in Section 4.3 indicates that this redefinition is perfectly legal.

Practice 3. Evaluate $\int_{1.3}^{3.4} \lfloor x \rfloor\, dx$.

Calculus is the study of derivatives and integrals, their meanings and their applications. The Fundamental Theorem of Calculus demonstrates how differentiation and integration are closely related processes: integration is really anti-differentiation, the inverse of differentiation.

Applications: The Future

Calculus is important for many reasons, but students are usually required to study calculus because they will need to *apply* calculus concepts in a variety of fields. Most applied problems in integral calculus require the following steps to get from a real-life problem to a numerical answer:

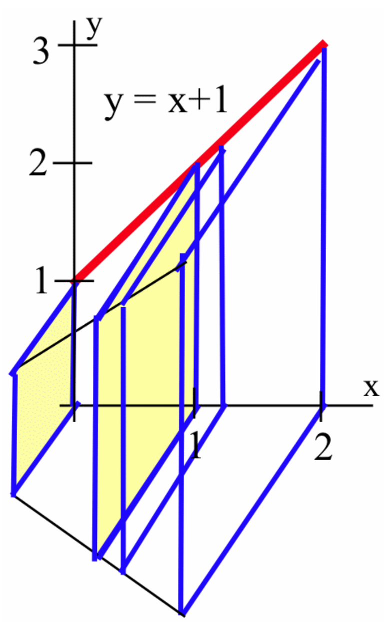
$$\boxed{\text{applied problem}} \xrightarrow{1} \boxed{\text{Riemann sum}} \xrightarrow{2} \boxed{\text{definite integral}} \xrightarrow{3} \boxed{\text{number}}$$

Step 1 is absolutely vital. If we can not translate the ideas of an applied problem into an area or a Riemann sum or a definite integral, then we can not use integral calculus to solve the problem. For a few special types of applied problems, we will be able to move directly from the problem to an integral, but usually it will be easier to first break the problem into smaller pieces and to build a Riemann sum. Section 4.7 and all of Chapter 5 focus on translating different types of applied problems into Riemann sums and definite integrals. **Computers and calculators are seldom of any help with Step 1.**

Step 2 is usually easy. If we have a Riemann sum $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ on an interval $[a,b]$, then the limit of the sum (as $n \to \infty$) is simply the definite integral $\int_a^b f(x)\,dx$.

Step 3 can be handled in several ways.

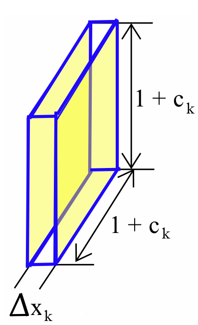
- If the function f is relatively simple, we may be able to find an antiderivative for f (using techniques from Section 4.6 and Chapter 8) and then apply FTC^2 to get a numerical answer.
- If the function f is more complicated, then integral tables or computers (Section 4.8) may help us find an antiderivative for f, in which case we can apply FTC^2 to get a numerical answer.
- If we cannot find an antiderivative for f, we can compute approximate numerical answers for the definite integral using various approximation methods (Sections 4.9 and 8.7); we typically employ computers to carry out the heavy-duty arithmetic.



Usually any difficulties in solving an applied problem arise in the first and third steps. There are techniques and details to master and understand, but it is also important to keep in mind where these techniques and details fit into the bigger picture.

The next Example illustrates these steps for the problem of finding a volume of a solid. We will explore techniques for finding volumes of solids in greater detail in Chapter 5.

Example 5. Find the volume of the solid shown in the margin for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)



Solution. Step 1: Going from the figure to a Riemann sum. If we break the solid into n "slices" with cuts perpendicular to the x-axis (and the xy-plane) using a partition $\mathcal{P}$ with cuts at x_1, x_2, x_3, ..., x_{n-1} (like slicing a block of cheese or a loaf of bread), then the volume of the original solid is equal to the sum of the volumes of the "slices."

The volume of the k-th slice is *approximately* equal to the volume of a thin, rectangular box:

$$(\text{height}) \cdot (\text{base}) \cdot (\text{thickness}) \approx (c_k + 1)\,(c_k + 1) \cdot \Delta x_k$$

where c_k is any chosen value between x_{k-1} and x_k. Therefore:

$$\text{total volume} = \sum_{k=1}^{n} (\text{volume of the } k\text{-th slice}) = \sum_{k=1}^{n} (c_k+1)^2 \, \Delta x_k$$

which is a Riemann sum.

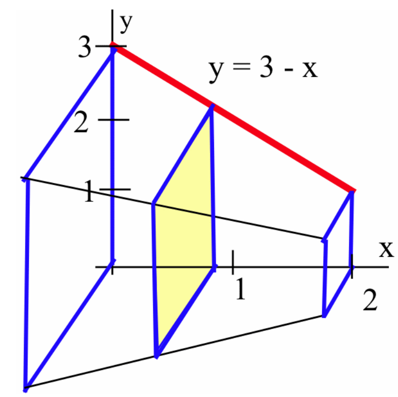
Step 2: Going from the Riemann sum to a definite integral.
We can improve the Riemann sum approximation of the total volume from Step 1 by taking thinner slices (making all of the Δx_k smaller and smaller) so that the mesh of the partition $\mathcal{P}$ approaches 0:

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{k=1}^{n} (c_k+1)^2 \, \Delta x_k = \int_0^2 (x+1)^2 \, dx = \int_0^2 \left[x^2+2x+1\right] \, dx$$

Step 3: Going from the definite integral to a numerical answer.
We can now use FTC^2 to evaluate the integral: $F(x) = \frac{1}{3}x^3 + x^2 + x$ is an antiderivative of x^2+2x+1 (check this by differentiating $F(x)$), so:

$$\int_0^2 \left[x^2+2x+1\right] \, dx = \left[\frac{1}{3}x^3+x^2+x\right]_0^2$$
$$= \left[\frac{1}{3}\cdot 2^3+2^2+2\right] - \left[\frac{1}{3}\cdot 0^3+0^2+0\right] = \frac{26}{3}$$

The volume of the solid shape is exactly $\frac{26}{3}$ cubic inches. ◀



Practice 4. Find the volume of the solid shape in the margin figure for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)

Leibniz's Rule For Differentiating Integrals

If the endpoint of an integral is a function of x rather than simply x, then we need to use the Chain Rule together with FTC^1 to calculate the derivative of the integral. For example:

$$A'(x) = f(x) \qquad \Rightarrow \qquad \frac{d}{dx}\left[A\left(x^2\right)\right] = A'(x)\cdot 2x = f\left(x^2\right)\cdot 2x$$

We can generalize this result by applying the Chain Rule to the derivative of the integral:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = \frac{d}{dx}\left[A\left(g(x)\right)\right] = f\left(g(x)\right)\cdot g'(x)$$

and combine this with some integral properties to further extend FTC^1.

Leibniz's Rule

If f is a continuous function, $A(x) = \int_a^x f(t)\,dt$
and $g_1(x)$ and $g_2(x)$ are both differentiable functions

then $\frac{d}{dx}\left[\int_{g_1(x)}^{g_2(x)} f(t)\,dt\right] = f\left(g_2(x)\right)\cdot g_2'(x) - f\left(g_1(x)\right)\cdot g_1'(x)$

Proof. Assume for simplicity that f, g_1 and g_2 are continuous on $(-\infty, \infty)$ and let c be any number. Then:

$$\begin{aligned}\int_{g_1(x)}^{g_2(x)} f(t)\,dt &= \int_c^{g_2(x)} f(t)\,dt + \int_{g_1(x)}^{c} f(t)\,dt \\ &= \int_c^{g_2(x)} f(t)\,dt - \int_c^{g_1(x)} f(t)\,dt\end{aligned}$$

Now apply the preceding result. □

Example 6. If a is any constant, compute the derivatives $\dfrac{d}{dx}\left[\int_a^{5x} t^2\,dt\right]$, $\dfrac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right]$ and $\dfrac{d}{dw}\left[\int_{\pi w}^{\sin w} z^3\,dz\right]$.

Solution. Applying Leibniz's Rule:

$$\begin{aligned}\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right] &= (5x)^2 \cdot 5 = 125x^2 \\ \frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right] &= \cos(x^2) \cdot 2x = 2x\cos(x^2) \\ \frac{d}{dw}\left[\int_{\pi w}^{\sin(w)} z^3\,dz\right] &= (\sin(w))^3 \cdot \cos(w) - (\pi w)^3 \cdot \pi\end{aligned}$$

The last quantity simplifies to $\sin^3(w)\cos(w) - \pi^4 w^3$. ◀

Practice 5. Compute $\dfrac{d}{dx}\left[\int_0^{x^3} \sin(t)dt\right]$.

4.5 Problems

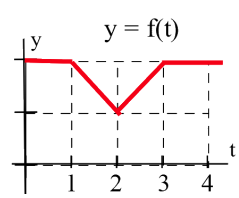
In Problems 1–2, (a) Use FTC[2] to find a formula for $A(x)$, differentiate $A(x)$ to obtain a formula for $A'(x)$, and evaluate $A'(x)$ at $x = 1, 2$ and 3. (b) Use FTC[1] to evaluate $A'(x)$ at $x = 1, 2$ and 3.

1. $A(x) = \int_0^x 3t^2\,dt$
2. $A(x) = \int_1^x (1+2t)\,dt$

In Problems 3–8, compute $A'(1)$, $A'(2)$ and $A'(3)$.

3. $A(x) = \int_0^x 2t\,dt$
4. $A(x) = \int_1^x 2t\,dt$
5. $A(x) = \int_{-3}^x 2t\,dt$
6. $A(x) = \int_0^x \left(3 - t^2\right)\,dt$
7. $A(x) = \int_0^x \sin(t)\,dt$
8. $A(x) = \int_1^x |t-2|\,dt$

In 9–12, $A(x) = \int_0^x f(t)\,dt$, with $f(t)$ given graphically. Evaluate $A'(1)$, $A'(2)$ and $A'(3)$.

9.


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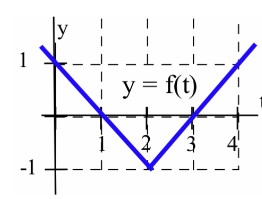

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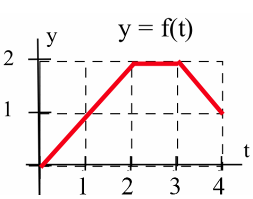

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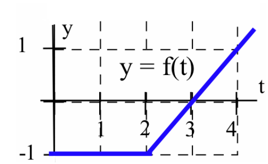


In 13–33, verify that $F(x)$ is an antiderivative of the integrand and use FTC^2 to evaluate the integral.

13. $\displaystyle\int_0^1 2x\,dx, \quad F(x) = x^2 + 5$

14. $\displaystyle\int_1^4 3x^2\,dx, \quad F(x) = x^3 + 2$

15. $\displaystyle\int_1^3 x^2\,dx, \quad F(x) = \frac{1}{3}x^3$

16. $\displaystyle\int_0^3 \left[x^2 + 4x - 3\right]\,dx, \; F(x) = \frac{1}{3}x^3 + 2x^2 - 3x$

17. $\displaystyle\int_1^5 \frac{1}{x}\,dx, \quad F(x) = \ln(x)$

18. $\displaystyle\int_2^5 \frac{1}{x}\,dx, \quad F(x) = \ln(x) + 4$

19. $\displaystyle\int_{\frac{1}{2}}^3 \frac{1}{x}\,dx, \quad F(x) = \ln(x)$

20. $\displaystyle\int_1^3 \frac{1}{x}\,dx, \quad F(x) = \ln(x) + 2$

21. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx, \quad F(x) = \sin(x)$

22. $\displaystyle\int_0^{\pi} \sin(x)\,dx, \quad F(x) = -\cos(x)$

23. $\displaystyle\int_0^1 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$

24. $\displaystyle\int_1^4 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$

25. $\displaystyle\int_1^7 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$

26. $\displaystyle\int_1^4 \frac{1}{2\sqrt{x}}\,dx, \quad F(x) = \sqrt{x}$

27. $\displaystyle\int_1^9 \frac{1}{2\sqrt{x}}\,dx, \quad F(x) = \sqrt{x}$

28. $\displaystyle\int_2^5 \frac{1}{x^2}\,dx, \quad F(x) = -\frac{1}{x}$

29. $\displaystyle\int_{-2}^3 e^x\,dx, \quad F(x) = e^x$

30. $\displaystyle\int_0^3 \frac{2x}{1+x^2}\,dx, \quad F(x) = \ln(1+x^2)$

31. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2(x)\,dx, \quad F(x) = \tan(x)$

32. $\displaystyle\int_1^e \ln(x)\,dx, \quad F(x) = x\cdot\ln(x) - x$

33. $\displaystyle\int_0^3 2x\sqrt{1+x^2}\,dx, \quad F(x) = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}$

For 34–48, find an antiderivative of the integrand and use FTC^2 to evaluate the definite integral.

34. $\displaystyle\int_2^5 3x^2\,dx$

35. $\displaystyle\int_{-1}^2 x^2\,dx$

36. $\displaystyle\int_1^3 \left[x^2 + 4x - 3\right]\,dx$

37. $\displaystyle\int_1^e \frac{1}{x}\,dx$

38. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,dx$

39. $\displaystyle\int_{25}^{100} \sqrt{x}\,dx$

40. $\displaystyle\int_3^5 \sqrt{x}\,dx$

41. $\displaystyle\int_1^{10} \frac{1}{x^2}\,dx$

42. $\displaystyle\int_1^{1000} \frac{1}{x^2}\,dx$

43. $\displaystyle\int_0^1 e^x\,dx$

44. $\displaystyle\int_{-2}^2 \frac{2x}{1+x^2}\,dx$

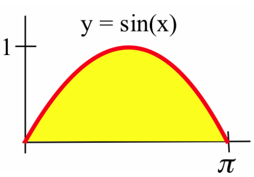
45. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2(x)\,dx$

46. $\displaystyle\int_0^1 e^{2x}\,dx$

47. $\displaystyle\int_3^3 \sin(x)\cdot\ln(x)\,dx$

48. $\displaystyle\int_2^4 (x-2)^3\,dx$

In 49–54, find the area of the shaded region.

49.


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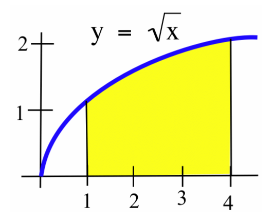

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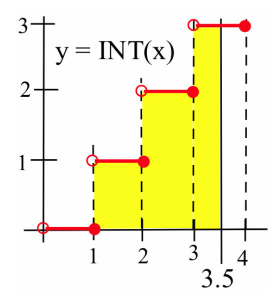

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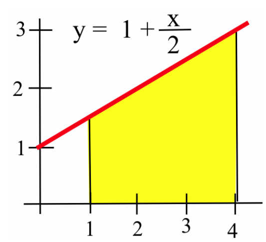

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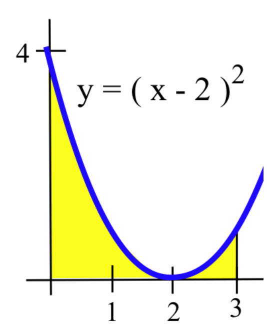

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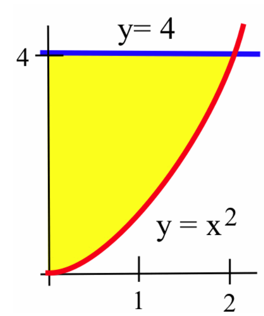


55. Given that $A'(x) = \tan(x)$, find $\mathbf{D}(A(3x))$, $\mathbf{D}\left(A(x^2)\right)$ and $\mathbf{D}(A(\sin(x)))$.

56. Given that $B'(x) = \sec(x)$, find $\mathbf{D}(B(3x))$, $\mathbf{D}\left(B(x^2)\right)$ and $\mathbf{D}(B(\sin(x)))$.

In 57–68, apply Leibniz's Rule.

57. $\dfrac{d}{dx}\left[\int_1^{5x} \sqrt{1+t}\,dt\right]$

58. $\dfrac{d}{dx}\left[\int_2^{x^2} \sqrt{1+t}\,dt\right]$

59. $\dfrac{d}{dx}\left[\int_0^{\sin(x)} \sqrt{1+t}\,dt\right]$

60. $\dfrac{d}{dx}\left[\int_1^{2+3x} \left(t^2+5\right)\,dt\right]$

61. $\dfrac{d}{dx}\left[\int_0^{1-2x} \left(3t^2+2\right)\,dt\right]$

62. $\dfrac{d}{dx}\left[\int_x^{9} \left(3t^2+2\right)\,dt\right]$

63. $\dfrac{d}{dx}\left[\int_x^{\pi} \cos(3t)\,dt\right]$

64. $\dfrac{d}{dx}\left[\int_{7x}^{\pi} \cos(2t)\,dt\right]$

65. $\dfrac{d}{dx}\left[\int_x^{x^2} \tan(t)\,dt\right]$

66. $\dfrac{d}{dx}\left[\int_0^{\pi} \cos(3t)\,dt\right]$

67. $\dfrac{d}{dx}\left[\int_2^{\ln(x)} 5t\cdot\cos(3t)\,dt\right]$

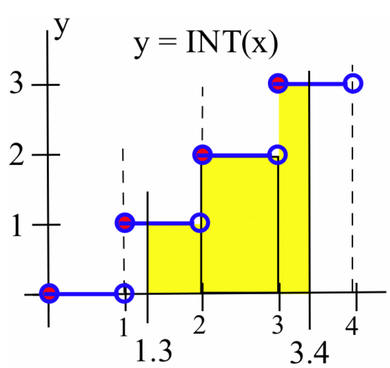
68. $\dfrac{d}{dx}\left[\int_0^{\pi} \tan(7t)\,dt\right]$

69. $\dfrac{d}{dy}\left[\int_0^{y^2} \tan(\theta)\,d\theta\right]$

4.5 Practice Answers

1. $A(1) = 1$, $A(2) = 1.5$, $A(3) = 1$, $A(4) = 1$; $A'(x) = g(x)$ so $A'(1) = g(1) = 1$, $A'(2) = g(2) = 0$, $A'(3) = -1$, $A'(4) = 1$.

2. $F(x) = x^3 - x$ is an antiderivative of $f(x) = 3x^2 - 1$ so:
$$\int_1^3 \left[3x^2-1\right]\,dx = \left[x^3-x\right]_1^3 = \left[3^3-3\right] - \left[1^3-1\right] = 24$$
$F(x) = x^3 - x + 7$ is another antiderivative of $f(x) = 3x^2 - 1$ so:
$$\int_1^3 \left[3x^2-1\right]\,dx = \left[x^3-x+7\right]_1^3 = \left[3^3-3+7\right] - \left[1^3-1+7\right] = 24$$
No matter which antiderivative of $f(x) = 3x^2 - 1$ you use, the value of the definite integral $\int_1^3 \left[3x^2-1\right]\,dx$ is 24.



3. Because $f(x) = \lfloor x \rfloor$ is not continuous on $[1.3, 3.4]$ we cannot use the Fundamental Theorem of Calculus. Instead, we can think of the definite integral as an area (see margin figure) and compute:
$$\int_{1.3}^{3.4} \lfloor x \rfloor\,dx = 3.9$$

4. First break the solid into "slices" and approximate the volume of the k-th slice by $(3-c_k)^2 \cdot \Delta x_k$ where c_k is any point in the k-th subinterval. Next add up these approximate volumes to get a Riemann Sum:
$$\sum_{k=1}^{n} (3-c_k)^2 \cdot \Delta x_k$$

and then take the limit of these Riemann sums as the mesh of the partitions approaches 0 (and $n \to \infty$, where n is the number of subintervals in the partition):

$$\begin{aligned}
\lim_{\|\mathcal{P}\|\to 0}\left[\sum_{k=1}^{n}(3-c_k)^2\cdot\Delta x_k\right] &= \int_0^2 (3-x)^2\,dx \\
&= \int_0^2 \left(9-6x+x^2\right)\,dx \\
&= \left[9x-3x^2+\frac{1}{3}x^3\right]_0^2 \\
&= \left[18-12+\frac{8}{3}\right]-[0-0+0]=\frac{26}{3}
\end{aligned}$$

5. $\dfrac{d}{dx}\left[\displaystyle\int_0^{x^3}\sin(t)\,dt\right]=\sin(x^3)\cdot\dfrac{d}{dx}\left[x^3\right]=3x^2\sin(x^3)$

4.6 Finding Antiderivatives

In order to use the second part of the Fundamental Theorem of Calculus, we need an antiderivative of the integrand, but sometimes it is not easy to find one. This section collects some of the information we already know about general properties of antiderivatives and about antiderivatives of particular functions. It shows how to use this information to find antiderivatives of more complicated functions and introduces a "change of variable" technique to make that job easier.

Indefinite Integrals and Antiderivatives

Antiderivatives arise so often that there is a special notation to indicate the antiderivative of a function:

> $\int f(x)\,dx$, read as "the **indefinite integral** of f" or as "the antiderivatives of f," represents the collection (or family) of all functions whose derivatives are f.

If you've been wondering why we called $\int_a^b f(t)\,dt$ a **definite** integral, now you know. A definite integral has specific upper and lower limits, while an indefinite integral does not.

If F is an antiderivative of f, then any member of the family $\int f(x)\,dx$ has the form $F(x)+C$ for some constant C. We write $\int f(x)\,dx = F(x)+C$, where C represents an arbitrary constant. There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an *infinite* number of antiderivatives and each has the form $F(x)+C$, which means there are many ways to write a particular indefinite integral and some of them may look very different. You can check that $F(x)=\sin^2(x)$, $G(x)=-\cos^2(x)$ and $H(x)=2\sin^2(x)+\cos^2(x)$ all have the same derivative, $f(x)=2\sin(x)\cos(x)$, so the indefinite integral of $2\sin(x)\cos(x)$, $\int 2\sin(x)\cos(x)\,dx$, can be written in several ways: $\sin^2(x)+C$ or $-\cos^2(x)+K$ or $2\sin^2(x)+\cos^2(x)+C$.

Practice 1. Verify that $\int 2\tan(x)\cdot\sec^2(x)\,dx = \tan^2(x)+C$ and that $\int 2\tan(x)\cdot\sec^2(x)\,dx = \sec^2(x)+K$.

Properties of Antiderivatives (Indefinite Integrals)

These sum, difference and constant-multiple properties follow directly from corresponding properties for derivatives.

> If f and g are integrable functions, then
>
> - $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
> - $\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx$

Although we know general rules for *derivatives* of products and quotients, unfortunately there are no easy general patterns for *antiderivatives* of products and quotients—we will only be able to add one more general property to this list (in Section 8.2).

We already know antiderivatives for several important functions.

Constant Functions: $\int k\,dx = kx + C$

Powers of x: $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \dfrac{1}{x}\,dx = \ln|x| + C$

Exponential Functions: $\int e^x\,dx = e^x + C$

Trig Functions: $\int \cos(x)\,dx = \sin(x) + C$, $\int \sin(x)\,dx = -\cos(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$, $\int \csc^2(x)\,dx = -\cot(x) + C$

$\int \sec(x)\cdot\tan(x)\,dx = \sec(x) + C$, $\int \csc(x)\cdot\cot(x)\,dx = -\csc(x) + C$

All of these antiderivatives can be verified by differentiating. For $\int \frac{1}{x}\,dx$ you may be wondering about the presence of the absolute value signs in the antiderivative. If $x > 0$, you can check that:

$$\mathbf{D}\left(\ln(|x|)\right) = \mathbf{D}\left(\ln(x)\right) = \frac{1}{x}$$

If $x < 0$, then you can check that:

$$\mathbf{D}\left(\ln(|x|)\right) = \mathbf{D}\left(\ln(-x)\right) = \frac{-1}{-x} = \frac{1}{x}$$

When computing a definite integral of the form $\int_a^b \frac{1}{x}\,dx$, either a and b will both be positive or both be negative, because the integrand is not defined at $x = 0$, so $x = 0$ cannot be included in the interval of integration.

Our list of antiderivatives of particular functions will grow in coming chapters and will eventually include antiderivatives of additional trigonometric functions, the inverse trigonometric functions, logarithms, rational functions and more. (See Appendix I.)

Antiderivatives of More Complicated Functions

Antiderivatives are very sensitive to small changes in the integrand, so we should be very careful.

Fortunately, an antiderivative can always be checked by differentiating, so even though we may not find the correct antiderivative, we should be able to determine whether or not an antiderivative candidate is actually an antiderivative.

Example 1. We know $\mathbf{D}\left(\sin(x)\right) = \cos(x)$, so $\int \cos(x)\,dx = \sin(x) + C$.
Find: (a) $\int \cos(2x+3)\,dx$ (b) $\int \cos(5x-7)\,dx$ (c) $\int \cos(x^2)\,dx$

Solution. (a) Because $\sin(x)$ is an antiderivative of $\cos(x)$, it is reasonable to hope that $\sin(2x+3)$ will be an antiderivative of $\cos(2x+3)$. Unfortunately, we see that $\mathbf{D}\left(\sin(2x+3)\right) = \cos(2x+3)\cdot 2$, exactly twice the result we want. Let's try again by modifying our "guess" to be half the original guess:

$$\mathbf{D}\left(\frac{1}{2}\sin(2x+3)\right) = \frac{1}{2}\cos(2x+3)\cdot 2 = \cos(2x+3)$$

which is what we want, so $\int \cos(2x+3)\,dx = \dfrac{1}{2}\sin(2x+3) + C$.

(b) $\mathbf{D}\left(\sin(5x-7)\right) = \cos(5x-7)\cdot 5$, so dividing the original guess by 5 we get $\mathbf{D}\left(\frac{1}{5}\sin(5x-7)\right) = \frac{1}{5}\cos(5x-7)\cdot 5 = \cos(5x-7)$ and conclude that $\int \cos(5x-7)\,dx = \dfrac{1}{5}\sin(5x-7) + C$.

(c) $\mathbf{D}\left(\sin(x^2)\right) = \cos(x^2)\cdot 2x$. It was easy enough in parts (a) and (b) to modify our "guesses" to eliminate the constants 2 and 5, but here the x is much harder to eliminate:

$$\begin{aligned}
\mathbf{D}\left(\frac{1}{2x}\sin\left(x^2\right)\right) &= \mathbf{D}\left(\frac{\sin(x^2)}{2x}\right) \\
&= \frac{2x\cdot\mathbf{D}\left(\sin(x^2)\right)-\sin(x^2)\cdot\mathbf{D}(2x)}{(2x)^2)} \\
&= \frac{(2x)^2\cos(x^2)-2\sin(x^2)}{(2x)^2} \\
&= \cos(x^2)-\frac{\sin(x^2)}{2x^2} \neq \cos(x^2)
\end{aligned}$$

Our guess did not check out — we're stuck. ◀

Advanced mathematical techniques beyond the scope of this text can show that $\cos(x^2)$ does not have an "elementary" antiderivative composed of polynomials, roots, trigonometric functions, exponential functions or their inverses.

The value of a definite integral of $\cos(x^2)$ could still be approximated as accurately as needed by using Riemann sums or one of the numerical techniques in Sections 4.9 and 8.7, but no matter how hard we try, we cannot find a concise formula for an antiderivative of $\cos(x^2)$ in order to use the Fundamental Theorem of Calculus. Even a simple-looking integrand can be very difficult. At this point, there is no quick way to tell the difference between an "easy" indefinite integral and a "difficult" or "impossible" one.

Getting the Constants Right

The previous example illustrated one technique for finding antiderivatives: "guess" the form of the answer, differentiate your "guess" and then modify your original "guess" so its derivative is exactly what you want it to be.

Example 2. Knowing that $\int \sec^2(x)\,dx = \tan(x)+C$ and $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}+C$, find (a) $\int \sec^2(3x+7)\,dx$ (b) $\int \frac{1}{\sqrt{5x+3}}\,dx$.

Solution. (a) If we "guess" an answer of $\tan(3x+7)$ and then differentiate it, we get $\mathbf{D}\left(\tan(3x+7)\right) = \sec^2(3x+7)\cdot\mathbf{D}(3x+7) = 3\sec^2(3x+7)$, which is three times what we want. If we divide our original guess by 3 and try again, we have:

$$\begin{aligned}
\mathbf{D}\left(\frac{1}{3}\tan(3x+7)\right) &= \frac{1}{3}\mathbf{D}\left(\tan(3x+7)\right) = \frac{1}{3}\sec^2(3x+7)\cdot 3 \\
&= \sec^2(3x+7)
\end{aligned}$$

so $\int \sec^2(3x+7)\,dx = \frac{1}{3}\tan(3x+7)+C$.

(b) If we "guess" $2\sqrt{5x+3}$ and then differentiate it, we get:

$$\mathbf{D}\left(2\,(5x+3)^{\frac{1}{2}}\right) = 2\cdot\frac{1}{2}\,(5x+3)^{-\frac{1}{2}}\,\mathbf{D}(5x+3) = 5\cdot(5x+3)^{-\frac{1}{2}}$$

which is five times what we want. Dividing our guess by 5 and differentiating, we have:

$$\mathbf{D}\left(\frac{2}{5}(5x+3)^{\frac{1}{2}}\right) = \frac{2}{5}\cdot\frac{1}{2}(5x+3)^{-\frac{1}{2}}\cdot 5 = \frac{1}{\sqrt{5x+3}}$$

so $\displaystyle\int \frac{1}{\sqrt{5x+3}}\,dx = \frac{2}{5}\sqrt{5x+3}+C$. ◀

Practice 2. Find $\displaystyle\int \sec^2(7x)\,dx$ and $\displaystyle\int \frac{1}{\sqrt{3x+8}}\,dx$.

The "guess and check" method is a very effective technique if you can make a good first guess, one that misses the desired result only by a constant multiple. In that situation, just divide the first guess by the unwanted constant multiple. If the derivative of your guess misses by something other than a constant multiple, then more drastic modifications are needed. Sometimes the next technique can help.

Making Patterns More Obvious: Changing the Variable

Successful integration is mostly a matter of recognizing patterns. The "change of variable" technique can make some underlying patterns of an integral easier to recognize. Essentially, the technique involves rewriting an integral that is originally in terms of one variable, say x, in terms of another variable, say u, with the hope that it will be easier to find an antiderivative of the new integrand.

For example, we can rewrite $\displaystyle\int \cos(5x+1)\,dx$ by setting $u = 5x+1$. Then $\cos(5x+1)$ becomes $\cos(u)$ but we must also convert the dx in the original integral. We know that $\dfrac{du}{dx} = 5$, so rewriting this last expression in differential notation, we get $du = 5\,dx$; isolating dx yields $dx = \frac{1}{5}\,du$ so:

We first discussed differential notation in Section 2.8; although you may not have used them much in differential calculus, you will now use them extensively.

$$\int \cos(5x+1)\,dx = \int \cos(u)\cdot\frac{1}{5}\,du = \frac{1}{5}\int \cos(u)\,du$$

This new integral is easier:

$$\frac{1}{5}\int \cos(u)\,du = \frac{1}{5}\sin(u)+C$$

but our original problem was in terms of x and our answer is in terms of u, so we must "resubstitute" using the relationship $u = 5x+1$:

$$\frac{1}{5}\sin(u)+C = \frac{1}{5}\sin(5x+1)+C$$

We can now conclude that:

$$\int \cos(5x+1)\,dx = \frac{1}{5}\sin(5x+1)+C$$

As always, you can check this result by differentiating.

We can summarize the steps of this "change of variable" (or "u-substitution") method as:

Often u is set equal to some "interior" part of the original integrand function.

- set a new variable, say u, equal to some function of the original variable x
- calculate the differential du in terms of x and dx
- rewrite the original integral in terms of u and du
- integrate the new integral to get an answer in terms of u
- resubstitute for u to get a result in terms of the original variable x

Example 3. Make the suggested change of variable, rewrite each integral in terms of u and du, and evaluate the integral.

(a) $\displaystyle\int \cos(x) \cdot e^{\sin(x)}\, dx \quad \text{with} \quad u = \sin(x)$

(b) $\displaystyle\int \frac{2x}{5+x^2}\, dx \quad \text{with} \quad u = 5 + x^2$

Solution. (a) $u = \sin(x) \Rightarrow du = \cos(x)\, dx$ and $e^{\sin(x)} = e^u$:

$$\int \cos(x) e^{\sin(x)}\, dx = \int e^u\, du = e^u + C = e^{\sin(x)} + C$$

(b) $u = 5 + x^2 \Rightarrow du = 2x\, dx$, so:

$$\int \frac{2x}{5+x^2}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln\left|5 + x^2\right| + C$$

Because $5 + x^2 > 0$, we can also write the answer as $\ln\left(5 + x^2\right)$. ◀

In each example, the change of variable did not find the antiderivative, but it did make the pattern of the integrand more obvious, which in turn made it easier to determine an antiderivative.

Practice 3. Make the suggested change of variable, rewrite each integral in terms of u and du and evaluate the integral.

(a) $\displaystyle\int (7x+5)^3\, dx \quad \text{with} \quad u = 7x + 5$

(b) $\displaystyle\int 3x^2 \cdot \sin\left(x^3 - 1\right)\, dx \quad \text{with} \quad u = x^3 - 1$

The previous examples have supplied a suggested substitution, but in the future *you* will need to decide what u should equal. Unfortunately there are no rules that guarantee your choice will lead to an easier integral—sometimes you will need to resort to trial and error until you find a particular u-substitution that works for your integrand. There is, however, a "rule of thumb" that frequently results in easier integrals. Even though the following suggestion comes with no guarantees, it is often worth trying.

A "Rule of Thumb" for Changing the Variable

If part of the integrand consists of a composition of functions, $f(g(x))$, try setting $u = g(x)$, the "inner" function.

The key to becoming skilled at selecting a good u and correctly making the substitution is **practice**.

If part of the integrand is being raised to a power, try setting u equal to the part being raised to the power. For example, if the integrand includes $(3 + \sin(x))^5$, try $u = 3 + \sin(x)$. If part of the integrand involves a trigonometric (or exponential or logarithmic) function of another function, try setting u equal to the "inside" function: if the integrand includes the function $\sin\left(3 + x^2\right)$, try $u = 3 + x^2$.

Example 4. Select a u for each integrand and rewrite the associated integral in terms of u and du.

(a) $\displaystyle\int \cos(3x)\sqrt{2+\sin(3x)}\,dx$ (b) $\displaystyle\int \frac{5e^x}{2+e^x}\,dx$ (c) $\displaystyle\int e^x \cdot \sin\left(e^x\right)\,dx$

Solution. (a) If $u = 2 + \sin(3x)$, $du = 3\cos(3x)\,dx \Rightarrow \frac{1}{3}\,du = \cos(3x)\,dx$ so the integral becomes $\displaystyle\int \frac{1}{3}\sqrt{u}\,du$. (b) With $u = 2 + e^x \Rightarrow du = e^x\,dx$, the integral becomes $\displaystyle\int \frac{5}{u}\,du$. (c) With $u = e^x \Rightarrow du = e^x\,dx$, the integral becomes $\displaystyle\int \sin(u)\,du$. ◀

Changing Variables with Definite Integrals

If we need to change variables in a *definite* integral, we have two choices:

- First work out the corresponding *indefinite* integral and then use that antiderivative and FTC2 to evaluate the definite integral.
- Change variables in the definite integral, which requires changing the limits of integration from x limits to u limits.

For the second option, if the original integral had endpoints $x = a$ and $x = b$, and we make the substitution $u = g(x) \Rightarrow du = g'(x)\,dx$, then the new integral will have endpoints $u = g(a)$ and $u = g(b)$:

$$\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Example 5. Evaluate $\displaystyle\int_0^1 (3x-1)^4\,dx$.

Solution. Using the first option with $u = 3x - 1 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$, the corresponding indefinite integral becomes:

$$\int (3x-1)^4\,dx = \int u^4 \cdot \frac{1}{3}\,du = \frac{1}{3}\cdot\frac{1}{5}u^5 + C = \frac{1}{15}\,(3x-1)^5 + C$$

We now use this result to evaluate the original definite integral:

$$\int_0^1 (3x-1)^4\,dx = \left[\frac{1}{15}(3x-1)^5\right]_0^1 = \left[\frac{1}{15}\cdot 2^5\right] - \left[\frac{1}{15}\cdot(-1)^5\right]$$
$$= \frac{32}{15} - \frac{-1}{15} = \frac{33}{15} = \frac{11}{5}$$

For the second option, we make the same substitution $u = 3x-1 \Rightarrow \frac{1}{3}\,du = dx$ while also computing $x = 0 \Rightarrow u = 3\cdot 0 - 1 = -1$ and $x = 1 \Rightarrow u = 3\cdot 1 - 1 = 2$:

Both options require roughly the same amount of work and computation. In practice you should choose the option that seems easiest for you and poses the least risk of error.

$$\int_{x=0}^{x=1} (3x-1)^4\,dx = \int_{u=-1}^{u=2} \frac{1}{3}u^4\,du = \frac{1}{3}\cdot\frac{1}{5}u^5\Big|_{-1}^{2} = \frac{2^5}{15} - \frac{(-1)^5}{15} = \frac{33}{15}$$

We arrive at the same answer either way. ◀

Practice 4. If the original integrals in Example 4 had endpoints (a) $x = 0$ to $x = \pi$ (b) $x = 0$ to $x = 2$ or (c) $x = 0$ to $x = \ln(3)$, then the new integrals should have what endpoints?

Special Transformations for $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$

The integrals of $\sin^2(x)$ and $\cos^2(x)$ arise often, and we can find their antiderivatives with the help of some trigonometric identities. Solving the first identity in the margin for $\sin^2(x)$, we get:

$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and solving the second identity for $\cos^2(x)$, we get:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Integrating the first of these new identities yields:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we can also write:

$$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

In practice, it's easier to remember the new trig identities and use them to work out these antiderivatives, rather than memorizing the antiderivatives directly.

Similarly, using $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ yields:

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C = \frac{1}{2}x + \frac{1}{2}\sin(x)\cos(x) + C$$

4.6 Problems

For Problems 1–4, put $f(x) = x^2$ and $g(x) = x$ to verify the inequality.

1. $\int_1^2 f(x) \cdot g(x)\,dx \neq \left(\int_1^2 f(x)\,dx\right)\left(\int_1^2 g(x)\,dx\right)$
2. $\int_1^2 \frac{f(x)}{g(x)}\,dx \neq \frac{\int_1^2 f(x)\,dx}{\int_1^2 g(x)\,dx}$
3. $\int_0^1 f(x) \cdot g(x)\,dx \neq \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right)$
4. $\int_1^4 \frac{f(x)}{g(x)}\,dx \neq \frac{\int_1^4 f(x)\,dx}{\int_1^4 g(x)\,dx}$

For 5–14, use the suggested u to find du and rewrite the integral in terms of u and du. Then find an antiderivative in terms of u and, finally, rewrite your answer in terms of x.

5. $\int \cos(3x)\,dx, \quad u = 3x$
6. $\int \sin(7x)\,dx, \quad u = 7x$
7. $\int e^x \sin(2 + e^x)\,dx, \quad u = 2 + e^x$
8. $\int e^{5x}\,dx, \quad u = 5x$
9. $\int \cos(x)\sec^2(\sin(x))\,dx, \quad u = \sin(x)$
10. $\int \frac{\cos(x)}{\sin(x)}\,dx, \quad u = \sin(x)$
11. $\int \frac{5}{3 + 2x}\,dx, \quad u = 3 + 2x$
12. $\int x^2\left(5 + x^3\right)^7\,dx, \quad u = 5 + x^3$
13. $\int x^2 \sin\left(1 + x^3\right)\,dx, \quad u = 1 + x^3$
14. $\int \frac{e^x}{1 + e^x}\,dx, \quad u = 1 + e^x$

For 15–26, use the change-of-variable technique to evaluate the indefinite integral.

15. $\int \cos(4x)\,dx$
16. $\int e^{3x}\,dx$
17. $\int x^3\left(5 + x^4\right)^{11}\,dx$
18. $\int x \cdot \sin\left(x^2\right)\,dx$
19. $\int \frac{3x^2}{2 + x^3}\,dx$
20. $\int \frac{\sin(x)}{\cos(x)}\,dx$
21. $\int \frac{\ln(x)}{x}\,dx$
22. $\int x\sqrt{1 + x^2}\,dx$
23. $\int (1 + 3x)^7\,dx$
24. $\int \frac{1}{x} \cdot \sin(\ln(x))\,dx$
25. $\int e^x \cdot \sec\left(e^x\right) \cdot \tan\left(e^x\right)\,dx$
26. $\int \frac{1}{\sqrt{x}}\cos\left(\sqrt{x}\right)\,dx$

In 27–42, evaluate the integral.

27. $\int_0^{\frac{\pi}{2}} \cos(3x)\,dx$
28. $\int_0^{\pi} \cos(4x)\,dx$
29. $\int_0^1 e^x \cdot \sin\left(2 + e^x\right)\,dx$
30. $\int_0^1 e^{5x}\,dx$
31. $\int_{-1}^1 x^2\left(1 + x^3\right)^5\,dx$
32. $\int_0^1 x^4\left(x^5 - 1\right)^{10}\,dx$
33. $\int_0^2 \frac{5}{3 + 2x}\,dx$
34. $\int_0^{\ln(3)} \frac{e^x}{1 + e^x}\,dx$
35. $\int_0^1 x\sqrt{1 - x^2}\,dx$
36. $\int_2^5 \frac{2}{1 + x}\,dx$
37. $\int_0^1 \sqrt{1 + 3x}\,dx$
38. $\int_0^1 \frac{1}{\sqrt{1 + 3x}}\,dx$
39. $\int \sin^2(5x)\,dx$
40. $\int \cos^2(3x)\,dx$
41. $\int \left[\frac{1}{2} - \sin^2(x)\right]\,dx$
42. $\int \left[e^x + \sin^2(x)\right]\,dx$
43. Find the area under one arch of $y = \sin^2(x)$.
44. Evaluate $\int_0^{2\pi} \sin^2(x)\,dx$.

In 45–53, expand the integrand first.

45. $\int \left(x^2 + 1\right)^3\,dx$
46. $\int \left(x^3 + 5\right)^2\,dx$
47. $\int \left(e^x + 1\right)^2\,dx$
48. $\int \left(x^2 + 3x - 2\right)^2\,dx$
49. $\int (x^2 + 1)(x^3 + 5)\,dx$
50. $\int (7 + \sin(x))^2\,dx$
51. $\int e^x\left(e^x + e^{3x}\right)\,dx$
52. $\int (2 + \sin(x))\sin(x)\,dx$
53. $\int \sqrt{x}\left(x^2 + 3x - 2\right)\,dx$

In 54–64, divide, then find an antiderivative.

54. $\int \frac{x+1}{x}\,dx$

55. $\int \frac{3x}{x+1}\,dx$

56. $\int \frac{x-1}{x+2}\,dx$

57. $\int \frac{x^2-1}{x+1}\,dx$

58. $\int \frac{2x^2-13x+15}{x-1}\,dx$

59. $\int \frac{2x^2-13x+18}{x-1}\,dx$

60. $\int \frac{2x^2-13x+11}{x-1}\,dx$

61. $\int \frac{x+2}{x-1}\,dx$

62. $\int \frac{e^x+e^{3x}}{e^x}\,dx$

63. $\int \frac{x+4}{\sqrt{x}}\,dx$

64. $\int \frac{\sqrt{x}+3}{x}\,dx$

The definite integrals in 65–70 involve areas associated with parts of circles; use your knowledge of circles and their areas to evaluate them. (Suggestion: Sketch a graph of the integrand function.)

65. $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

66. $\int_{0}^{1} \sqrt{1-x^2}\,dx$

67. $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

68. $\int_{-4}^{0} \sqrt{16-x^2}\,dx$

69. $\int_{-1}^{1} \left[2+\sqrt{1-x^2}\right]\,dx$

70. $\int_{0}^{2} \left[3-\sqrt{1-x^2}\right]\,dx$

4.6 Practice Answers

1. $\mathbf{D}\left(\tan^2(x)+C\right) = 2\tan^1(x)\cdot\mathbf{D}\left(\tan(x)\right) = 2\tan(x)\sec^2(x)$
$\mathbf{D}\left(\sec^2(x)+C\right) = 2\sec^1(x)\cdot\mathbf{D}\left(\sec(x)\right) = 2\sec(x)\cdot\sec(x)\tan(x)$

2. We know $\mathbf{D}\left(\tan(x)\right) = \sec^2(x)$, so it is reasonable to try $\tan(7x)$: $\mathbf{D}\left(\tan(7x)\right) = \sec^2(7x)\cdot\mathbf{D}(7x) = 7\sec^2(7x)$, a result seven times the result we want, so divide the original "guess" by 7 and try again:

$$\mathbf{D}\left(\frac{1}{7}\tan(7x)\right) = \frac{1}{7}\sec^2(7x)\cdot 7 = \sec^2(7x)$$

so $\int \sec^2(7x)\,dx = \frac{1}{7}\tan(7x)+C$.
$\mathbf{D}\left((3x+8)^{\frac{1}{2}}\right) = \frac{1}{2}(3x+8)^{-\frac{1}{2}}\,\mathbf{D}(3x+8) = \frac{3}{2}(3x+8)^{-\frac{1}{2}}$ so multiply our original "guess" by $\frac{2}{3}$:

$$\mathbf{D}\left(\frac{2}{3}(3x+8)^{\frac{1}{2}}\right) = \frac{2}{3}\cdot\frac{1}{2}\cdot(3x+8)^{-\frac{1}{2}}\cdot\mathbf{D}(3x+8) = \frac{2}{3}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{3x+8}}$$

hence $\int \frac{1}{\sqrt{3x+8}}\,dx = \frac{2}{3}\sqrt{3x+8}+C$.

3. (a) $u = 7x+5 \Rightarrow du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$ so:

$$\int (7x+5)^3\,dx = \int u^3\cdot\frac{1}{7}\,du = \frac{1}{7}\cdot\frac{1}{4}u^4+C = \frac{1}{28}(7x+5)^4+C$$

(b) $u = x^3-1 \Rightarrow du = 3x^2\,dx$ so $\int \sin(x^3-1)\cdot 3x^2\,dx$ becomes:

$$\int \sin(u)\,du = -\cos(u)+C = -\cos\left(x^3-1\right)+C$$

4. (a) $u = 2+\sin(3x)$ so $x=0 \Rightarrow u = 2+\sin(3\cdot 0) = 2$ and $x=\pi \Rightarrow u = 2+\sin(3\pi) = 2$. (This integral is now easy; why?)

(b) $u = 2+e^x$ so $x=0 \Rightarrow u = 2+e^0 = 3$ and $x=2 \Rightarrow u = 2+e^2$

(c) $u = e^x$ so $x=0 \Rightarrow u = e^0 = 1$ and $x=\ln(3) \Rightarrow u = e^{\ln(3)} = 3$

4.7 First Applications of Definite Integrals

The development of calculus by Newton and Leibniz was a vital step in the advancement of pure mathematics, but Newton also advanced the sciences and applied mathematics. Not only did he discover theoretical results, he immediately used those results to answer important questions about gravity and motion. The success of these applications of mathematics to the physical sciences helped establish what we now take for granted: mathematics can and should be used to answer questions about the world.

Newton applied mathematics to the outstanding problems of his day, problems primarily in the field of physics. During the intervening 300-plus years, thousands upon thousands of people have continued these theoretical and applied traditions, using mathematics to help develop our understanding of the physical and biological sciences, as well as the behavioral sciences and economics. Mathematics is still used to answer new questions in physics and engineering, but it is also important for modeling ecological processes, for understanding the behavior of DNA, for determining how the brain works, and even for devising financial strategies. The mathematics you are learning now can help you become part of this tradition, and you might even use it to add to our understanding of the world.

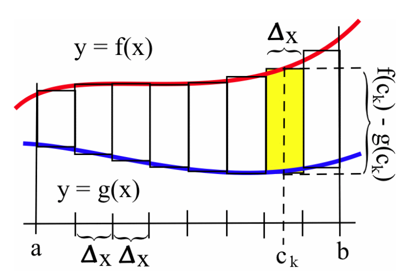
It is important to understand the special applications of integration we will study in case you need to use those particular applications. But it is also important that you understand the *process* of building models with integrals so you can apply that process to other situations in a variety of fields of study. Conceptually, converting an applied problem to a Riemann sum is the most valuable step.

Typically, it is also the most challenging.

Area between Two Curves

We have already used integrals to find the area between the graph of a function and the horizontal axis. We can also use integrals to find the area between the graphs of two functions.

If $f(x) \geq g(x)$ for all x in $[a, b]$, then we can approximate the area between the graphs of f and g by partitioning the interval $[a, b]$ and forming a Riemann sum (see margin). The height of each rectangle is $f(c_k) - g(c_k)$ so the area of the k-th rectangle is:



$$(\text{height}) \cdot (\text{base}) = [f(c_k) - g(c_k)] \cdot \Delta x_k$$

and an approximation of the total area is given by

$$\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k$$

which is a Riemann sum.

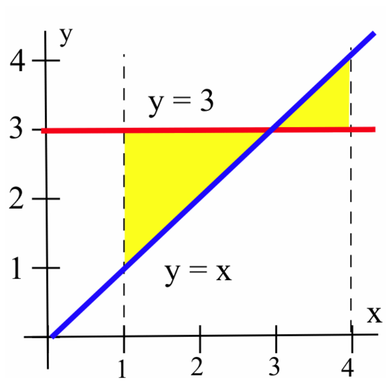
The limit of this Riemann sum, as the mesh of the partitions approaches 0, is a definite integral:

$$\int_a^b [f(x) - g(x)]\, dx$$

We will sometimes use an arrow to indicate "the limit of the Riemann sum as the mesh of the partitions approaches 0," writing:

$$\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k \longrightarrow \int_a^b [f(x) - g(x)]\, dx$$

If	$T(x) \geq B(x)$ for $a \leq x \leq b$
then	the area of the region bounded by the graphs of the "top" function $T(x)$, the "bottom" function $B(x)$, and the lines $x = a$ and $x = b$ is given by: $\displaystyle\int_a^b [T(x) - B(x)]\, dx$



Example 1. Find the area bounded between the graphs of $f(x) = x$ and $g(x) = 3$ for $1 \leq x \leq 4$.

Solution. It is clear from the margin figure that the area between f and g is 2.5 square units. Using the integration procedure above, we need to identify a "top" function and a "bottom" function. For $1 \leq x \leq 3$, $g(x) = 3 \geq x = f(x)$ so the area of the left-hand triangle is given by the integral:

$$\int_1^3 [3 - x]\, dx = \left[3x - \frac{1}{2}x^2\right]_1^3 = \left[9 - \frac{9}{2}\right] - \left[3 - \frac{1}{2}\right] = 2$$

For the interval $3 \leq x \leq 4$, $g(x) = 3 \leq x = f(x)$ so the area of the right-hand triangle is given by the integral:

$$\int_3^4 [x - 3]\, dx = \left[\frac{1}{2}x^2 - 3x\right]_3^4 = [8 - 12] - \left[\frac{9}{2} - 9\right] = \frac{1}{2}$$

Adding these two areas, we get $2 + 0.5 = 2.5$. ◀

If we had mindlessly integrated in the previous Example without consulting a graph:

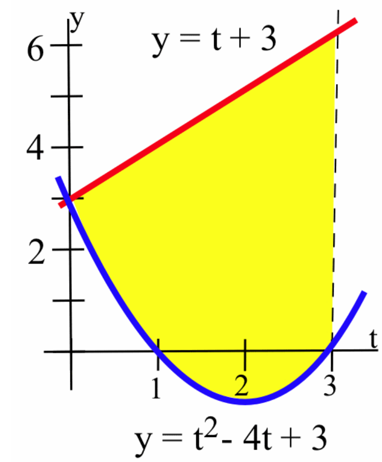
$$\int_1^4 [3 - x]\, dx = \left[3x - \frac{1}{2}x^2\right]_1^4 = [12 - 8] - \left[3 - \frac{1}{2}\right] = \frac{3}{2}$$

we would have arrived at an incorrect answer.

Graphing the region in question to determine which function is on "top" and which is on "bottom" is often crucial to getting the right answer to a problem involving the area between two curves.

Practice 1. Use integrals and the graphs of $f(x) = 1 + x$ and $g(x) = 3 - x$ to determine the area between the graphs of f and g for $0 \leq x \leq 3$.

Example 2. Objects A and B start from the same location at the same time and travel along the same path with respective velocities $v_A(t) = t+3$ and $v_B(t) = t^2 - 4t + 3$ meters per second (see margin). How far ahead is A after 3 seconds? After 5 seconds?



Solution. From the graph, it appears that $v_A(t) \geq v_B(t)$, at least for $0 \leq t \leq 3$, but for the second question we need to know whether this holds for $3 \leq t \leq 5$ as well. Setting $v_A(t) = v_B(t)$ to see where the graphs intersect:

$$t+3 = t^2 - 4t + 3 \quad \Rightarrow \quad t^2 - 5t = 0 \quad \Rightarrow \quad t = 0 \ \text{ or } \ t = 5$$

Checking that $v_A(1) = 4 > 0 = v_B(1)$ (or referring to the graph), we can conclude that $v_A(t) \geq v_B(t)$ on the interval $[0, 5]$.

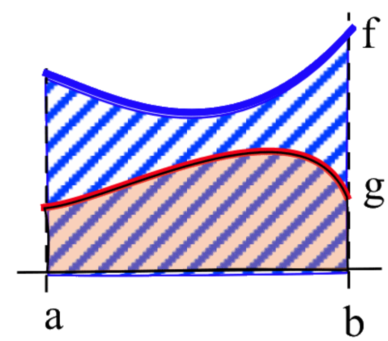
Because $v_A(t) \geq v_B(t)$, the "area" between the graphs of v_A and v_B over an interval $[0, x]$ represents the distance between the objects after x seconds. After three seconds, the distance apart is:

$$\int_0^3 [v_A(t) - v_B(t)]\, dt = \int_0^3 \Big[(t+3) - (t^2 - 4t + 3)\Big]\, dt = \int_0^3 \Big[5t - t^2\Big]\, dt$$
$$= \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^3 = \left[\frac{45}{2} - 9\right] - [0 - 0] = \frac{27}{2}$$

or 13.5 meters. After five seconds, the distance apart is

$$\int_0^5 [v_A(t) - v_B(t)]\, dt = \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^5 = \frac{125}{6}$$

or approximately 20.83 meters. ◀

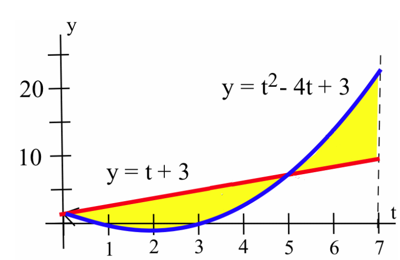


If $f(x) \geq g(x) \geq 0$ on an interval $[a, b]$ (as illustrated in the margin figure), we could have used a simpler geometric argument that the area between the graphs of f and g is just the area below the graph of f **minus** the area below the graph of g:

$$\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx$$

which agrees with our previous result. We took a different approach at the beginning of this section, however, because it provides a nice (yet simple) example of translating a geometric or physical problem into a Riemann sum and then into a definite integral.

Example 3. Find the area of the shaded region in the margin figure.



Solution. These are the same two functions from our previous Example; in our previous solution we observed that $t+3 \geq t^2 - 4t + 3$ for $0 \leq t \leq 5$, and it is straightforward to check that $t + 3 \leq t^2 - 4t + 3$ for $t \geq 5$ (and, in particular, for $5 \leq t \leq 7$).

We therefore need to split our problem into two pieces and subtract the "bottom" function from the "top" function on each interval. The area of the left region is:

$$\int_0^5 \left[(t+3)-(t^2-4t+3)\right]\,dt = \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^5 = \frac{125}{6}$$

(as worked out in the previous example), while the area of the region on the right is:

$$\int_5^7 \left[(t^2-4t+3)-(t+3)\right]\,dt = \left[\frac{1}{3}t^3 - \frac{5}{2}t^2\right]_5^7 = \frac{38}{3}$$

so the total area is $\frac{125}{6} + \frac{38}{3} = \frac{67}{2} = 33.5$. ◀

Average Value of a Function

We compute the **average** (or **mean value**) of n numbers, $a_1, a_2, \ldots, a_n$ by adding them up and dividing by n:

A "bar" above a quantity typically indicates the **mean** of that quanitity.

$$\text{average} = \overline{a} = \frac{1}{n}\sum_{k=1}^{n} a_k$$

but computing the average value of a *function* requires an integral.

To estimate the average value of f on the interval $[a,b]$, we can partition $[a,b]$ into n equally long subintervals of length $\Delta x = \frac{b-a}{n}$, then choose a value c_k in each subinterval, and find the average of the function values $f(c_k)$ at those n points:

$$\overline{f} = \text{average of } f \;\approx\; \frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} \;=\; \sum_{k=1}^{n} f(c_k)\cdot\frac{1}{n}$$

While this last term resembles a Riemann sum, it does not have the form $\sum f(c_k)\cdot\Delta x_k$, because $\frac{1}{n} \neq \Delta x = \frac{b-a}{n}$. But multiplying and dividing by $b-a$ yields:

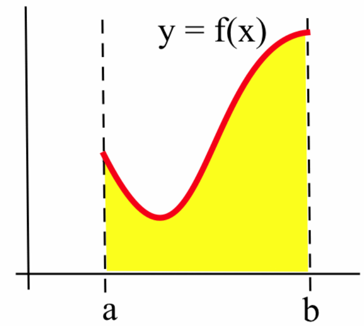
$$\sum_{k=1}^{n} f(c_k)\cdot\frac{1}{n} = \sum_{k=1}^{n} f(c_k)\cdot\frac{b-a}{n}\cdot\frac{1}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\cdot\frac{b-a}{n}$$

This last (Riemann) sum converges to a definite integral:

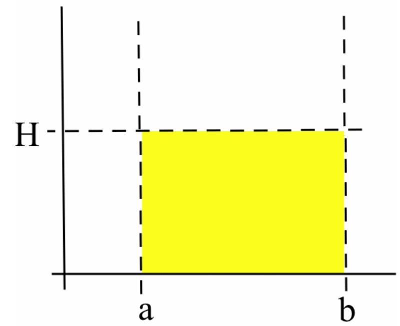
$$\frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\cdot\frac{b-a}{n} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\cdot\Delta x \longrightarrow \frac{1}{b-a}\int_a^b f(x)\,dx$$

as the number of subintervals n gets larger and the mesh, $\Delta x = \frac{b-a}{n}$, approaches 0.

> **Definition: Average (Mean) Value of a Function**
> The **average value** of an integrable function f on $[a, b]$ is
>
> $$\frac{1}{b-a}\int_a^b f(x)\,dx$$



The average value of a positive function has a nice geometric interpretation. Imagine that the area under f (see margin) represents a liquid trapped above by the graph of f and on the other sides by the x-axis and the lines $x = a$ and $x = b$. If we remove the "lid" (the graph of f), the liquid would settle into the shape of a rectangle with the same area as the region under the graph of f. If the height of this rectangle is H, then the area of the rectangle is $H \cdot (b - a)$, so:



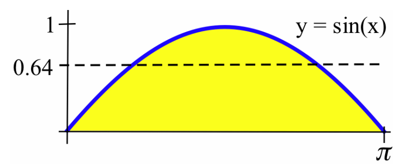
$$H \cdot (b-a) = \int_a^b f(x)\,dx \quad \Rightarrow \quad H = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The average value of a positive function f is the height H of the rectangle whose area is the same as the area under f.

Example 4. Find the average value of $\sin(x)$ on the interval $[0, \pi]$.

Solution. Using our definition, the average value is:

$$\frac{1}{\pi - 0}\int_0^\pi \sin(x)\,dx = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^\pi = \frac{1}{\pi}[(1)-(-1)] = \frac{2}{\pi} \approx 0.6366$$

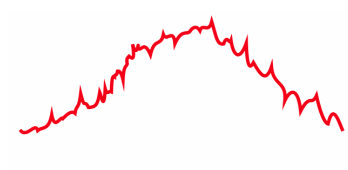


A rectangle with height $\dfrac{2}{\pi} \approx 0.64$ on the interval $[0, \pi]$ encloses the same area as one arch of the sine curve. ◀

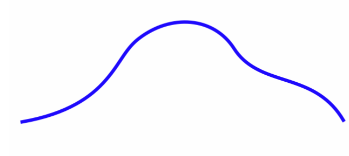
If the interval in the previous Example had been $[0, 2\pi]$, the average value would be 0. (Why?)

Practice 2. During a nine-hour work day, the production rate at time t hours was $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

Function averages, involving means as well as more complicated techniques, are used to "smooth" data so that underlying patterns become more obvious and to remove high frequency "noise" from signals. In these situations, the value of the original function f at a point is replaced by some "average of f" over an interval including that point. If f is the graph of rather jagged data (see margin), then the 10-year average of f is the integral:



"noisy" signal

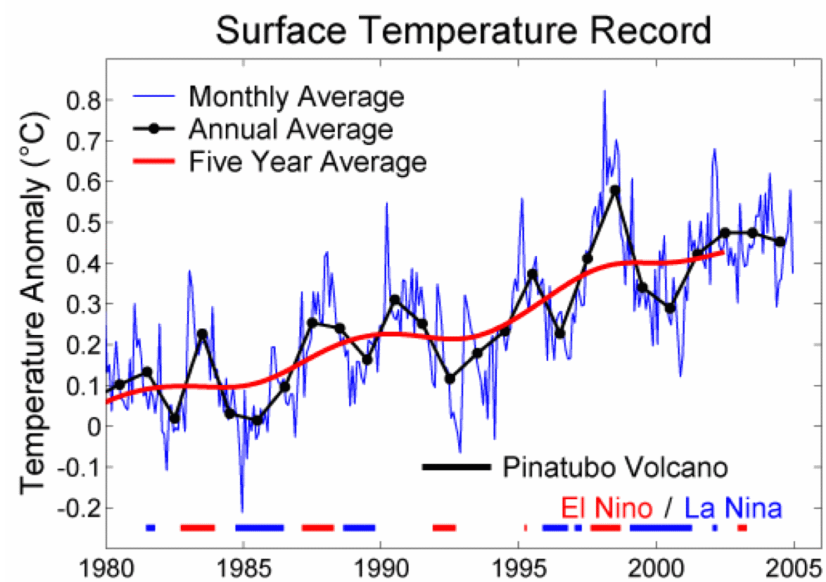


"smoothed" signal

$$g(x) = \frac{1}{10}\int_{x-5}^{x+5} f(t)\,dt$$

an average of f over a timespan of five years on either side of x.

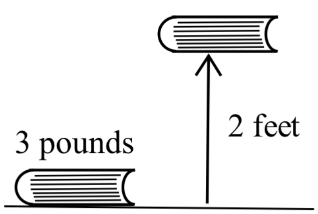
The figure below shows the graphs of a monthly average (rather "noisy" data) of surface-temperature data, an annual average (still rather "jagged") and a five-year average (a much smoother function):



This "moving average" of "noisy" data is frequently used with data such as weather information and stock prices.

Typically this "moving average" function reveals a pattern much more clearly than the original data.

Work

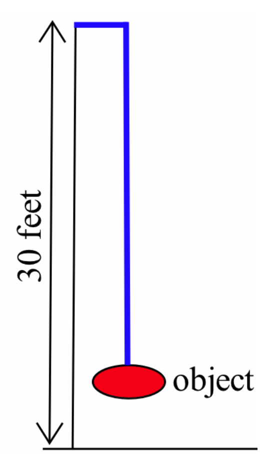


In physics, the amount of work done on an object is defined as the force applied to the object times the displacement of the object (the distance the object is moved while the force is applied). Or, more succinctly:

$$\boxed{\text{work} = (\text{force}) \cdot (\text{displacement})}$$

If you lift a three-pound book two feet, then the force is 3 pounds (the weight of the book), and the displacement is 2 feet, so you have done (3 pounds) · (2 feet) = 6 foot-pounds of work. When the applied force and the displacement are both constants, calculating work is simply a matter of multiplication.

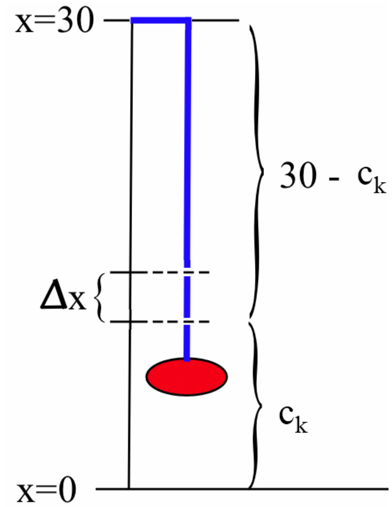
Practice 3. How much work is done lifting a 10-pound object from the ground to the top of a 30-foot building?



If either the force or the displacement varies, however, we need to use integration.

Example 5. How much work is done lifting a 10-pound object from the ground to the top of a 30-foot building using a cable that weighs 2 pounds per foot?

Solution. This is more challenging situation. We know the work needed to move the object is $(10)(30) = 300$ foot-pounds, but once we start pulling the cable onto the roof, we need to do less and less work to pull the remaining part of the cable.



Let's partition the cable into small increments so the displacement of each small piece of the cable is roughly constant. If we break the cable into n small pieces, each piece has length $\Delta x = \frac{30}{n}$, so its weight (the force required to move it) is:

$$(\Delta x \text{ ft}) \cdot \left(2 \frac{\text{lbs}}{\text{ft}}\right) = 2\Delta x \text{ lbs}$$

If this small piece of cable is initially c_k feet above the ground, then its displacement is $30 - c_k$ feet, so the work done on this small piece is $2(30 - c_k)\Delta x$ ft-lbs and the total work done on the entire cable is (approximately):

$$\sum_{k=1}^{n} 2(30 - c_k)\Delta x \quad \longrightarrow \quad \int_0^{30} 2(30 - x)\, dx$$

Once again we have formed a Riemann sum, which converges to a definite integral as we chop the cable into smaller and smaller pieces. This integral represents the work needed to lift the cable to the roof:

$$\int_0^{30} 2(30 - x)\, dx = \int_0^{30} (60 - 2x)\, dx = 60x - x^2 \Big|_0^{30}$$
$$= [1800 - 900] - [0 - 0] = 900 \text{ ft-lbs}$$

so the total work required to lift the object and the cable to the roof is $300 + 900 = 1200$ ft-lbs. ◀

Practice 4. Suppose the building in Example 5 is 50 feet tall and the cable weighs 3 pounds per foot.

(a) Compute the work done raising the object from the ground to a height of 10 feet.

(b) From a height of 10 feet to a height of 20 feet.

The situation in the previous Example and Practice problems is but one of many that arise when computing work. We will examine others in Section 5.4.

Summary

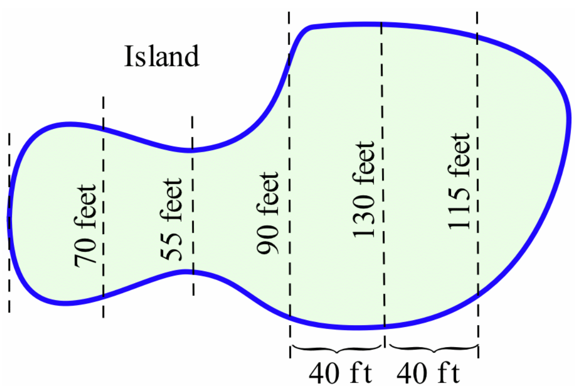
The area, average and work applications in this section merely introduce a few of the many applications of definite integrals. They illustrate the pattern of moving from an applied problem to a Riemann sum, to a definite integral and, finally, to a numerical answer. We will explore many more applications in Chapter 5.

4.7 Problems

In Problems 1–4, use the values in the table below to estimate the indicated areas.

x	f(x)	g(x)	h(x)
0	5	2	5
1	6	1	6
2	6	2	8
3	4	2	6
4	3	3	5
5	2	4	4
6	2	5	2

1. Estimate the area between f and g for $1 \le x \le 4$.
2. Estimate the area between f and g for $1 \le x \le 6$.
3. Estimate the area between f and h for $0 \le x \le 4$.
4. Estimate the area between g and h for $0 \le x \le 6$.
5. Estimate the area of the island in the figure below.



6. Estimate the area of the island in figure above if the distances between the lines is 50 feet instead of 40 feet.

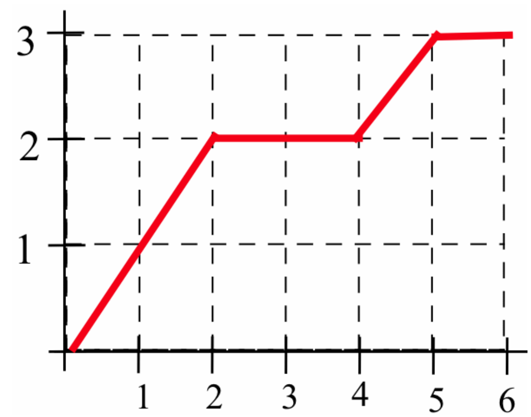
In Problems 7–18, sketch a graph of each function and find the area between the graphs of f and g for x in the given interval.

7. $f(x) = x^2 + 3$, $g(x) = 1$, $-1 \le x \le 2$
8. $f(x) = x^2 + 3$, $g(x) = 1 + x$, $0 \le x \le 3$
9. $f(x) = x^2$, $g(x) = x$, $0 \le x \le 2$
10. $f(x) = 4 - x^2$, $g(x) = x + 2$, $0 \le x \le 2$
11. $f(x) = \dfrac{1}{x}$, $g(x) = x$, $1 \le x \le e$
12. $f(x) = \sqrt{x}$, $g(x) = x$, $0 \le x \le 4$
13. $f(x) = x + 1$, $g(x) = \cos(x)$, $0 \le x \le \dfrac{\pi}{4}$
14. $f(x) = (x-1)^2$, $g(x) = x + 1$, $0 \le x \le 3$
15. $f(x) = e^x$, $g(x) = x$, $0 \le x \le 2$
16. $f(x) = \cos(x)$, $g(x) = \sin(x)$, $0 \le x \le \dfrac{\pi}{4}$
17. $f(x) = 3$, $g(x) = \sqrt{1 - x^2}$, $0 \le x \le 1$
18. $f(x) = 2$, $g(x) = \sqrt{4 - x^2}$, $-2 \le x \le 2$

In Problems 19–22, use the values of f in the table at the beginning of the page to estimate the average value of f on the indicated interval.

19. $[0.5, 4.5]$ 20. $[0.5, 6.5]$ 21. $[1.5, 3.5]$ 22. $[3.5, 6.5]$

In 23–26, find the average value of the function whose graph appears below on the given interval.

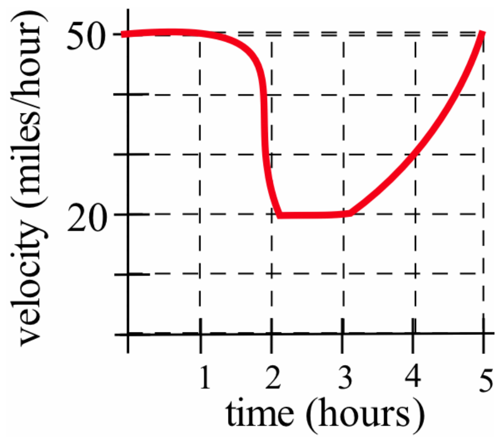


23. $[0, 2]$ 24. $[0, 4]$ 25. $[1, 6]$ 26. $[4, 6]$

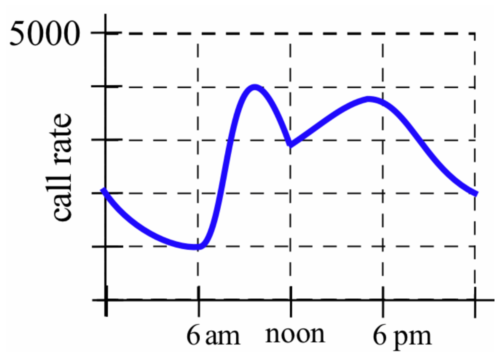
In Problems 27–32, find the average value of the given function on the indicated interval.

27. $f(x) = 2x + 1$, $0 \le x \le 4$
28. $f(x) = x^2$, $0 \le x \le 2$
29. $f(x) = x^2$, $1 \le x \le 3$
30. $f(x) = \sqrt{x}$, $0 \le x \le 4$
31. $f(x) = \sin(x)$, $0 \le x \le \pi$
32. $f(x) = \cos(x)$, $0 \le x \le \pi$
33. Calculate the average value of $f(x) = \sqrt{x}$ on $[0, C]$ for $C = 1, 9, 81, 100$. What is the pattern?
34. Calculate the average value of $f(x) = x$ on $[0, C]$ for $C = 1, 10, 80, 100$. What is the pattern?

35. The figure below shows the velocity of a car during a five-hour trip.
   (a) Estimate how far the car traveled.
   (b) At what constant velocity should you drive in order to travel the same distance in five hours?



36. The figure below shows the number of telephone calls per minute at a large company. Estimate the average number of calls per minute:
   (a) from 8:00 a.m. to 5:00 p.m.
   (b) from 9:00 a.m. to 1:00 p.m.



37. (a) How much work is done lifting a 20-pound bucket from the ground to the top of a 30-foot building with a cable that weighs three pounds per foot?
   (b) How much work is done lifting the same bucket from the ground to a height of 15 feet with the same cable?

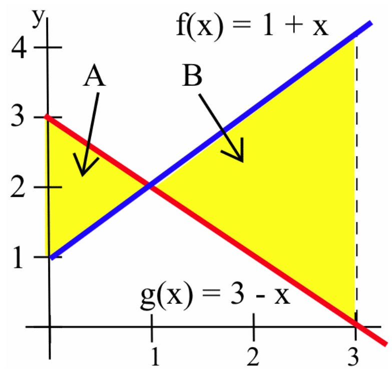
38. (a) How much work is done lifting a 60-pound chair from the ground to the top of a 20-foot building with a cable that weighs 1 pound per foot?
   (b) How much work is done lifting the same chair from the ground to a height of 5 feet with the same cable?

39. (a) How much work is done lifting a 10-pound calculus book from the ground to the top of a 30-foot building with a cable that weighs 2 pounds per foot?
   (b) From the ground to a height of 10 feet?
   (c) From a height of 10 feet to a height of 20 feet?

40. How much work is done lifting an 80-pound injured child to the top of a 20-foot hole using a stretcher weighing 14 pounds and a cable that weighs 1 pound per foot?

41. How much work is done lifting a 60-pound injured child to the top of a 15-foot hole using a stretcher weighing 10 pounds and a cable that weighs 2 pound per foot?

42. How much work is done lifting a 120-pound injured adult to the top of a 30-foot hole using a stretcher weighing 10 pounds and a cable that weighs 2 pound per foot?



4.7 Practice Answers

1. Referring to a graph (see margin figure) and using geometry: $A = \frac{1}{2}(2)(1) = 1$ and $B = \frac{1}{2}(4)(2) = 4$ so the total area is $1 + 4 = 5$.
   Referring to a graph of the functions and using integrals:
   $$A = \int_0^1 [(3-x)-(1+x)]\,dx = \int_0^1 [2-2x]\,dx$$
   $$= \left[2x - x^2\right]_0^1 = [2-1]-[0-0] = 1$$
   $$B = \int_1^3 [(1+x)-(3-x)]\,dx = \int_1^3 [2x-2]\,dx$$
   $$= \left[x^2 - 2x\right]_1^3 = [9-6]-[1-2)] = 4$$
   which also results in a total area of $1 + 4 = 5$.

2. Using the average value formula:
   $$\frac{1}{9-0}\int_0^9 \left[5+\sqrt{t}\right]\,dt = \frac{1}{9}\int_0^9 \left[5+t^{\frac{1}{2}}\right]\,dt = \frac{1}{9}\left[5t + \frac{2}{3}t^{\frac{3}{2}}\right]_0^9$$
   $$= \frac{1}{9}\left[\left(45+\frac{2}{3}\cdot 27\right)-(0+0)\right] = \frac{45+18}{9} = 7$$
   so the average hourly production rate is 7 cars per hour.

3. $(\text{force})\cdot(\text{displacement}) = (10 \text{ pounds})\cdot(30 \text{ feet}) = 300 \text{ foot-pounds}$

4. (a) The work required to move the object a distance of 10 feet is $(10 \text{ pounds})\cdot(10 \text{ feet}) = 100$ foot-pounds. The work required to move the top 10 feet of the cable onto the roof is:
   $$\int_0^{10} (10-x)\cdot 3\,dx = \left[30x - \frac{3}{2}x^2\right]_0^{10} = [300-150]-[0] = 150 \text{ ft-lbs}$$
   and the force required to move the remaining 40 feet of cable is:
   $$(40 \text{ ft})\cdot\left(3\,\frac{\text{lbs}}{\text{ft}}\right)(10 \text{ ft}) = 1200 \text{ ft-lbs}$$
   so the total work required is $100 + 150 + 1200 = 1450$ foot-pounds.

   (b) The work required to move the object a distance of 10 feet is again $(10 \text{ pounds})\cdot(10 \text{ feet}) = 100$ foot-pounds. The work required to move the top 10 feet of the cable onto the roof is again 150 foot-pounds, and the force required to move the remaining 30 feet of cable is:
   $$(30 \text{ ft})\cdot\left(3\,\frac{\text{lbs}}{\text{ft}}\right)(10 \text{ ft}) = 900 \text{ ft-lbs}$$
   so the total work required is $100 + 150 + 900 = 1150$ foot-pounds.

4.8 Using Tables (and Technology) to Find Antiderivatives

Appendix I shows patterns for many antiderivatives — some of which you should already know based on your work in this chapter. Many reference books and Web sites contain far more than the ones listed in the appendix. A table of integrals helps you while you are learning calculus and serves as a reference later when you are using calculus.

Think of an integral table as a dictionary: something to use when you need to spell a challenging word or need the meaning of a new word. It would be difficult to write a report if you had to look up the spelling of *every* word, and it will be difficult to learn and use calculus if you have to look up every antiderivative. Tables of antiderivatives are limited by necessity and often take longer to use than finding an antiderivative from scratch, but they can also be very valuable and useful.

This section shows how to transform some integrals into forms found in Appendix I and how to use "recursion" formulas found in integral tables. The first Examples and Practice problems illustrate some of the techniques used to change an integral into a standard form.

These techniques are useful whether that standard form resides in a table or in your head.

Example 1. Use Appendix I to find $\displaystyle\int \frac{1}{9+x^2}\,dx$.

Solution. The integrand is a rational function, and the first entry you see listed in the "Rational Functions" section of Appendix I should be:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

This resembles the pattern we need, so replacing the a with 3 we have:

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{3^2+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

You can (and should) check this answer by differentiating. ◀

Appendix I (like some other integral tables) omits the "$+C$" arbitrary constant for conciseness, but you need to remember to include it when using the results of the table to find an indefinite integral.

Practice 1. Use Appendix I to find $\displaystyle\int \frac{1}{25+x^2}\,dx$ and $\displaystyle\int \frac{1}{25-x^2}\,dx$.

Notice that a small change in the form of the integrand (from $+$ to $-$ here) can lead to a very different result.

Example 2. Use Appendix I to find $\displaystyle\int \frac{1}{5+x^2}\,dx$.

Solution. The integrand is again a rational function, and the general form is the same as in the previous Example:

$$\int \frac{1}{5+x^2}\,dx = \int \frac{1}{(\sqrt{5})^2+x^2}\,dx = \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

but here we needed to put $a=\sqrt{5}$. ◀

The constant in the denominator of this integrand was not a perfect square, but the process is exactly the same — even if the result looks a bit "messier" due to the presence of the radical.

Practice 2. Use Appendix I to find $\displaystyle\int \frac{1}{7+x^2}\,dx$ and $\displaystyle\int \frac{1}{7-x^2}\,dx$.

We often need to perform some algebraic manipulations to change an integrand into one that exactly matches a pattern in the table.

Example 3. Use Appendix I to find $\displaystyle\int \frac{1}{9+4x^2}\,dx$.

Solution. The integrand is again a rational function, and the general form resembles the one used in the previous Examples, but here we have a $4x^2$ where we only see x^2 in the table pattern. To get the integrand in the form we want, we can factor a 4 out of the denominator:

$$\begin{aligned}\int \frac{1}{9+4x^2}\,dx &= \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{3}{2}\right)^2+x^2}\,dx \\ &= \frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot\arctan\left(\frac{x}{\frac{3}{2}}\right)+C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

Another approach involves a change of variable. First write:

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{3^2+(2x)^2}\,dx$$

We have $2x$ where we would like to see a standalone variable. To get that pattern, put $u = 2x \Rightarrow du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$:

$$\begin{aligned}\int \frac{1}{3^2+(2x)^2}\,dx &= \int \frac{1}{3^2+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C \\ &= \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

which yields the same result as our previous method. ◀

Practice 3. Use Appendix I to find $\displaystyle\int \frac{1}{25+9x^2}\,dx$ and $\displaystyle\int \frac{1}{25-9x^2}\,dx$.

Sometimes a change of variable is absolutely necessary.

Example 4. Use Appendix I to find $\displaystyle\int \frac{e^x}{9+e^{2x}}\,dx$.

Solution. Here the integrand is *not* a rational function, but we can transform it into one by using the substitution $u = e^x \Rightarrow du = e^x\,dx$ so that $u^2 = (e^x)^2 = e^{2x}$:

$$\begin{aligned}\int \frac{e^x}{9+e^{2x}}\,dx &= \int \frac{1}{3^2+(e^x)^2}\cdot e^x\,dx = \int \frac{1}{3^2+u^2}\,du \\ &= \frac{1}{3}\arctan\left(\frac{u}{3}\right)+C = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right)+C\end{aligned}$$

If you don't see the exact pattern you need in an integral table, try a substitution first. ◀

Practice 4. Evaluate $\displaystyle\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$ and $\displaystyle\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$.

How should you recognize whether algebra or a change of variable is needed? Experience and practice, practice, practice.

Using "Recursion" Formulas

A **recursion formula** gives one antiderivative in terms of another antiderivative. Usually the new antiderivative is somehow simpler than the original one. For example, the first recursion formula for a trigonometric function listed in Appendix I states:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

We will develop this formula from scratch in Problem 25 of Section 8.2. For now, you can check that it works by comparing the derivative of your answer to the original integrand for an integration problem that uses this — or any other — recursion formula.

This formula would allow us to write $\int \sin^8(x)\,dx$, for instance, in terms of $\int \sin^6(x)\,dx$, which should (theoretically, at least) be easier to compute than the original integral.

Example 5. Use a recursion formula to evaluate $\int \sin^4(x)\,dx$.

Solution. Applying the formula given in the discussion above:

$$\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx$$

This new integral is one we already know how to evaluate:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + K$$

Putting this together with the result of the recursion formula, we get:

$$\begin{aligned}\int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + C \\ &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C\end{aligned}$$

We could have included the "$+K$" here but then the result at the next stage would have included the constant terms

$$\cdots + \frac{3}{4}K + C$$

which is also an arbitrary constant.

We could have used Appendix I to find $\int \sin^2(x)\,dx$ instead — or even applied the recursion formula a second time to rewrite $\int \sin^2(x)\,dx$ in terms of $\int \sin^0(x)\,dx = \int 1\,dx$. ◀

Practice 5. Use Appendix I to evaluate $\int \cos^4(x)\,dx$ and $\int \cos^4(7x)\,dx$.

Using Technology

Many Web sites (such as Wolfram|Alpha, `www.wolframalpha.com`), computer programs (wxMaxima is a good free one) and calculators (such as the TI-89 or TI-Nspire CAS) feature computer algebra systems that can find antiderivatives of a wide variety of functions. For example, typing `integral sin^4(x)` into Wolfram|Alpha yields:

$$\int \sin^4(x)\,dx = \frac{1}{32}\left(12x - 8\sin(2x) + \sin(4x)\right) + \text{constant}$$

which (applying some trig idenities) agrees with our result above.

Although technology can help us find an antiderivative and evaluate a definite integral, in an application problem **you** still need to set up the Riemann sum that leads to the definite integral.

4.8 Problems

In Problems 1–48, use patterns and recursion formulas from the integral table in Appendix I as necessary (along with any other antiderivatives and integration techniques you already know) to evaluate each integral.

1. $\displaystyle\int \frac{1}{4+x^2}\,dx$

2. $\displaystyle\int \frac{5}{4+x^2}\,dx$

3. $\displaystyle\int \left[2x+\frac{2}{25+x^2}\right]dx$

4. $\displaystyle\int \frac{1}{4-x^2}\,dx$

5. $\displaystyle\int \frac{2}{9-x^2}\,dx$

6. $\displaystyle\int \left[\cos(x)+\frac{3}{25-x^2}\right]dx$

7. $\displaystyle\int \frac{1}{3+x^2}\,dx$

8. $\displaystyle\int \frac{5}{7+x^2}\,dx$

9. $\displaystyle\int \left[e^x+\frac{7}{2+x^2}\right]dx$

10. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

11. $\displaystyle\int \frac{3}{\sqrt{5-x^2}}\,dx$

12. $\displaystyle\int \frac{3}{\sqrt{4-x^2}}\,dx$

13. $\displaystyle\int \frac{1}{4+25x^2}\,dx$

14. $\displaystyle\int \frac{2}{\sqrt{9-16x^2}}\,dx$

15. $\displaystyle\int \frac{5}{\sqrt{1-4x^2}}\,dx$

16. $\displaystyle\int \sec(x+5)\,dx$

17. $\displaystyle\int \frac{2}{\sqrt{1+9x^2}}\,dx$

18. $\displaystyle\int x\cdot\sec(2x^2+7)\,dx$

19. $\displaystyle\int \ln(x+1)\,dx$

20. $\displaystyle\int \ln(3x-1)\,dx$

21. $\displaystyle\int 3x\cdot\ln(5x^2+7)\,dx$

22. $\displaystyle\int e^x\ln\left(e^x-3\right)\,dx$

23. $\displaystyle\int \cos(x)\cdot\ln\left(\sin(x)\right)\,dx$

24. $\displaystyle\int \frac{2}{\sqrt{x^2-9}}\,dx$

25. $\displaystyle\int \sqrt{4+x^2}\,dx$

26. $\displaystyle\int \sqrt{9+x^2}\,dx$

27. $\displaystyle\int \sqrt{16+x^2}\,dx$

28. $\displaystyle\int_0^1 \frac{1}{4+x^2}\,dx$

29. $\displaystyle\int_1^3 \left[2x+\frac{2}{25+x^2}\right]dx$

30. $\displaystyle\int_0^2 \frac{2}{9-x^2}\,dx$

31. $\displaystyle\int_{-1}^1 \frac{1}{3+x^2}\,dx$

32. $\displaystyle\int_0^1 \left[e^x+\frac{7}{2+x^2}\right]dx$

33. $\displaystyle\int_1^2 \frac{3}{\sqrt{5-x^2}}\,dx$

34. $\displaystyle\int_0^1 \frac{1}{4+25x^2}\,dx$

35. $\displaystyle\int_0^{0.1} \frac{5}{\sqrt{1-4x^2}}\,dx$

36. $\displaystyle\int_0^1 \frac{1}{\sqrt{9-4x^2}}\,dx$

37. $\displaystyle\int_0^6 \ln(x+1)\,dx$

38. $\displaystyle\int_0^3 3x\cdot\ln(5x^2+7)\,dx$

39. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cdot\ln\left(2+\sin(x)\right)\,dx$

40. $\displaystyle\int_0^2 \sqrt{4+x^2}\,dx$

41. $\displaystyle\int_{-3}^3 \sqrt{9+x^2}\,dx$

42. $\displaystyle\int_0^1 \sqrt{16+x^2}\,dx$

43. $\displaystyle\int \sin^3(x)\,dx$

44. $\displaystyle\int \cos^3(x)\,dx$

45. $\displaystyle\int \cos^5(x)\,dx$

46. $\displaystyle\int \sec^5(x)\,dx$

47. $\displaystyle\int x^2\cos(x)\,dx$

48. $\displaystyle\int x^2\sin^5(x)\,dx$

49. Before doing any calculations, predict which you expect to be larger:
    - the average value of $\sin(x)$ on $[0, \pi]$
    - the average value of $\sin^2(x)$ on $[0, \pi]$

    Then calculate each average to see if your prediction was correct.
50. Find the area of the region bounded by the graph of $f(x) = \ln(x)$, the x-axis and the lines $x = 1$ and $x = C$ when $C = e$, 10, 100 and 200.
51. Find the average value of $f(x) = \ln(x)$ on the interval $1 \le x \le C$ when $C = e$, 10, 100, 200.
52. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

    (a) $\displaystyle\int_0^1 e^x\,dx$ (b) $\displaystyle\int_0^1 xe^x\,dx$ (c) $\displaystyle\int_0^1 x^2e^x\,dx$
53. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

    (a) $\displaystyle\int_1^2 e^x\,dx$ (b) $\displaystyle\int_1^2 xe^x\,dx$ (c) $\displaystyle\int_1^2 x^2e^x\,dx$
54. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

    (a) $\displaystyle\int_0^\pi \sin(x)\,dx$

    (b) $\displaystyle\int_0^\pi x\sin(x)\,dx$

    (c) $\displaystyle\int_0^\pi x^2\sin(x)\,dx$
55. Evaluate $\displaystyle\int_0^C \frac{2}{1+x^2}\,dx$ for $C = 1$, 10, 20 and 30. Before doing the calculation, estimate the value of the integral when $C = 40$.

4.8 Practice Answers

1. The integral $\displaystyle\int \frac{1}{25+x^2}\,dx$ resembles the pattern from Example 1:

$$\int \frac{1}{25+x^2}\,dx = \int \frac{1}{5^2+x^2}\,dx = \frac{1}{5}\arctan\left(\frac{x}{5}\right) + C$$

The integrand in $\displaystyle\int \frac{1}{25-x^2}\,dx$ is also a rational function, but we need a different pattern from Appendix I (see margin) with $a = 5$:

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$$

$$\int \frac{1}{25-x^2}\,dx = \int \frac{1}{5^2-x^2}\,dx = \frac{1}{10}\ln\left|\frac{x+5}{x-5}\right| + C$$

2. The integral $\displaystyle\int \frac{1}{7+x^2}\,dx$ matches the pattern in Example 2:

$$\int \frac{1}{7+x^2}\,dx = \int \frac{1}{(\sqrt{7})^2+x^2}\,dx = \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

For $\displaystyle\int \frac{1}{7-x^2}\,dx$ we need the pattern in the margin with $a = \sqrt{7}$:

$$\int \frac{1}{7-x^2}\,dx = \int \frac{1}{(\sqrt{7})^2-x^2}\,dx = \frac{1}{2\sqrt{7}}\ln\left|\frac{x+\sqrt{7}}{x-\sqrt{7}}\right| + C$$

3. For the integral $\displaystyle\int \frac{1}{25+9x^2}\,dx$ we can factor 9 from the denominator:

$$\int \frac{1}{25+9x^2}\,dx = \int \frac{1}{9\left(\frac{25}{9}+x^2\right)}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{5}{3}\right)^2+x^2}\,dx$$
$$= \frac{1}{9}\cdot\frac{1}{\frac{5}{3}}\arctan\left(\frac{x}{\frac{5}{3}}\right)+C = \frac{1}{15}\arctan\left(\frac{3x}{5}\right)+C$$

and proceed as before. We could proceed similarly for $\displaystyle\int \frac{1}{25-9x^2}\,dx$ or we could substitute $u=3x$ (see margin):

$u = 3x \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$

$$\int \frac{1}{25-9x^2}\,dx = \int \frac{1}{5^2-(3x)^2}\,dx = \int \frac{1}{5^2-u^2}\cdot\frac{1}{3}\,du$$
$$= \frac{1}{3}\cdot\frac{1}{2\cdot 5}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{30}\cdot\ln\left|\frac{3x+5}{3x-5}\right|+C$$

4. For $\displaystyle\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$, first use the substitution in the margin:

$u = \sin(x) \Rightarrow du = \cos(x)\,dx$

$$\int \frac{\cos(x)}{25+\sin^2(x)}\,dx = \int \frac{1}{25+u^2}\,du$$

followed by the result of the first part of Practice 1:

$$\int \frac{1}{25+u^2}\,du = \frac{1}{5}\arctan\left(\frac{u}{5}\right)+C = \frac{1}{5}\arctan\left(\frac{\sin(x)}{5}\right)+C$$

For $\displaystyle\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$ use the same substitution, followed by the result from the second part of Practice 1:

$$\int \frac{1}{25-u^2}\,du = \frac{1}{10}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{10}\ln\left|\frac{\sin(x)+5}{\sin(x)-5}\right|+C$$

5. For $\displaystyle\int \cos^4(x)\,dx$ we need the recursion formula:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x)+\frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

with $n=4$:

$$\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\int \cos^2(x)\,dx$$
$$= \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\int \left[\frac{1}{2}+\frac{1}{2}\cos(2x)\right]dx$$
$$= \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\left[\frac{1}{2}x+\frac{1}{4}\sin(2x)\right]+C$$

$u = 7x \Rightarrow du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$

$\int \cos^4(7x)\,dx = \frac{1}{7}\int \cos^4(u)\,du$

For $\displaystyle\int \cos^4(7x)\,dx$, first use a substitution (see margin) and then the result of the previous integration:

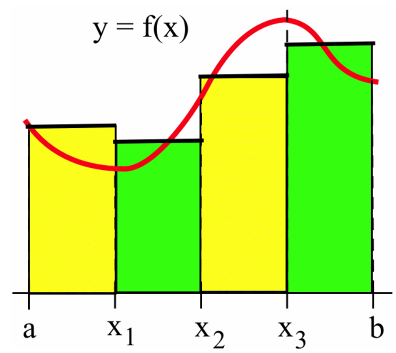
$$\int \cos^4(7x)\,dx = \frac{1}{28}\cos^3(7x)\sin(7x)+\frac{3}{28}\left[\frac{1}{2}(7x)+\frac{1}{4}\sin(14x)\right]+C$$

4.9 Approximating Definite Integrals

The Fundamental Theorem of Calculus tells how to calculate the exact value of a definite integral *if* the integrand is continuous and *if* we can find a formula for an antiderivative of the integrand. In practice, however, we may need to compute the definite integral of a function for which we only have table values or a graph—or of a function that does not have an elementary antiderivative. This section presents several techniques for getting approximate numerical values for definite integrals without using antiderivatives. Mathematically, exact answers are preferable and satisfying, but for most applications a numerical answer accurate to several digits is just as useful.

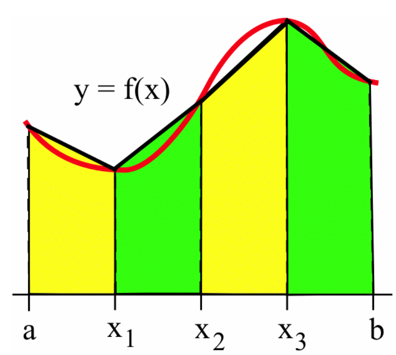
The ideas behind these methods are geometric and rather simple, but using the methods to get good approximations typically requires lots of arithmetic, something calculators and computers are very good (and quick) at doing.

The General Approach

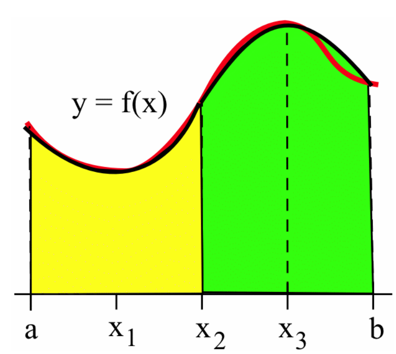


The methods in this section approximate the definite integral of a function f by partitioning the interval of integration and building an "easy" function with values close to those of f on each interval, then evaluating the definite integrals of the "easy" functions exactly. If the "easy" functions are close to f, then the sum of the definite integrals of the "easy" functions should be close to the definite integral of f.

The **Left**, **Right** and **Midpoint Rules** approximate f with horizontal lines on each partition interval so the "easy" functions are constant functions, and the approximating regions are rectangles (see top margin figure). The **Trapezoidal Rule** approximates f with slanted lines, so the "easy" functions are linear and the approximating regions are trapezoids (see middle margin figure). Finally, **Simpson's Rule** approximates f with parabolas, so the "easy" functions are quadratic polynomials (see bottom margin figure).



The Left and Right approximation rules are simply Riemann sums with the point c_k in the k-th subinterval chosen to be the left or right endpoint of that subinterval. They typically require a large number of computations to get even mediocre approximations to the definite integral of f and are seldom used in practice. Along with the Midpoint Rule (which chooses each c_k to be the midpoint of the k-th subinterval), they are discussed near the end of the Problems for this section.



All of these methods partition the interval $[a,b]$ into n subintervals of equal width, so each subinterval has width $h = \Delta x_k = \dfrac{b-a}{n}$. The points of the partition are $x_0 = a$, $x_1 = a+h$, $x_2 = a+2\cdot h$, $x_3 = a+3\cdot h$, and so on. The k-th point in the partition is given by the formula $x_k = a + k\cdot h$ and the last (n-th) point is thus:

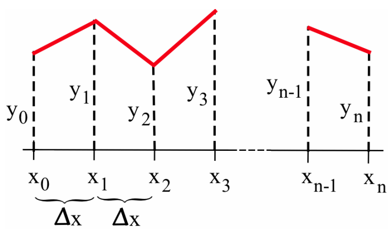
$$x_n = a + n\cdot h = a + n\left(\frac{b-a}{n}\right) = a + b - a = b$$

Approximating a Definite Integral Using Trapezoids

If the graph of f is curved, then slanted lines typically come closer to the graph of f than horizontal ones do. These slanted lines lead to trapezoidal approximating regions.

The area of a trapezoid is (base) · (average height) so the area of the first trapezoid in the margin figure is:

See Problem 29.



$$(\Delta x) \cdot \left(\frac{y_0 + y_1}{2} \right)$$

Similarly, the areas of the next few trapezoids are:

$$(\Delta x) \cdot \left(\frac{y_1 + y_2}{2} \right), \quad (\Delta x) \cdot \left(\frac{y_2 + y_3}{2} \right), \quad (\Delta x) \cdot \left(\frac{y_3 + y_4}{2} \right)$$

and so on, with the area of the last region being

$$(\Delta x) \cdot \left(\frac{y_{n-1} + y_n}{2} \right)$$

The sum of these n trapezoidal areas is:

$$\begin{aligned} T_n &= (\Delta x) \left(\frac{y_0 + y_1}{2} \right) + (\Delta x) \left(\frac{y_1 + y_2}{2} \right) + (\Delta x) \left(\frac{y_2 + y_3}{2} \right) + \cdots \\ &\qquad + (\Delta x) \left(\frac{y_{n-1} + y_n}{2} \right) \\ &= \left(\frac{\Delta x}{2} \right) \left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n) \right] \\ &= \left(\frac{h}{2} \right) \left[y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n \right] \\ &= \left(\frac{h}{2} \right) \left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n) \right] \end{aligned}$$

Each $f(x_k)$ value, except the first ($k = 0$) and the last ($k = n$), is the right-endpoint height of one trapezoid and the left-endpoint height of the next, so it shows up in the calculation for two trapezoids and is multiplied by 2 in the formula for the trapezoidal approximation.

Trapezoidal Approximation Rule

If f is integrable on $[a, b]$ and $[a, b]$ is partitioned into n subintervals of width $h = \frac{b-a}{n}$

then the Trapezoidal approximation of $\int_a^b f(x)\,dx$ is:

$$T_n = \frac{h}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

Example 1. Compute T_4, the Trapezoidal approximation of $\int_1^3 f(x)\,dx$ for $n = 4$, with the values of f in the margin table.

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Solution. The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$ so:

$$\begin{aligned} T_4 &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\ &= \frac{0.5}{2}\left[4.2 + 2(3.4) + 2(2.8) + 2(3.6) + (3.2)\right] = (0.25)(27) = 6.75 \end{aligned}$$

so we can say that $\int_1^3 f(x)\,dx \approx 6.75$. ◀

Let's see how well the Trapezoidal Rule approximates an integral whose value we can compute exactly:

$$\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}[27-1] = \frac{26}{3} \approx 8.6666667$$

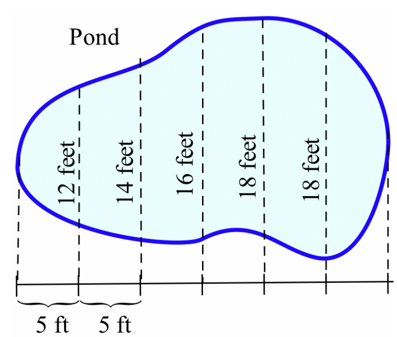
Example 2. Calculate T_4 for $\int_1^3 x^2\,dx$.

Solution. The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$ so:

$$\begin{aligned} T_4 &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\ &= \frac{0.5}{2}\left[(1.0)^2 + 2(1.5)^2 + 2(2.0)^2 + 2(2.5)^2 + (3.0)^2\right] \\ &= (0.25)\left[1 + 2(2.25) + 2(4) + 2(6.25) + 9\right] = 8.75 \end{aligned}$$

which is within 0.1 of the exact answer. Larger values for n give better approximations: $T_{20} = 8.67$ and $T_{100} = 8.6668$. ◀

Practice 1. On a summer day, the level of the pond shown in the margin fell 0.1 feet because of evaporation. Use the Trapezoidal Rule to approximate the surface area of the pond and then estimate how much water evaporated.



Approximating a Definite Integral Using Parabolas

If the graph of f is curved, the slanted lines from the Trapezoidal Rule may not fit the graph of f as closely as we would like, requiring a large number of subintervals to achieve a good approximation of the definite integral. Curves typically fit the graph of f better than straight lines in such situations, and the easiest nonlinear curves we know are parabolas.

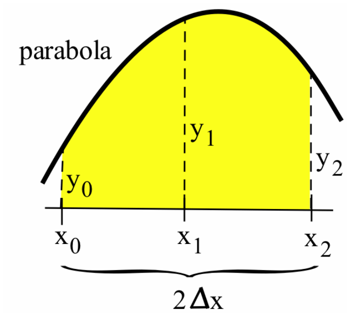
Just as we need two points to determine an equation of a line, we will need three points to determine an equation of a parabola.

This parabolic method is known as **Simpson's Rule**, named after British mathematician and inventor Thomas Simpson (1710–1761); Germans call it *Kepler'sche Fassregel*, after Johannes Kepler, who developed it 100 years before Simpson.

Calling these points (x_0, y_0), (x_1, y_1) and (x_2, y_2), the area under a parabolic region with evenly spaced x_k values (see margin) is:

This result is not obvious; see Problem 32 for the necessary algebra.

$$(2\Delta x) \cdot \left[\frac{y_0 + 4y_1 + y_2}{6} \right] = \frac{\Delta x}{3} [y_0 + 4y_1 + y_2]$$



Taking the subintervals in pairs, the areas of the next few parabolic regions are:

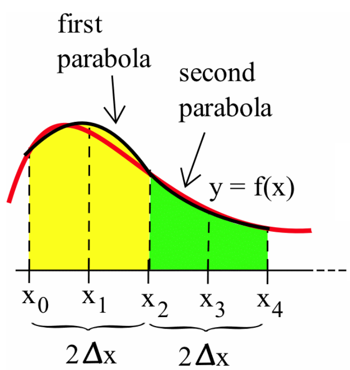
$$\frac{\Delta x}{3} [y_2 + 4y_3 + y_4], \quad \frac{\Delta x}{3} [y_4 + 4y_5 + y_6], \quad \frac{\Delta x}{3} [y_6 + 4y_7 + y_8]$$

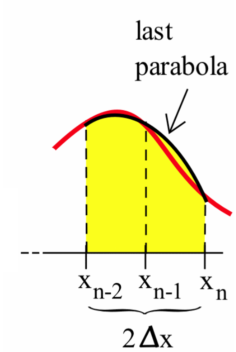
and so on, with the area of the last pair of regions being:

$$\frac{\Delta x}{3} [y_{n-2} + 4y_{n-1} + y_n]$$

so the sum of all n parabolic areas (see margin) is:

$$\begin{aligned} S_n &= \frac{\Delta x}{3} [y_0 + 4y_1 + y_2] + \frac{\Delta x}{3} [y_2 + 4y_3 + y_4] + \cdots + \frac{\Delta x}{3} [y_{n-2} + 4y_{n-1} + y_n] \\ &= \left(\frac{h}{3}\right) [(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 \cdots + y_{n-2} + 4y_{n-1} + y_n] \\ &= \left(\frac{h}{3}\right) [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-1} + 4y_{n-1} + y_n)] \\ &= \left(\frac{h}{3}\right) [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\ &\qquad + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \end{aligned}$$





In order to use **pairs** of subintervals, the number n of subintervals must be **even**. Notice that the coefficient pattern for the area under a single parabolic region is 1–4–1, but when we put several parabolas next to each other, they share some edges and the pattern becomes 1–4–2–4–2–$\cdots$–2–4–1 with the shared edges getting counted twice.

Parabolic Approximation Rule (Simpson's Rule)

If f is integrable on $[a, b]$ and $[a, b]$ is partitioned into n subintervals of width $h = \frac{b-a}{n}$

then the Parabolic approximation of $\int_a^b f(x)\,dx$ is:

$$S_n = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Example 3. Calculate S_4, the Simpson's Rule approximation of $\int_1^3 f(x)\,dx$ for the function f with values in the margin table.

Solution. The step size is $h = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{1}{2}$, so:

$$\begin{aligned} S_4 &= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{\frac{1}{2}}{3}\left[4.2 + 4(3.4) + 2(2.8) + 4(3.6) + (3.2)\right] = \frac{1}{6}(41) = \frac{41}{6} \end{aligned}$$

or approximately 6.833. ◀

Example 4. Calculate S_4 for $\displaystyle\int_1^3 2^x\,dx$.

Solution. As in the previous Examples, $h = \frac{b-a}{n} = 0.5$ and $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$ and $x_4 = 3$.

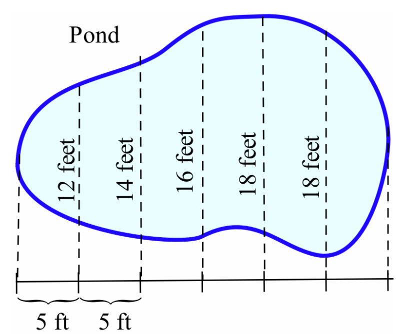
$$\begin{aligned} S_4 &= \frac{h}{3}\cdot\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{\frac{1}{2}}{3}\cdot\left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right] \\ &= \left(\frac{1}{6}\right)\left[2^1 + 4\left(2^{1.5}\right) + 2\left(2^2\right) + 4\left(2^{2.5}\right) + \left(2^3\right)\right] \\ &= \left(\frac{1}{6}\right)\left[2 + 4(2\sqrt{2}) + 2(4) + 4(4\sqrt{2}) + 8\right] = \left(\frac{1}{6}\right)\left[18 + 20\sqrt{2}\right] \end{aligned}$$

or approximately 8.656854. The exact value of the integral is:

$$\int_1^3 2^x\,dx = \left[\frac{2^x}{\ln(2)}\right]_1^3 = \frac{8}{\ln(2)} - \frac{2}{\ln(2)} = \frac{6}{\ln(2)} \approx 8.65617024533$$

Larger values of n give better approximations: $S_{20} = 8.656171$ and $S_{100} = 8.656170$. ◀

Practice 2. Use Simpson's Rule to estimate the surface area of the pond in the margin figure.



Which Method Is Best?

The most difficult and time-consuming part of these approximations, whether done by hand or by computer, is the evaluation of the function at the x_k values. For n subintervals, all of the methods require about the same number of function evaluations. The table on the next page illustrates how closely each method approximates $\displaystyle\int_1^5 \frac{1}{x}\,dx = \ln(5) \approx 1.609437912$ using several values of n. The results in the table also show how quickly the actual error shrinks as the value of n increases: just doubling n from 4 to 8 cuts the actual error of the Simpson's Rule approximation of this definite integral by a factor of 9—a good reward for our extra work.

The "error bounds" in the third column are discussed below.

method	approximation	error bound	actual error
T_4	1.683333333	0.6666666	0.07389542
S_4	1.622222222	0.5333333	0.01278431
L_4	2.083333333	2.0000000	0.47389542
R_4	1.283333333	2.0000000	0.32610458
M_4	1.574603175	0.3333333	0.03483474
T_8	1.628968254	0.1666666	0.01953034
S_8	1.610846561	0.0333333	0.00140865
L_8	1.828968254	1.0000000	0.21953034
R_8	1.428968254	1.0000000	0.18046966
M_8	1.599844394	0.0833333	0.00959352
T_{20}	1.612624844	0.0266667	0.00318693
S_{20}	1.609486789	0.0008533	0.00004888
L_{20}	1.692624844	0.4000000	0.08318693
R_{20}	1.532624844	0.4000000	0.07681307
M_{20}	1.607849324	0.0133333	0.00158859

Notice that for each value of n, the Simpson's Rule approximation S_n has the smallest error, and that the error for the Midpoint Rule approximation M_n (discussed in the Problems) is roughly half the error for the Trapezoidal Rule T_n. L_n and R_n denote the Left and Right approximations, respectively.

How Good Are the Approximations?

The approximation rules are valuable by themselves, but they are particularly useful because we can find "error bound" formulas that guarantee how close these approximations come to the exact values of the definite integral. It is useful to know that the value of an integral is "about 3.7," but we can have more confidence in our approximation if we know that value is "within 0.0001 of 3.7." Then we can decide if our approximation is good enough for the job at hand or if we need to improve it.

We can also solve the formulas for the error bounds provided below to determine how many subintervals we need to guarantee that our approximation is within some specified distance of the exact answer. There is no reason to use 1000 subintervals if 18 will give the needed accuracy. Unfortunately, the formulas for the error bounds require information about the derivatives of the integrands, so we cannot use these error bound formulas for the approximations of integrals of functions defined only by tables or graphs — or of continuous (hence integrable) functions that fail to have continuous derivatives.

The "error bound" formula for the Trapezoidal Rule approximation given at the top of the next page is just a "guarantee": the actual error is guaranteed to be no larger than the error bound. In fact, the actual error is usually much smaller than the error bound (compare the error bounds with the actual error for T_4, T_8 and T_{20} in the table above to see this principle in action).

The word "error" does not indicate a mistake, it simply means the deviation or distance of the approximate answers from the exact answer.

Error Bound for Trapezoidal Approximation

If f'' is continuous on $[a, b]$ and $|f''(x)| \leq B_2$
then the "error" of the T_n approximation of $\int_a^b f(x)\,dx$ satisfies:

$$|\text{"error"}| = \left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{(b-a)^3 B_2}{12n^2}$$

While it's possible to prove this error bound formula using mathematics you've already learned, the proof is highly technical and sheds little or no insight into the workings of the Trapezoidal Rule, so we (like the authors of most calculus books) have omitted it.

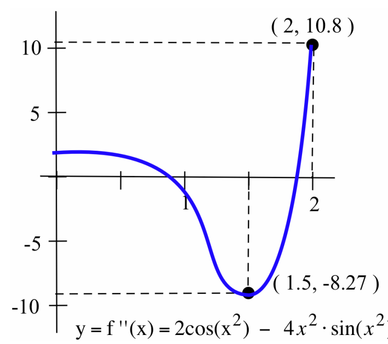
Example 5. You can be certain that the T_{10} approximation of $\int_0^2 \sin(x^2)\,dx$ is within what distance of the exact value of the integral?

Solution. We know that $b - a = 2$, $n = 10$ and $f(x) = \sin\left(x^2\right)$, so $f''(x) = -4x^2 \cdot \sin\left(x^2\right) + 2 \cdot \cos\left(x^2\right)$ is continuous on $[0, 2]$.

Practice your differentiation skills by verifying this.

We now need an "upper bound" for $|f''(x)|$. If $f''(x)$ is differentiable (it is here) then we could use the techniques of Chapter 3 to find its maximum value on $[0, 2]$ but that would require finding a *third* derivative of f, as well as some challenging algebra. Using the triangle inequality and the facts that $-1 \leq \sin(\theta) \leq 1$ and $-1 \leq \cos(\theta) \leq 1$, we can conclude:

$$|f''(x)| = \left|-4x^2 \cdot \sin\left(x^2\right) + 2 \cdot \cos\left(x^2\right)\right| \leq 4 \cdot 2^2 \cdot 1 + 2 \cdot 1 = 18$$



so we could take $B_2 = 18$. We can do a bit better, however, by consulting a graph of $f''(x)$ on $[0, 2]$ (see margin); it appears clear from the graph that $|f''(x)| \leq 11$, so we take $B_2 = 11$ instead.

Using these values for a, b, n and B_2 in the "error bound" formula:

$$|\text{"error"}| = \left|\int_0^2 \sin(x^2)\,dx - T_{10}\right| \leq \frac{2^3 \cdot 11}{12 \cdot 10^2} = \frac{88}{1200} = \frac{11}{150} < 0.074$$

so we can be certain that our T_{10} approximation of the definite integral is within 0.074 of the exact value:

$$T_{10} - 0.074 \leq \int_0^2 \sin(x^2)\,dx \leq T_{10} + 0.074$$

Computing $T_{10} = 0.7959247$, we can be certain that the value of the integral $\int_0^2 \sin(x^2)\,dx$ is somewhere between 0.722 and 0.870. ◀

Notice that (a bound for) the "error" depends on three things: the size of the interval of integration (the bigger the interval, the bigger the potential error); the number of subintervals in the partition (the more subintervals, the smaller the potential error); and the size of the second derivative of the integrand. We've already seen that the second derivative of a function is related to the concavity of its graph — later on we will learn that the second derivative helps measure the "curvature" of the graph of f; it should make sense that the more "curvy" a function is, the less effective a linear approximation technique would be.

Practice 3. Find an error bound for the T_{12} approximation of $\int_2^5 \frac{1}{x}\,dx$.

Example 6. How large must n be to be certain that T_n is within 0.001 of $\int_0^2 \sin(x^2)\,dx$?

Solution. Here we know the "allowable error" of 0.001 and we must find n. From Example 5 we know that $b-a=2$ and $B_2=11$, so we want the error bound to be less than the allowable error of 0.001:

$$\frac{2^3\cdot 11}{12\cdot n^2}<0.001 \quad\Rightarrow\quad \frac{12\cdot n^2}{88}>1000 \quad\Rightarrow\quad n^2>\frac{88000}{12}$$
$$\Rightarrow\quad n>\sqrt{\frac{22000}{3}}\approx 85.6$$

Because n must be an integer, we can take $n=86$. Computing $T_{86}\approx 0.80465$, we can be certain that the exact value of the integral is between 0.80365 and 0.80565. ◀

As often happens, T_{86} is even closer than 0.001 to the exact value of the integral:

$$\left|T_{86}-\int_0^2 \sin\left(x^2\right)\,dx\right|\approx 0.00012$$

Practice 4. Determine how large n must be in order to ensure that T_n is within 0.001 of $\displaystyle\int_2^5 \frac{1}{x}\,dx$.

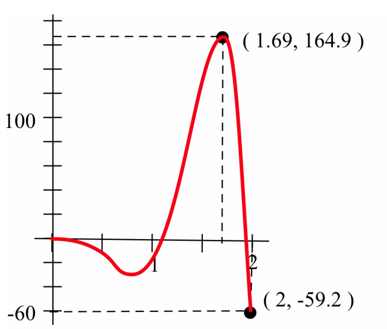
Error Bound for Simpson's Parabolic Approximation

If $f^{(4)}$ is continuous on $[a,b]$ and $\left|f^{(4)}(x)\right|\le B_4$

then the "error" of the S_n approximation of $\int_a^b f(x)\,dx$ satisfies:

$$|\text{"error"}|=\left|\int_a^b f(x)\,dx-S_n\right|\le\frac{(b-a)^5B_4}{180n^4}$$

Example 7. Find an error bound for the S_{10} approximation of $\displaystyle\int_0^2\sin(x^2)\,dx$.



$f^{(4)}(x)=(16x^4-12)\sin(x^2)-48x^2\cos(x^2)$

Solution. We have $b-a=2$, $n=10$ and $f(x)=\sin(x^2)$, so $f^{(4)}(x)=(16x^4-12)\sin(x^2)-48x^2\cos(x^2)$ is continuous on $[0,2]$. From a graph of $f^{(4)}(x)$ on $[0,2]$ (see margin), we can estimate that $B_4=165$, so

$$|\text{"error"}|=\left|\int_0^2\sin(x^2)\,dx-S_{10}\right|\le\frac{2^5\cdot 165}{180\cdot 10^4}=\frac{5280}{1800000}<0.003$$

and we can be certain that our S_{10} approximation of $\int_0^2\sin(x^2)\,dx$ is within 0.003 of the exact value:

$$S_{10}-0.003\le\int_0^2\sin(x^2)\,dx\le S_{10}+0.003$$

Computing $S_{10}=0.80537615$, we are certain that the exact value of $\int_0^2\sin(x^2)\,dx$ is between 0.80237615 and 0.80837615. Notice that we achieved a much narrower guarantee using S_{10} compared to using T_{10} to approximate the same integral. ◀

Example 8. Determine how large n must be to ensure that S_n is within 0.001 of the exact value of $\displaystyle\int_0^2\sin(x^2)\,dx$?

Solution. We want the "error bound" to be less than 0.001 and need to find n. We know that $b - a = 2$ and $B_4 = 165$

$$\frac{2^5 \cdot 165}{180 \cdot n^4} < 0.001 \quad \Rightarrow \quad \frac{180 \cdot n^2}{5280} > 1000 \quad \Rightarrow \quad n^2 > \frac{5280000}{180} = \frac{88000}{3}$$

$$\Rightarrow \quad n > \sqrt[4]{\frac{88000}{3}} \approx 13.09$$

Because n must be an even integer, we can take $n = 14$ and be certain that S_{14} is within 0.001 of $\int_0^2 \sin(x^2)\,dx$. ◀

As we have come to expect, S_{14} is even closer than 0.001 to the exact value of the integral; using advanced methods, we can show that:

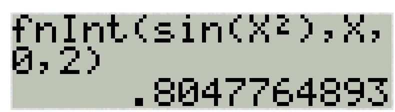
$$\left| \int_0^2 \sin(x^2)\,dx - S_{14} \right| \approx 0.00015$$

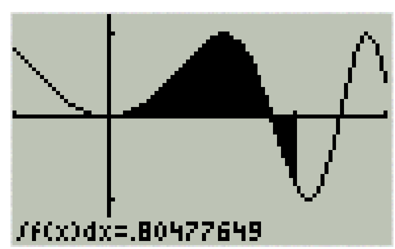
Alternative Methods

In Section 8.7 and in Chapter 10, you will learn how to approximate a function f over an entire interval $[a, b]$ using a single polynomial $p(x)$ of degree n; you can then approximate $\int_a^b f(x)\,dx$ with $\int_a^b p(x)\,dx$, which is relatively easy to compute. One advantage of this method is that (once we have found $p(x)$), we only need to evaluate another polynomial ($P(x)$ where $P'(x) = p(x)$) at two values ($P(a)$ and $P(b)$) to compute $\int_a^b p(x)\,dx \approx \int_a^b f(x)\,dx$ and we can get better approximations by increasing n and using polynomials of higher and higher degree; using the Trapezoidal Rule or Simpson's Rule requires us to evaluate $f(x)$ at $n+1$ points. A disadvantage of this approach is that our original $f(x)$ must have n continuous derivatives, which is not always the case, and we need to be able to compute those n derivatives at a single point. Most textbooks on Numerical Analysis offer more sophisticated techniques for approximating definite integrals.

Using Technology

If you have written even the most basic computer code, you should be able to write a program to compute any Trapezoidal Rule or Simpson's Rule approximation you want (accurate up to the floating-point limitations of the machine running your code). If you have a graphing calculator, it likely has one or more numerical integration utilities (see the margin for TI-83 output). The Web site Wolfram | Alpha (`www.wolframalpha.com`) can approximate definite integrals to any desired accuracy; typing `integral sin(x^2) from x=0 to x=2` yields:





$$\int_0^2 \sin\left(x^2\right) dx = \sqrt{\frac{\pi}{2}}\; S\left(2\sqrt{\frac{2}{\pi}}\right) \approx 0.804776$$

Wolfram | Alpha can also be used to quickly apply Simpson's Rule:

```
use Simpson's rule sin(x^2) from 0 to 2 with 10 intervals
```

yields an approximation of 0.804811 for $\int_0^2 \sin(x^2)\,dx$.

4.9 Problems

1. Use the values in the table below left to approximate $\int_2^6 f(x)\,dx$ by calculating T_4 and S_4.
2. Use the values in the table below left to approximate $\int_2^6 f(x)\,dx$ by calculating T_8 and S_8.

x	$f(x)$
2.0	2.1
2.5	2.7
3.0	3.8
3.5	2.3
4.0	0.3
4.5	-1.8
5.0	-0.9
5.5	0.5
6.0	2.2

x	$g(x)$
-3.0	4.2
-2.5	1.8
-2.0	0.7
-1.5	1.5
-1.0	3.4
-0.5	4.3
0.0	3.5
0.5	-0.3
1.0	-4.6

3. Use the values in the table above right to approximate $\int_{-3}^1 g(x)\,dx$ by calculating T_8 and S_8.
4. Use the values in the table above right to approximate $\int_{-3}^1 g(x)\,dx$ by calculating T_4 and S_4.

For Problems 5–10, calculate (a) T_4, (b) S_4 and (c) the exact value of the integral.

5. $\displaystyle\int_1^3 x\,dx$
6. $\displaystyle\int_0^2 [1-x]\,dx$
7. $\displaystyle\int_{-1}^1 x^2\,dx$
8. $\displaystyle\int_2^6 \frac{1}{x}\,dx$
9. $\displaystyle\int_0^\pi \sin(x)\,dx$
10. $\displaystyle\int_0^1 \sqrt{x}\,dx$

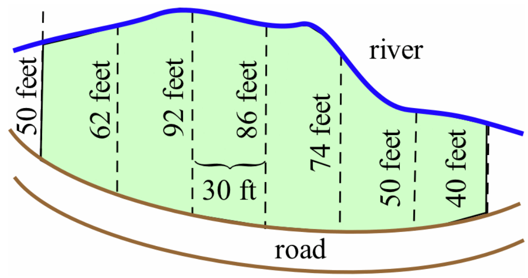
For Problems 11–16, calculate (a) T_6 and (b) S_6.

11. $\displaystyle\int_0^2 \frac{1}{1+x^3}\,dx$
12. $\displaystyle\int_1^2 2^x\,dx$
13. $\displaystyle\int_{-1}^1 \sqrt{4-x^2}\,dx$
14. $\displaystyle\int_0^1 e^{-x^2}\,dx$
15. $\displaystyle\int_1^4 \frac{\sin(x)}{x}\,dx$
16. $\displaystyle\int_0^1 \sqrt{1+\sin(x)}\,dx$

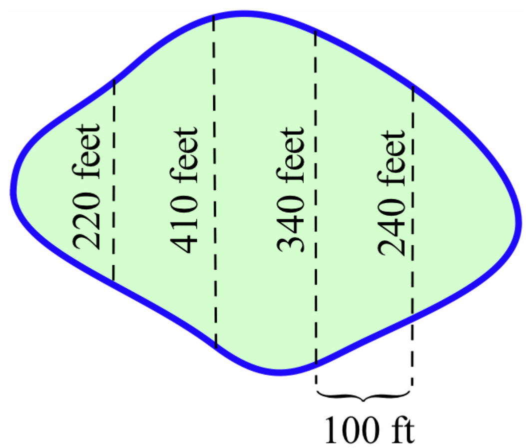
For 17–22, calculate (a) the error bound for T_4, (b) the error bound for S_4, (c) the value of n so that the error bound for T_n is less than 0.001, and (d) the value of n so that the error bound for S_n is less than 0.001.

17. $\displaystyle\int_1^3 x\,dx$
18. $\displaystyle\int_0^2 [1-x]\,dx$
19. $\displaystyle\int_{-1}^1 x^3\,dx$
20. $\displaystyle\int_2^6 \frac{1}{x}\,dx$
21. $\displaystyle\int_0^\pi \sin(x)\,dx$
22. $\displaystyle\int_0^1 \sqrt{x}\,dx$

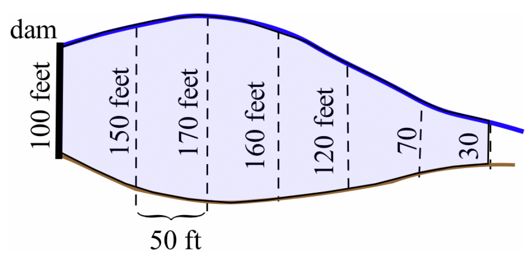
23. Estimate the area of the piece of land located between the river and the road in the figure below.



24. Estimate the area of the island in the figure below.



25. Estimate the volume of water in the reservoir shown below if the average depth is 22 feet.



26. The table below left shows the speedometer readings (in feet per minute) for a car at one-minute intervals. Estimate how far the car traveled (a) during the first 5 minutes of the trip and (b) during the first 10 minutes of the trip.

t	$v(t)$	t	$v(t)$
0	0	6	5200
1	2000	7	4400
2	3000	8	3000
3	5000	9	2000
4	5000	10	1200
5	6000		

t	$v(t)$	t	$v(t)$
0	0	6	520
1	420	7	440
2	540	8	360
3	300	9	260
4	500	10	180
5	580		

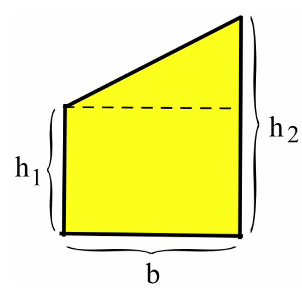
27. The table above right shows the speed (in feet per minute) of a jogger at one-minute intervals. Estimate how far the jogger ran during her workout.

28. Use the error-bound formula for Simpson's Rule to show that the parabolic approximation gives the exact value of $\int_a^b f(x)\,dx$ if $f(x) = Ax^3 + Bx^2 + Cx + D$ is a polynomial of degree 3 or less.

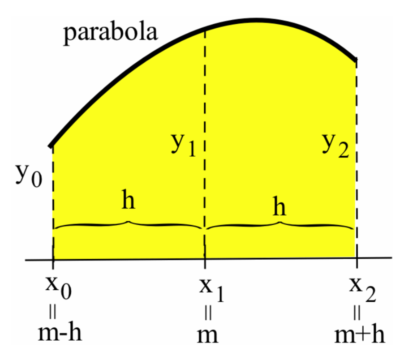
29. A trapezoidal region with base b and heights h_1 and h_2 (assume $h_1 \neq h_2$) can be cut into a rectangle with base b and height h_1 and a triangle with base b and height $h_1 - h_2$ (see figure at right). Show that the sum of the area of the rectangle and the area of the triangle is $b \cdot \left[\frac{h_1 + h_2}{2}\right]$.

30. Let $f(m)$ be the minimum value of f on the interval $[x_0, x_1]$, $f(M)$ be the maximum value of f on $[x_0, x_1]$, and $h = x_1 - x_0$. Show that:
$$h \cdot f(m) \leq b \cdot \left[\frac{f(x_0) + f(x_1}{2}\right] \leq h \cdot f(M)$$
and use this result to show that the trapezoidal approximation is between the lower and upper Riemann sums for f. Because the limit (as $h \to 0$) of these Riemann sums is $\int_a^b f(x)\,dx$, conclude that the limit of the trapezoidal sums must equal $\int_a^b f(x)\,dx$.

31. Let $f(m)$ be the minimum value of f on the interval $[x_0, x_2]$, $f(M)$ the maximum of f on $[x_0, x_2]$ and $h = x_1 - x_0 = x_2 - x_1$. Show that the value
$$2h \cdot \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right]$$
is between $2h \cdot f(m)$ and $2h \cdot f(M)$ and use this result to show that the parabolic approximation of $\int_a^b f(x)\,dx$ is between the lower and upper Riemann sums for f. Conclude that the limit of the parabolic sums must equal $\int_a^b f(x)\,dx$.



32. This problem guides you through the steps to show that the area under a parabolic region (see margin) with evenly spaced x_k values (which, for the purposes of this problem we will call $x_0 = m - h$, $x_1 = m$ and $x_2 = m + h$) is:
$$\frac{h}{3} \cdot [f(x_0) + 4f(x_1) + f(x_2)] = \frac{h}{3} \cdot [y_0 + 4y_1 + y_2]$$



(a) For $f(x) = Ax^2 + Bx + C$, verify that:
$$\int_{m-h}^{m+h} f(x)\,dx = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{m-h}^{m+h} = 2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch$$

(b) Expand each of the polynomials:

$$\begin{aligned} y_0 = f(m-h) &= A(m-h)^2 + B(m-h) + C \\ y_1 = f(m) &= Am^2 + Bm + C \\ y_2 = f(m+h) &= A(m+h)^2 + B(m+h) + C \end{aligned}$$

and use the results to verify that:

$$\begin{aligned} \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] &= 2h\left[\frac{f(m-h) + 4f(m) + f(m+h)}{6}\right] \\ &= 2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch \end{aligned}$$

(c) Compare the results of parts (a) and (b) to conclude that for any quadratic function $f(x) = Ax^2 + Bx + C$:

$$\int_{m-h}^{m+h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Left-Endpoint, Right-Endpoint and Midpoint Rules

The rectangular approximation methods approximate an integrand with horizontal lines, so that the approximating regions are rectangles and the sum of the areas of these rectangular regions is a Riemann sum. The Left- and Right-Endpoint Rules are easy to understand and use, but they typically require a very large number of subintervals to ensure good approximations of a definite integral. The Midpoint Rule uses the value of the integrand at the midpoint of each subinterval: if these midpoint values of f are available (for example, when f is given by a formula) then the Midpoint Rule is often more efficient than the Trapezoidal rule. The rectangular approximation rules are:

$$\begin{aligned} L_n &= h \cdot \left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right] \\ R_n &= h \cdot \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right] \\ M_n &= h \cdot \left[f(c) + f(c+h) + f(c+2h) + \cdots + f(c+(n-1)h)\right] \end{aligned}$$

where $c = x_0 + \frac{h}{2}$ so that the points c, $c+h$, $c+2h$, etc. are the midpoints of the subintervals. The "error bounds" for these methods are:

$$\begin{aligned} |\text{"error" for } L_n \text{ or } R_n| &\le \frac{(b-a)^2 B_1}{2n} \\ |\text{"error" for } M_n| &\le \frac{(b-a)^3 B_2}{24n^2} \end{aligned}$$

where $B_1 \ge |f'(x)|$ on $[a,b]$ and $B_2 \ge |f''(x)|$ on $[a,b]$. Notice that the error bound for M_n is half the error bound of T_n, the trapezoidal approximation.

For Problems 33–38, calculate (a) L_4, (b) R_4, (c) M_4 and (d) the exact value of the integral.

33. $\displaystyle\int_1^3 x\,dx$

34. $\displaystyle\int_0^2 [1-x]\;dx$

35. $\displaystyle\int_{-1}^1 x^2\,dx$

36. $\displaystyle\int_2^6 \frac{1}{x}\,dx$

37. $\displaystyle\int_0^\pi \sin(x)\,dx$

38. $\displaystyle\int_0^1 \sqrt{x}\,dx$

39. Show that the Trapezoidal approximation is the average of the Left- and Right-Endpoint approximations: $T_n = \frac{1}{2}\left(L_n + R_n\right)$.

The integrals in Problems 40–43 will arise in applications from Chapter 5. Use technology to approximate each integral by applying Simpson's Rule with $n = 10$ and $n = 40$ to approximate their values. (Is S_{40} very different from S_{10}?)

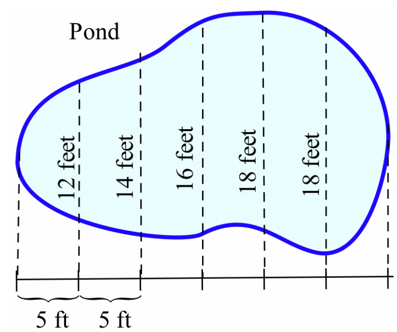
40. $\displaystyle\frac{1}{\sqrt{2\pi}}\int_{-2}^2 e^{-\frac{1}{2}x^2}\,dx$

41. $\displaystyle\int_{-1}^2 \sqrt{1+4x^2}\,dx$

42. $\displaystyle\int_0^\pi \sqrt{1+\cos^2(x)}\,dx$

43. $\displaystyle\int_0^{2\pi} \sqrt{16\sin^2(t)+9\cos^2(t)}\,dt$

4.9 Practice Answers



1. Using the Trapezoidal Rule to approximate the pond's surface area:

$$T \approx \frac{5\text{ ft}}{2}\cdot\left[(0+2\cdot 12+2\cdot 14+2\cdot 16+2\cdot 18+2\cdot 18+0)\;\text{ft}\right] = 390\text{ ft}^2$$

so the volume is $(\text{surface area})(\text{depth}) \approx \left(390\text{ ft}^2\right)(0.1\text{ ft}) = 39\text{ ft}^3$.

2. Using Simpson's Rule to approximate the pond's surface area:

$$S \approx \frac{5\text{ ft}}{3}\cdot\left[(0+4\cdot 12+2\cdot 14+4\cdot 16+2\cdot 18+4\cdot 18+0)\;\text{ft}\right] \approx 413\text{ ft}^2$$

3. $b-a=3$, $n=12$ and $f(x)=\frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} \Rightarrow f''(x)=\frac{2}{x^3}$, so on the interval $[2,5]$:

$$\left|f''(x)\right| = \left|\frac{2}{x^3}\right| \le \frac{2}{2^3} = \frac{1}{4}$$

We can therefore take $B_2 = \frac{1}{4}$, so:

$$|\text{error}| \le \frac{(b-a)^3\cdot B_2}{12n^2} \le \frac{3^3\cdot\frac{1}{4}}{12(12)^2} = \frac{27}{6912} \approx 0.004$$

4. We want:

$$|\text{error}| \le \frac{(b-a)^3\cdot B_2}{12n^2} \le \frac{3^3\cdot\frac{1}{4}}{12\cdot n^2} = \frac{27}{48n^2} < 0.001$$

so solving for n:

$$\frac{48n^2}{27} > 1000 \Rightarrow n^2 > \frac{27000}{48} = \frac{1125}{2} \Rightarrow n > \sqrt{562.5} \approx 23.7$$

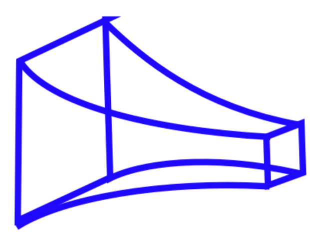
Using $n = 24$ will work. We can be certain that T_{24} is within 0.001 of the exact value of the integral. (We cannot guarantee that T_{23} is within 0.001 of the exact value of the integral, but it probably is.)

5
Applications of Definite Integrals

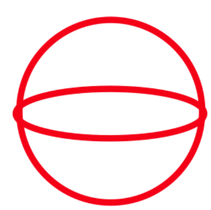
The previous chapter introduced the concepts of a definite integral as an "area" and as a limit of Riemann sums, demonstrated some of the properties of integrals, introduced some methods to compute values of definite integrals, and began to examine a few of their uses. This chapter focuses on several common applications of definite integrals.

An obvious goal of the chapter is to enable you to use integration when you encounter these particular applications later in mathematics or in other fields. A deeper goal is to illustrate the process of going from a problem to an integral, a process much broader than these particular applications. If you understand the process, then you can understand the use of integrals in many other fields and can even develop the integrals needed to solve problems in new areas. Another goal is to give you additional practice evaluating definite integrals.

Each section in this chapter follows the same basic format. First we describe a problem and present some background information. Then we approximate the solution to the basic problem using a Riemann sum. An exact answer comes from taking a limit of the Riemann sum, and we get a definite integral. After looking at several examples of the same basic application, we will examine some variations.



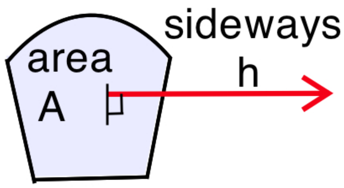
5.1 *Volumes by Slicing*



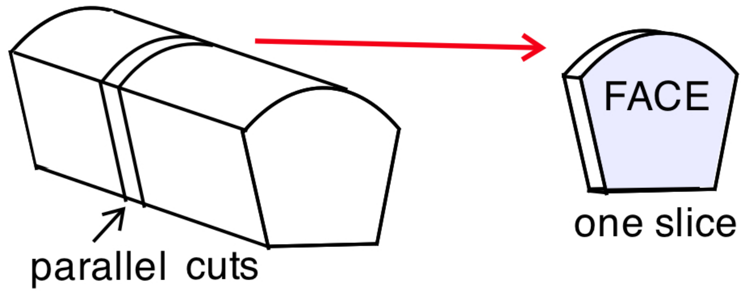
The previous chapter emphasized a geometric interpretation of definite integrals as "areas" in two dimensions. This section emphasizes another geometrical use of integration, computing volumes of solid three-dimensional objects such as those shown in the margin.

Our basic approach will involve cutting the whole solid into thin "slices" whose volumes we can approximate, adding the volumes of these "slices" together (to get a Riemann sum), and finally obtaining an exact answer by taking a limit of those sums to get a definite integral.

The Building Blocks: Right Solids



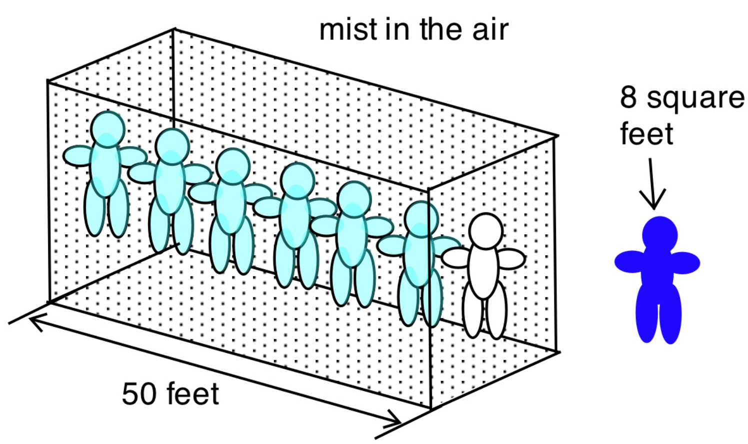
A **right solid** is a three-dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A (see margin). We call the region A a **face** of the solid and use the word "right" to indicate that the movement occurs along a line perpendicular—at a right angle—to the plane of A. Two parallel cuts produce one slice with two faces:



A slice has volume, and a face has area.

Example 1. Suppose a fine, uniform mist is suspended in the air and that every cubic foot of mist contains 0.02 ounces of water droplets. If you run 50 feet in a straight line through this mist, how wet do you get? Assume that the front (or a cross section) of your body has an area of 8 square feet.

Solution. As you run, the front of your body sweeps out a "tunnel" through the mist:

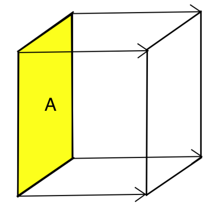


The volume of the "tunnel" is the area of the front of your body multiplied by the length of the tunnel:

$$\text{volume} = \left(8 \text{ ft}^2\right)(50 \text{ ft}) = 400 \text{ ft}^3$$

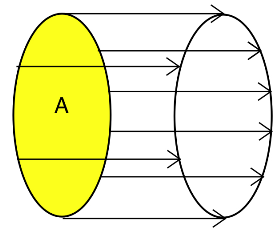
Because each cubic foot of mist held 0.02 ounces of water (which is now on you), you swept out a total of $\left(400 \text{ ft}^3\right)\left(0.02 \ \frac{\text{oz}}{\text{ft}^3}\right) = 8$ ounces of water. If the water were truly suspended and not falling, would it matter how fast you ran? ◀

If A is a rectangle, then the "right solid" formed by moving A along a line (see margin) is a 3-dimensional solid box B. The volume of B is:



$$(\text{area of } A)\,(\text{distance along the line}) = (\text{base})\,(\text{height})\,(\text{width})$$

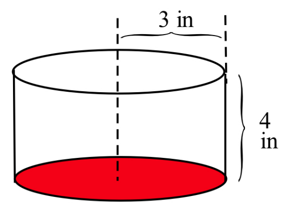
If A is a circle with radius r meters (see margin), then the "right solid" formed by moving A along a line a distance of h meters is a right circular cylinder with volume equal to:

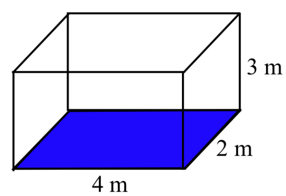


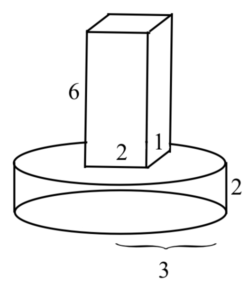
$$(\text{area of } A)\,(\text{distance along the line}) = \left[\pi\,(r\text{ ft})^2\right]\cdot[h\text{ ft}] = \pi r^2 h\text{ ft}^3$$

If we cut a right solid perpendicular to its axis (like slicing a block of cheese), then each face (cross-section) has the same two-dimensional shape and area. In general, if a 3-dimensional right solid B is formed by moving a 2-dimensional shape A along a line perpendicular to A, then the **volume** of B is defined to be:

$$(\text{area of } A)\cdot(\text{distance moved along the line perpendicular to } A)$$







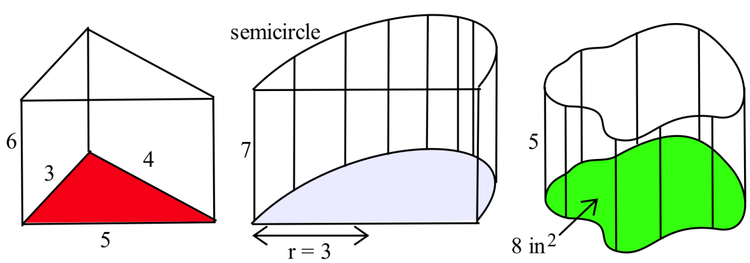
Example 2. Calculate the volumes of the right solids in the margin.

Solution. The cylinder is formed by moving the circular base with cross-sectional area $\pi r^2 = 9\pi$ in^2 a distance of 4 inches along a line perpendicular to the base, so the volume is $\left(9\pi\text{ in}^2\right)\cdot(4\text{ in}) = 36\pi\text{ in}^3$.

The volume of the box is $(\text{base area})\cdot(\text{distance base is moved}) = (8\text{ m}^2)\cdot(3\text{ m}) = 24\text{ m}^3$. We can also simply multiply "length times width times height" to get the same answer.

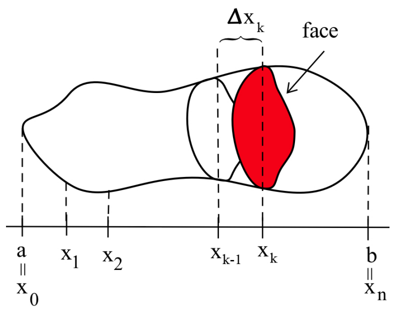
The last shape consists of two "easy" right solids with volumes $V_1 = \left(\pi\cdot 3^2\right)\cdot(2) = 18\pi\text{ cm}^3$ and $V_2 = (6)(1)(2) = 12\text{ cm}^3$, so the total volume is $(18\pi + 12)\text{ cm}^3 \approx 68.5\text{ cm}^3$. ◀

Practice 1. Calculate the volumes of the right solids shown below.



Volumes of General Solids

We can cut a general solid into "slices," each of which is "almost" a right solid if the cuts are close together. The volume of each slice will



then be approximately equal to the volume of a right solid, so we can approximate the total volume of the entire solid by adding up the approximate volumes of the right-solid "slices."

First we position an x-axis below the solid shape (see margin) and let $A(t)$ be the area of the face formed when we cut the solid perpendicular to the x-axis where $x = t$. If $\mathcal{P} = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ is a partition of $[a, b]$ and we cut the solid at each x_k, then each slice of the solid is "almost" a right solid and the volume of each slice is approximately

$$\text{(area of a face of the slice) (thickness of the slice)} \approx A(x_k) \cdot \Delta x_k$$

The total volume V of the solid is approximately equal to the sum of the volumes of the slices:

$$V = \sum \text{(volume of each slice)} \approx \sum A(x_k) \cdot \Delta x_k$$

which is a Riemann sum.

The limit, as the mesh of the partitions approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of $A(x)$:

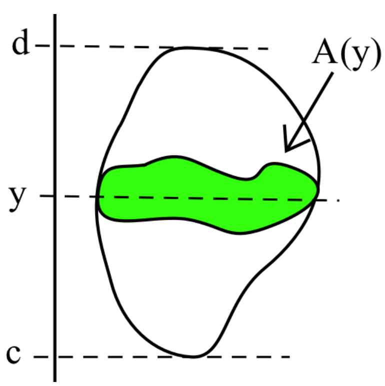
$$V \approx \sum A(x_k) \cdot \Delta x_k \longrightarrow \int_a^b A(x)\,dx$$

Volume by Slices Formula

If S is a solid and $A(x)$ is the area of the face formed by a cut at x made perpendicular to the x-axis

then the volume V of the part of S sitting above $[a, b]$ is:

$$V = \int_a^b A(x)\,dx$$

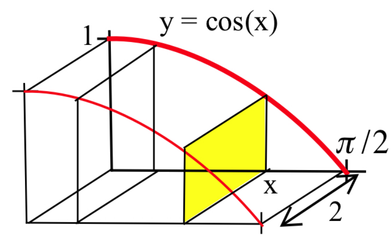


If S is a solid (see margin), and $A(y)$ is the area of a face formed by a cut at y perpendicular to the y-axis, then the volume of a slice with thickness Δy_k is approximately $A(y_k) \cdot \Delta y_k$. The volume of the part of S between cuts at $y = c$ and $y = d$ on the y-axis is therefore:

$$V = \int_c^d A(y)\,dy$$

Whether you slice a region with cuts perpendicular to the x-axis or cuts perpendicular to the y-axis depends on which slicing method results in slices with cross-sectional areas that are easiest to compute. Furthermore, slicing one way may result in a definite integral that is difficult to compute, while slicing the other way may result in a much easier definite integral (although you often can't tell which method will result in an easier integration process until you actually set up the integrals).

Example 3. For the solid shown in the margin, the cross-section formed by a cut at x is a rectangle with a base of 2 inches. (a) Find a formula for the approximate volume of the slice between x_{k-1} and x_k. (b) Compute the volume of the solid for x between 0 and $\frac{\pi}{2}$.



Solution. (a) The volume of a "slice" is approximately:

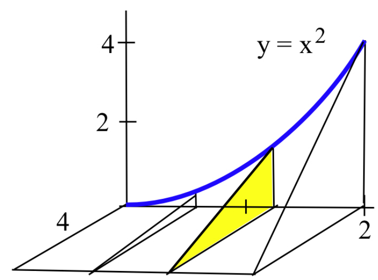
$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2 \text{ in}) \left(\cos(x_k) \text{ in}\right) \cdot (\Delta x_k \text{ in}) \\ &= 2\cos(x_k)\Delta x_k \text{ in}^3\end{aligned}$$

(b) If we cut the solid into n slices of equal thickness Δx and add up the approximate volumes of the slices, we get a Riemann sum

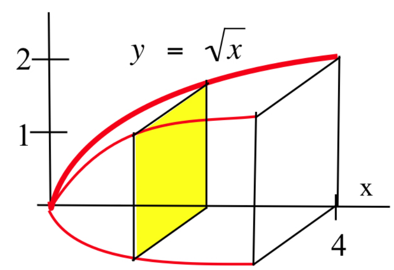
$$\sum_{k=1}^{n} 2\cos(x_k)\Delta x \longrightarrow \int_0^{\frac{\pi}{2}} 2\cos(x)\,dx = 2\sin(x)\Big|_0^{\frac{\pi}{2}} = 2$$

so the volume of the solid is 2 in^3. ◀

Practice 2. For the solid shown in the margin, the face formed by a cut at x is a triangle with a base of 4 inches. (a) Find a formula for the approximate volume of the slice between x_{k-1} and x_k. (b) Use a definite integral to compute the volume of the solid for x between 1 and 2.



Example 4. For the solid shown in margin, each face formed by a cut at x is a square. Compute the volume of the solid.



Solution. The volume of a "slice" is approximately:

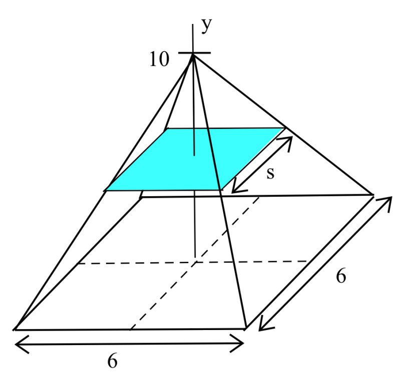
$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base})^2 \cdot (\text{thickness}) \\ &= (\sqrt{x_k})^2 \cdot \Delta x_k = x_k \cdot \Delta x_k\end{aligned}$$

Adding up the approximate volumes of n slices, we get a Riemann sum that approximates the volume of the entire solid:

$$\sum_{k=1}^{n} x_k \cdot \Delta x_k \longrightarrow \int_0^4 x\,dx = \frac{1}{2}x^2\Big|_0^4 = 8$$

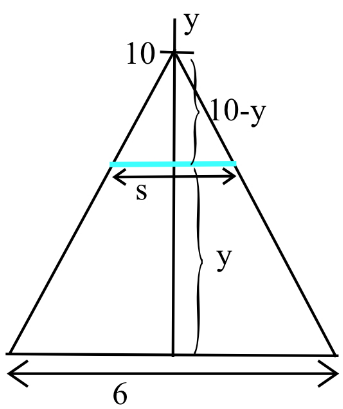
so the volume of the solid is 8. ◀

Example 5. Find the volume of the square-based pyramid shown in the margin.



Solution. Each cut perpendicular to the y-axis yields a square face, but in order to find the area of each square we need a formula for the length of one side s of the square as a function of y, the location of the cut. Using similar triangles (see margin), we know that:

$$\frac{s}{10-y} = \frac{6}{10} \quad \Rightarrow \quad s = \frac{6}{10}(10-y) = 6 - \frac{3}{5}y$$



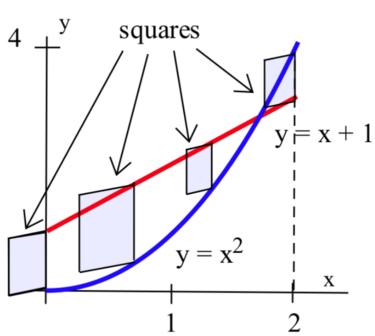
The rest of the solution is straightforward:

$$A(y) = (\text{side})^2 = \left[\frac{3}{5}(10-y)\right]^2 = \frac{9}{25}\left(100 - 20y + y^2\right)$$

so the volume of the solid is:

$$\begin{aligned} V &= \int_0^{10} A(y)\,dy = \int_0^{10} \frac{9}{25}\left(100-20y+y^2\right)\,dy \\ &= \frac{9}{25}\left[100y - 10y^2 + \frac{1}{3}y^3\right]_0^{10} \\ &= \frac{9}{25}\left[\left(1000 - 1000 + \frac{1000}{3}\right) - (0-0+0)\right] = 120 \end{aligned}$$

You may recall from geometry that the formula for the volume of a pyramid is $\frac{1}{3}Bh$ where B is the area of the base, which yields the same result as the definite integral: $\frac{1}{3}\left(6^2\right)(10) = 120$. ◀



Example 6. Form a solid with a base that is the region between the graphs of $f(x) = x+1$ and $g(x) = x^2$ for $0 \le x \le 2$ by building squares with heights (sides) equal to the vertical distance between the graphs of f and g (see margin). Find the volume of this solid.

Solution. The area of a square face is $A(x) = (\text{side})^2$ and the length of a side is either $f(x) - g(x)$ or $g(x) - f(x)$, depending on whether $f(x) \ge g(x)$ or $g(x) \ge f(x)$. We can express this side length as $|f(x) - g(x)|$ but the side length is squared in the area formula, so $A(x) = |f(x) - g(x)|^2 = (f(x) - g(x))^2$. Then:

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^2 (f(x)-g(x))^2\,dx = \int_0^2 \left[(x+1) - x^2\right]^2\,dx \\ &= \int_0^2 \left[1 + 2x - x^2 - 2x^3 + x^4\right]\,dx \\ &= \left[x + x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^2 \end{aligned}$$

which results in a volume of $\frac{26}{15}$. ◀

Wrap-Up

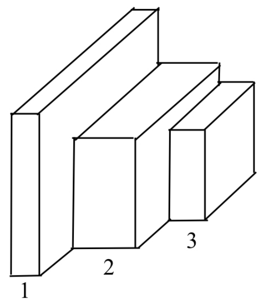
At first, all of these volumes may seem overwhelming — there are so many possible solids and formulas and different cases. If you concentrate on the differences, things can indeed seem very complicated.

Instead, focus on the pattern of cutting, finding areas of faces, volumes of slices, and adding those volumes. Then reason your way to a definite integral. Try to make cuts so the resulting faces have regular shapes (rectangles, triangles, circles) whose areas you can calculate easily. Try not to let the complexity of the whole solid confuse you. Sketch the shape of one face and label its dimensions. If you can find the area of one face in the middle of the solid, you can usually find the pattern for all of the faces—and then you can easily set up the integral.

5.1 Problems

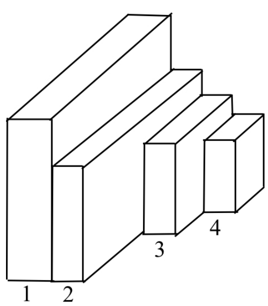
In Problems 1–5, compute the volume of the solid using the values provided in the table.

1.



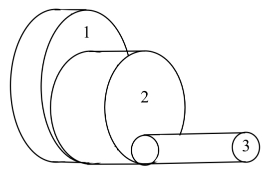
box	base	height	width
1	8	6	1
2	6	4	2
3	3	3	1

2.



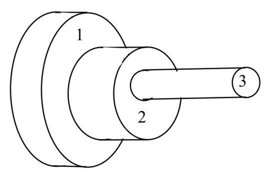
box	base	height	width
1	8	6	1
2	8	4	2
3	4	3	2
4	2	2	1

3.



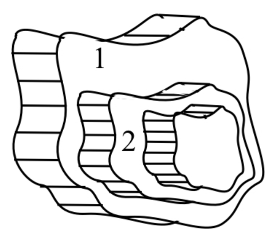
disk	radius	width
1	4	0.5
2	3	1.0
3	1	2.0

4.



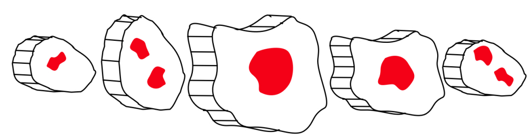
disk	diameter	width
1	8	0.5
2	6	1.0
3	2	2.0

5.



slice	face area	width
1	9	0.2
2	6	0.2
3	2	0.2

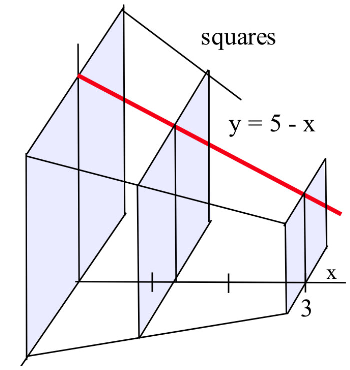
6. Five rock slices are embedded with mineral deposits. Use the information in the table to estimate the total rock volume.



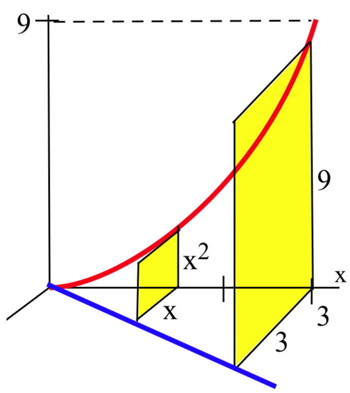
slice	face area	min. area	width
1	4	1	0.6
2	12	2	0.6
3	20	4	0.6
4	10	3	0.6
5	8	2	0.6

In Problems 7–12, represent the volume of each solid as a definite integral, then evaluate the integral.

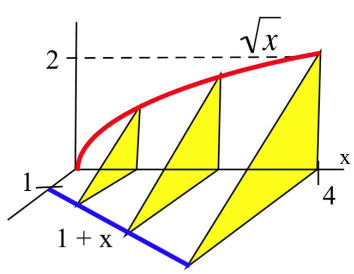
7. For $0 \le x \le 3$, each face is a square with height $5 - x$ inches.



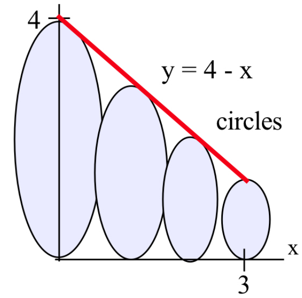
8. For $0 \le x \le 3$, each face is a rectangle with base x inches and height x^2 inches.



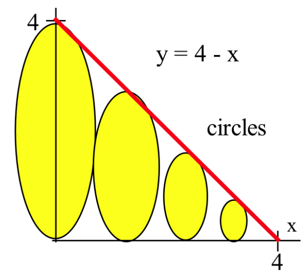
9. For $0 \le x \le 4$, each face is a triangle with base $x + 1$ m and height $\sqrt{x}$ m.



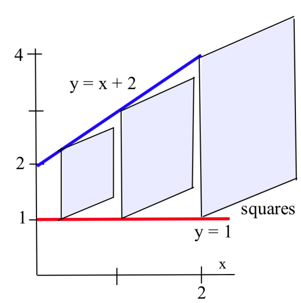
10. For $0 \le x \le 3$, each face is a circle with height (diameter) $4 - x$ m.



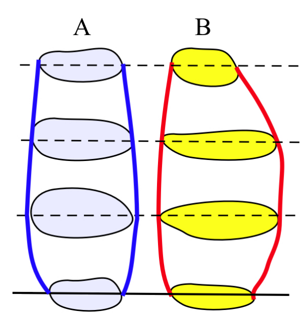
11. For $0 \le x \le 4$, each face is a circle with height (diameter) $4 - x$ m.



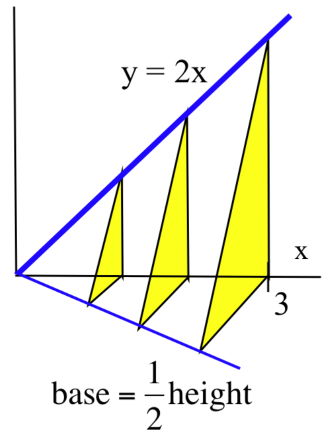
12. For $0 \le x \le 2$, each face is a square with a side extending from $y = 1$ to $y = x + 2$.

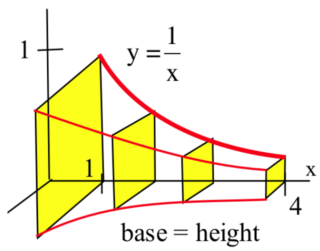


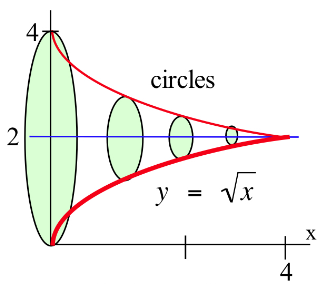
13. Suppose A and B are solids (see below) so that every horizontal cut produces faces of A and B that have equal areas. What can we conclude about the volumes of A and B? Justify your answer.

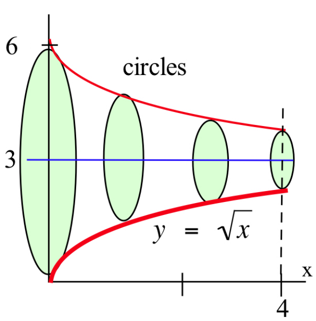


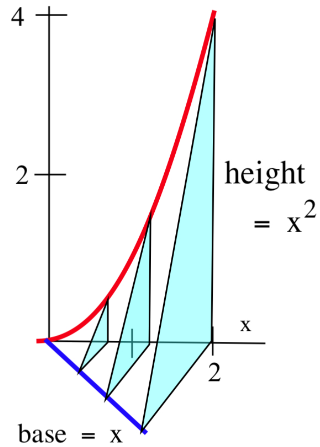
In 14–18, represent the volume of each solid as a definite integral, then evaluate the integral.

14. 

15. 

16. 

17. 

18. 

In 19–28, represent the volume of each solid as a definite integral, then evaluate the integral.

19. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.

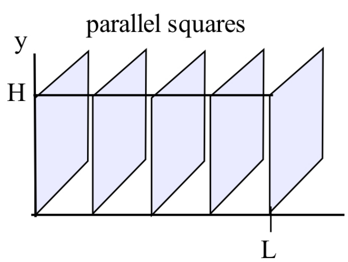
20. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.

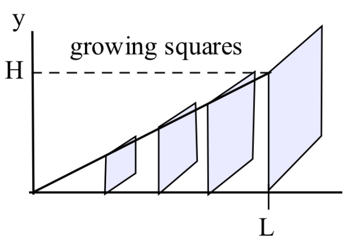
21. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are semicircles.

22. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and slices perpendicular to the base (and to the x-axis) are equilateral triangles.

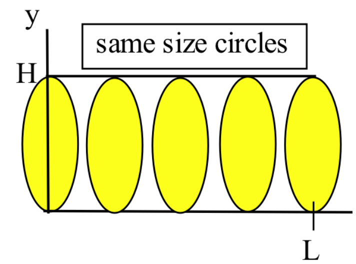
23. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 3 + x - x^2$, and slices perpendicular to the base (and to the x-axis) are:
    (a) squares.
    (b) semicircles.
    (c) rectangles twice as tall as they are wide.
    (d) isosceles right triangles with a hypotenuse in the base of the solid.

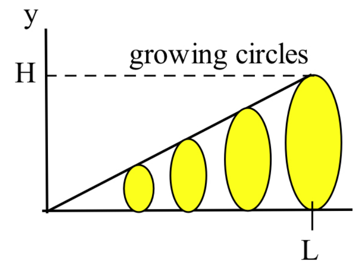
24. The base of a solid is the first-quadrant region bounded by the y-axis, the curve $y = \sin(x)$ and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are:
    (a) squares.
    (b) semicircles.
    (c) rectangles twice as tall as they are wide.
    (d) isosceles right triangles with a hypotenuse in the base of the solid.
25. The base of a solid is the region bounded by the x-axis, the y-axis and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
26. The base of a solid is the region bounded by the x-axis, the line $y = 3$ and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
27. The base of a solid is the region bounded below by the line $y = 1$, on the left by the line $x = 2$ and above by the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are semicircles.
28. The base of a solid is the region bounded below by $y = 1$, on the left by $x = 2$ and above by $y = 8 - x^2$, and slices perpendicular to the base (and to the x-axis) are semicircles.
29. Calculate (a) the volume of the right solid in the top figure (b) the volume of the "right cone" in the bottom figure and (c) the ratio of the "right cone" volume to the right solid volume.



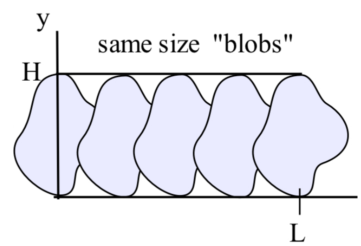


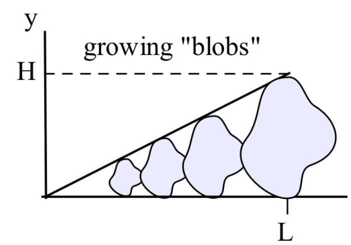
30. Calculate (a) the volume of the right solid in the top figure (b) the volume of the "right cone" in the bottom figure and (c) the ratio of the "right cone" volume to the right solid volume.





31. Calculate (a) the volume of the right solid in the top figure if each "blob" has area B (b) the volume of the "right cone" in the bottom figure, using "similar blobs" to conclude that the cross-section x units from the y-axis has area $A(x) = \frac{B}{L^2}x^2$ and (c) the ratio of the "right cone" volume to the right solid volume.





32. "Personal calculus": Describe a practical way to determine the volume of your hand and arm up to the elbow.

5.1 Practice Answers

1. triangular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2} \cdot 3 \cdot 4\right)(6) = 36$

   semicircular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2}\pi \cdot 3^2\right)(7) \approx 98.96$

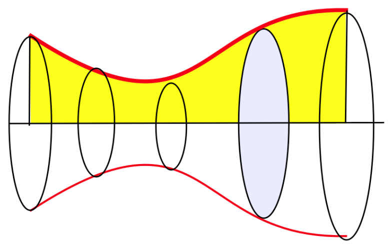
   "blob"-shaped base: $V = (\text{base area}) \cdot (\text{height}) = (8)\,(5) = 40 \text{ in}^3$

2. (a) The base of each triangular slice is 4 and the height is approximately ${x_k}^2$ so $A\,(x_k) \approx \frac{1}{2}(4)\left({x_k}^2\right) = 2{x_k}^2$ and the volume of the k-th slice is this approximately $2{x_k}^2 \cdot \Delta x_k$.

   (b) Adding up the approximate volumes of all n slices yields $\sum_{n=1}^{\infty} 2{x_k}^2 \cdot \Delta x_k$, which is a Riemann sum with limit:

$$\int_0^2 2x^2\,dx = \frac{2}{3}x^3\Big|_1^2 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

5.2 Volumes: Disks and Washers

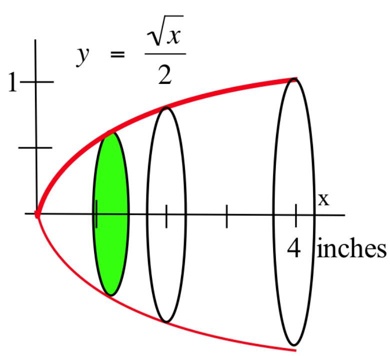


In the previous section, we computed volumes of solids for which we could determine the area of a cross-section or "slice." In this section, we restrict our attention to a special case in which the solid is generated by *rotating* a region in the xy-plane about a horizontal or vertical line. We call a solid formed in this way a **solid of revolution** and we call the line an **axis of rotation**.

If the axis of rotation coincides with a boundary of the region (as in the margin figure) then the cross-sections of the region perpendicular to the axis of rotation will be disks, making it relatively easy to find a formula for the area of a cross-section:

$$A(x) = \text{area of a disk} = \pi(\text{radius})^2$$

The radius is often a function of x, the location of the cross-section.



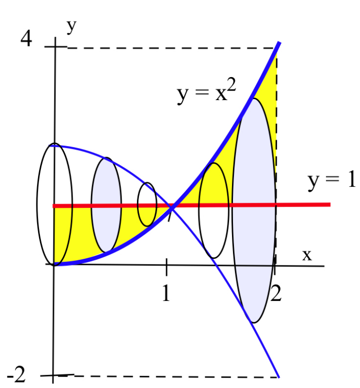
Example 1. Find the volume of the solid (shown in the margin) formed by rotating the region in the first quadrant bounded by the curve $y = \dfrac{\sqrt{x}}{2}$ and the line $x = 4$ about the x-axis.

Solution. Any slice perpendicular to the x-axis (and to the xy-plane) will yield a circular cross-section with radius equal to the distance between the curve $y = \dfrac{\sqrt{x}}{2}$ and the x-axis, so the volume of the region is given by:

$$V = \int_0^4 \pi \left[\frac{\sqrt{x}}{2}\right]^2 dx = \int_0^4 \pi \cdot \frac{x}{4}\, dx = \left.\frac{\pi}{8}x^2\right|_0^4 = 2\pi$$

or about 6.28 cubic inches. ◀

Sometimes the boundary curve intersects the axis of rotation.



Example 2. The region between the graph of $f(x) = x^2$ and the horizontal line $y = 1$ for $0 \le x \le 2$ is revolved about the horizontal line $y = 1$ to form a solid (see margin). Compute the volume of the solid.

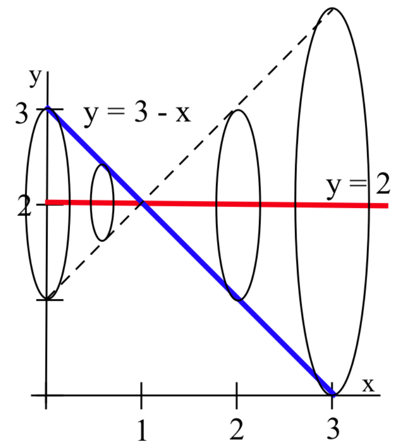
Solution. The margin figure shows cross-sections for several values of x, all of them disks. If $0 \le x \le 1$, then the radius of the disk is $r(x) = 1 - x^2$; if $1 \le x \le 2$, then $r(x) = x^2 - 1$. We could split up the volume computation into two separate integrals, using $A(x) = \pi\left[r(x)\right]^2 = \pi\left[1 - x^2\right]^2$ for $0 \le x \le 1$ and $A(x) = \pi\left[r(x)\right]^2 = \pi\left[x^2 - 1\right]^2$ for $1 \le x \le 2$, but:

$$\pi\left[x^2 - 1\right]^2 = \pi\left[-(1 - x^2)\right]^2 = \pi\left[1 - x^2\right]^2$$

for all x so we can instead compute the volume with a single integral:

$$V = \int_0^2 \pi \left[x^2 - 1\right]^2 \, dx = \pi \int_0^2 \left[x^4 - 2x^2 + 1\right] \, dx$$
$$= \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_0^2 = \pi \left[\frac{32}{5} - \frac{16}{3} + 2\right] = \frac{46\pi}{15}$$

or about 9.63. ◀



Practice 1. Find the volume of the solid formed by revolving the region between $f(x) = 3 - x$ and the horizontal line $y = 2$ about the line $y = 2$ for $0 \le x \le 3$ (see margin).

Volumes of Revolved Regions ("Disk Method")

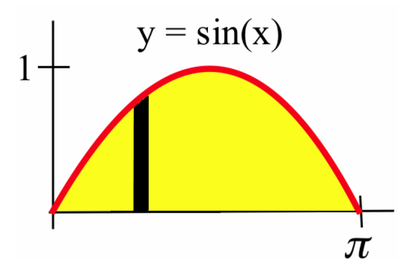
If	the region constrained by the graph of $y = f(x)$, the horizontal line $y = L$ and the interval $[a, b]$ is revolved about the horizontal line $y = L$
then	then the volume of the resulting solid is:

$$V = \int_a^b A(x) \, dx = \int_a^b \pi \, (\text{radius})^2 \, dx = \int_a^b \pi \left[f(x) - L\right]^2 \, dx$$

We often refer to this technique as the "disk" method because revolving a thin rectangular slice of the region (that we might use in a Riemann sum to approximate the area of the region) results in a disk. If the region between the graph of f and the x-axis ($L = 0$) is revolved about the x-axis, then the previous formula reduces to:

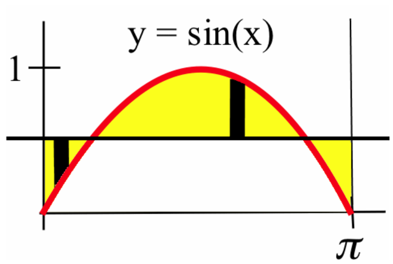
$$V = \int_a^b \pi \left[f(x)\right]^2 \, dx$$

Example 3. Find the volume generated when the region between one arch of the sine curve (for $0 \le x \le \pi$) and (a) the x-axis is revolved about the x-axis and (b) the line $y = \frac{1}{2}$ is revolved about the line $y = \frac{1}{2}$.



Solution. (a) The radius of each circular slice (see margin) is just the height of the function $y = \sin(x)$:

$$V = \int_0^\pi \pi \left[\sin(x)\right]^2 \, dx = \pi \int_0^\pi \sin^2(x) \, dx = \pi \int_0^\pi \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right] \, dx$$
$$= \pi \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_0^\pi = \pi \left[\frac{\pi}{2} - 0\right] - \pi \left[0 - 0\right] = \frac{\pi^2}{2} \approx 4.93$$

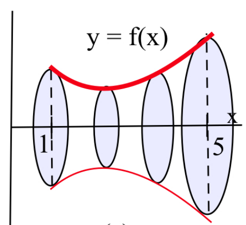


(b) Here we use the more general disk formula with $L = \frac{1}{2}$:

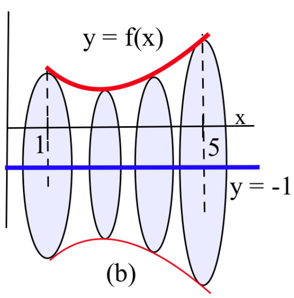
$$V = \int_0^{\pi} \pi \left[\sin(x) - \frac{1}{2}\right]^2 dx = \pi \int_0^{\pi} \left[\sin^2(x) - \sin(x) + \frac{1}{4}\right] dx$$
$$= \pi \int_0^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos(2x) - \sin(x) + \frac{1}{4}\right] dx$$
$$= \pi \left[\frac{3}{4}x - \frac{1}{4}\sin(2x) + \cos(x)\right]_0^{\pi}$$
$$= \pi \left[\frac{3\pi}{4} - 0 - 1\right] - \pi\left[0 - 0 + 1\right] = \frac{3\pi^2}{4} - 2\pi$$

or approximately 1.12. ◀

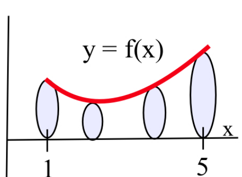
Practice 2. Find the volume generated when (a) the region between the parabola $y = x^2$ (for $0 \le x \le 2$) and the x-axis is revolved about the x-axis and (b) the region between the parabola $y = x^2$ (for $0 \le x \le 2$) and the line $y = 2$ is revolved about the line $y = 2$.



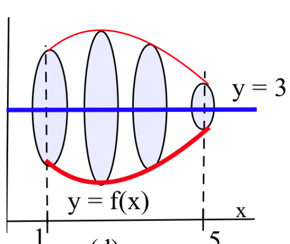
(a)



(b)



(c)



(d)

Example 4. Given that $\int_1^5 f(x)\,dx = 4$ and $\int_1^5 [f(x)]^2\,dx = 7$, represent the volume of each solid shown in the margin as a definite integral, and evaluate those integrals.

Solution. (a) Here the axis of rotation is $y = 0$ so:

$$V = \int_1^5 \pi\,(\text{radius})^2\,dx = \int_1^5 \pi\,[f(x)]^2\,dx = \pi \int_1^5 [f(x)]^2\,dx = 7\pi$$

(b) Here the axis of rotation is $y = -1$ so:

$$V = \int_1^5 \pi\,(\text{radius})^2\,dx = \int_1^5 \pi\,[f(x) - (-1)]^2\,dx$$
$$= \pi \int_1^5 [f(x) + 1]^2\,dx = \pi \int_1^5 \left[(f(x))^2 + 2f(x) + 1\right] dx$$
$$= \pi \left[\int_1^5 (f(x))^2\,dx + 2\int_1^5 f(x)\,dx + \int_1^5 1\,dx\right]$$
$$= \pi\left[7 + 2 \cdot 4 + (5 - 1)\right] = 19\pi$$

(c) This is not a solid of revolution, even though the cross-sections are disks. Each disk has diameter equal to the function height, so the radius of each disk is half that height, and the volume is:

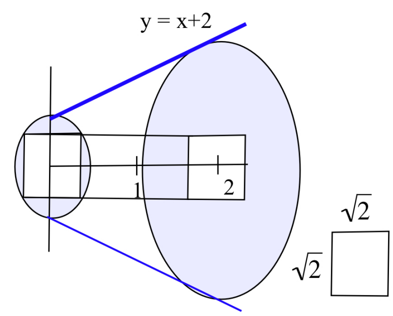
$$V = \int_1^5 \pi \left[\frac{f(x)}{2}\right]^2 dx = \frac{\pi}{4}\int_1^5 [f(x)]^2\,dx = \frac{\pi}{4} \cdot 7 = \frac{7\pi}{4}$$

The last one is left for you. ◀

Practice 3. Set up and evaluate an integral to compute the volume of the last solid shown in the margin.

Solids with Holes

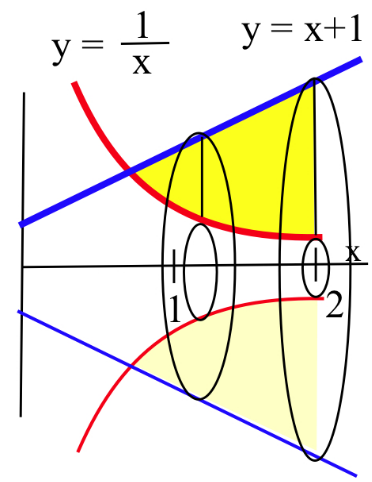
Some solids have "holes": for example, we might drill a cylindrical hole through a spherical solid (such as a ball bearing) to create a part for an engine. One approach involves using an integral (or using geometry) to compute the volume of the "outer" solid, then use another integral (or geometry) to compute the volume of the "hole" cut out of the original solid, and finally subtracting the second result from the first. You should be able to use this approach in the next problem.



Practice 4. Compute the volume of the solid shown in the margin.

A special case of a solid with a hole results from rotating a region bounded by two curves around an axis that does not intersect the region.

Example 5. Compute the volume of the solid shown in the margin.



Solution. The face for a slice made at x has area:

$$\begin{aligned} A(x) &= [\text{area of BIG circle}] - [\text{area of small circle}] \\ &= \pi\,[\text{BIG radius}]^2 - \pi\,[\text{small radius}]^2 \end{aligned}$$

Here the BIG radius is the distance from the line $y = x+1$ to the x-axis, or $R(x) = (x+1) - 0 = x+1$; similarly, the small radius is the distance from the curve $y = \frac{1}{x}$ to the x-axis, or $r(x) = \frac{1}{x} - 0 = \frac{1}{x}$, hence the cross-sectional area is:

$$A(x) = \pi\,[x+1]^2 - \pi\left[\frac{1}{x}\right]^2 = \pi\left[x^2 + 2x + 1 - \frac{1}{x^2}\right]$$

The curves intersect where:

$$\begin{aligned} x+1 = \frac{1}{x} \ &\Rightarrow\ x^2 + x = 1 \ \Rightarrow x^2 + x - 1 = 0 \\ &\Rightarrow\ x = \frac{-1 \pm \sqrt{1-4(-1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Clearly we need $x > 0$ for this region, so the left endpoint of integration must be $x = \frac{-1+\sqrt{5}}{2}$ while the right endpoint is $x = 2$, so the volume of the solid is:

$$V = \int_{\frac{-1+\sqrt{5}}{2}}^{2} \pi\left[x^2 + 2x + 1 - \frac{1}{x^2}\right] dx = \pi\left[\frac{1}{3}x^3 + x^2 + x + \frac{1}{x}\right]_{\frac{-1+\sqrt{5}}{2}}^{2}$$

$$= \pi\left[\frac{2^3}{3} + 2^2 + 2 + \frac{1}{2}\right] - \pi\left[\frac{1}{3}\left(\frac{-1+\sqrt{5}}{2}\right)^3 + \left(\frac{-1+\sqrt{5}}{2}\right)^2 + \frac{-1+\sqrt{5}}{2} + \frac{2}{-1+\sqrt{5}}\right]$$

which simplifies to $\frac{\pi}{6}\left[50 - 5\sqrt{5}\right] \approx 20.33$. ◀

The previous Example extends the "disk" method to a more general technique often called the "washer" method because a big disk with a smaller disk cut out of the middle resembles a washer (a small flat ring used with nuts and bolts).

Volumes of Revolved Regions ("Washer Method")

If the region constrained by the graphs of $y = f(x)$ and $y = g(x)$ and the interval $[a, b]$ is revolved about a horizontal line
then the volume of the resulting solid is:

$$V = \int_a^b \left[\pi\,(R(x))^2 - \pi\,(r(x))^2\right]\,dx$$

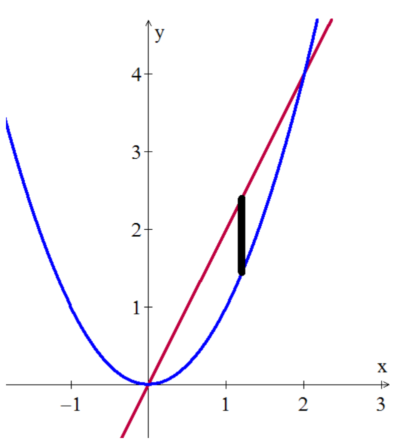
where $R(x)$ represents the distance from the axis of rotation to the farthest curve from that axis, and $r(x)$ represents the distance from the axis to the closest curve.

If $r(x) = 0$, the "washer" method becomes the "disk" method.

When applying the washer method, you should:

- graph the region
- draw a representative rectangular "slice" of that region
- check that revolving the slice about the axis of rotation results in a "washer"
- locate the limits of integration
- set up an integral
- evaluate the integral

If you are unable to find an antiderivative for the integrand of your integral, you can consult an integral table or use numerical methods to approximate the volume of the solid. You might also need to use numerical methods to locate where the boundary curves of the region intersect.



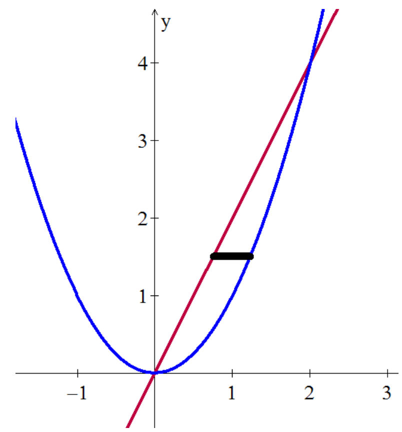
Example 6. Find the volume of the solid generated by rotating the region between the curves $y = 2x$ and $y = x^2$ about the (a) x-axis (b) y-axis (c) the line $x = -1$ (d) the line $y = 5$.

Solution. (a) The curves intersect where $x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$, so the limits of integration should involve $x = 0$ and $x = 2$. Revolving a vertical slice of the region with width Δx about the x-axis yields a "washer" with big radius $R(x) = 2x - 0 = 2x$ (the line $y = 2x$ is farthest from the x-axis) and small radius $r(x) =$

$x^2 - 0 = x^2$ (the parabola is closest to the x-axis when $0 \le x \le 2$). So the volume of the solid is:

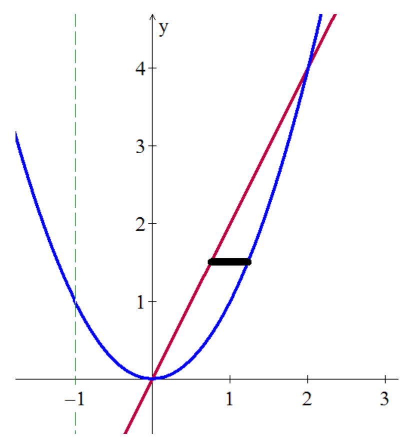
$$V = \int_0^2 \left[\pi(2x)^2 - \pi(x^2)^2\right] dx = \pi \int_0^2 \left[4x^2 - x^4\right] dx$$
$$= \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi \left[\frac{32}{3} - \frac{32}{5}\right] - \pi\left[0 - 0\right] = \frac{64\pi}{15}$$

(b) A vertical slice revolved around the y-axis does **not** result in a "washer" so instead we try slicing horizontally. A horizontal slice of thickness Δy revolved around the y-axis does result in a washer. The big radius is the x-distance from the parabola (where $x = \sqrt{y}$) to the y-axis (where $x = 0$) so $R(y) = \sqrt{y}$. Similarly, the small radius is the distance from the line (where $x = \frac{y}{2}$) to the y-axis (where $x = 0$), so $r(y) = \frac{y}{2}$. Because the variable of integration is now y, we need y-values for the limits of integration. At the lower intersection point of the two curves, $x = 0 \Rightarrow y = 0$; at the upper intersection point, $x = 2 \Rightarrow y = x^2 = 2^2 = 4$. So the volume of the solid is:



$$V = \int_{y=0}^{y=4} \left[\pi\left(\sqrt{y}\right)^2 - \pi\left(\frac{y}{2}\right)^2\right] dy = \pi \int_0^4 \left[y - \frac{1}{4}y^2\right] dy$$
$$= \pi \left[\frac{1}{2}y^2 - \frac{1}{12}y^3\right]_0^4 = \pi\left[8 - \frac{16}{3}\right] = \frac{8\pi}{3}$$

(c) This solid resembles the one from part (b), except now the radii are both bigger because the region (and the curves that form the boundary of the region) are farther away from the axis of rotation: $R(x) = \sqrt{y} - (-1) = \sqrt{y} + 1$ and $r(x) = \frac{y}{2} - (-1) = \frac{y}{2} + 1$:



$$V = \int_{y=0}^{y=4} \left[\pi\left(\sqrt{y}+1\right)^2 - \pi\left(\frac{y}{2}+1\right)^2\right] dy$$
$$= \pi \int_0^4 \left[\left(y + 2\sqrt{y} + 1\right) - \left(\frac{1}{4}y^2 + y + 1\right)\right] dy$$
$$= \pi \int_0^4 \left[2y^{\frac{1}{2}} - \frac{1}{4}y^2\right] dy = \pi \left[\frac{4}{3}y^{\frac{3}{2}} - \frac{1}{12}y^3\right]_0^4 = \pi\left[\frac{32}{3} - \frac{16}{3}\right] = \frac{16\pi}{3}$$

(d) For this solid, slicing the region vertically as in part (a) results in washers, but here the "near" and "far" roles of the curves are reversed: the parabola is farthest away from $y = 5$ while the line is closest. The radii are $R(x) = 5 - x^2$ and $r(x) = 5 - 2x$:

$$V = \int_{x=0}^{x=2} \pi\left[(5 - x^2)^2 - (5 - 2x)^2\right] dx = \frac{136\pi}{15}$$

The details of evaluating this definite integral are left to you. ◀

Practice 5. Find the volume of the solid generated by rotating the region between the curves $y = 2x$ and $y = x^2$ about the (a) the line $x = 5$ (b) the line $y = -5$.

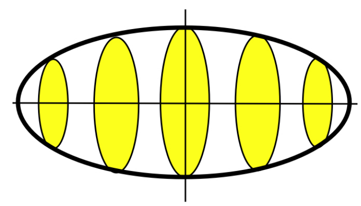
5.2 Problems

In Problems 1–12, find the volume of the solid generated when the region in the first quadrant bounded by the given curves is rotated about the x-axis.

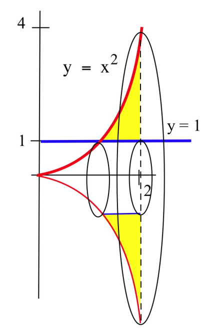
1. $y = x$, $x = 5$
2. $y = \sin(x)$, $x = \pi$
3. $y = \cos(x)$, $x = \frac{\pi}{3}$
4. $y = 3 - x$
5. $y = \sqrt{7 - x}$
6. $y = \sqrt[4]{9 - x}$
7. $y = 5 - x^2$
8. $x = 9 - y^2$
9. $x = 121 - y^2$
10. $x^2 + y^2 = 4$
11. $9x^2 + 25y^2 = 225$
12. $3x^2 + 5y^2 = 15$

In Problems 13–30, compute the volume of the solid formed when the region between the given curves is rotated about the specified axis.

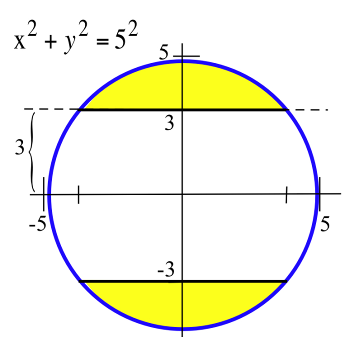
13. $y = x$, $y = x^4$ about the x-axis
14. $y = x$, $y = x^4$ about the y-axis
15. $y = x^2$, $y = x^4$ about the y-axis
16. $y = x^2$, $y = x^4$ about the x-axis
17. $y = x^2$, $y = x^3$ about the x-axis
18. $y = \sec(x)$, $y = 2\cos(x)$, $x = \frac{\pi}{3}$ about the x-axis
19. $y = \sec(x)$, $y = \cos(x)$, $x = \frac{\pi}{3}$ about the x-axis
20. $y = x$, $y = x^4$ about $y = 3$
21. $y = x$, $y = x^4$ about $y = -4$
22. $y = x$, $y = x^4$ about $x = -4$
23. $y = x$, $y = x^4$ about $x = 3$
24. $y = x$, $y = x^4$ about $x = 1$
25. $y = \sin(x)$, $y = x$, $x = 1$ about $y = 3$
26. $y = \sin(x)$, $y = x$, $x = \frac{\pi}{2}$ about $y = -2$
27. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $x = -2$
28. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $x = 4$
29. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $y = 2$
30. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $y = -\sqrt{3}$
31. Use calculus to compute the volume of a sphere of radius 2. (A sphere is formed when the region bounded by the x-axis and the top half of the circle $x^2 + y^2 = 2^2$ is revolved about the x-axis.)
32. Use calculus to determine the volume of a sphere of radius r. (Revolve the region bounded by the x-axis and the top half of the circle $x^2 + y^2 = r^2$ about the x-axis.)
33. Compute the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ is revolved around the x-axis (see figure below).



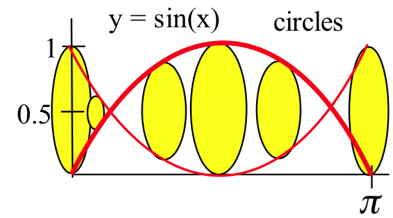
34. Compute the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved around the x-axis.
35. Compute the volume of the region shown below.



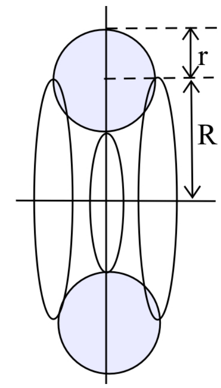
36. Compute the volume of a sphere of radius 5 with a hole of radius 3 drilled through its center.



37. Compute the volume of the region shown in the margin.



38. Determine the volume of the "doughnut" (called a "torus," see lower margin figure) generated by rotating a disk of radius r with center R units away from the x-axis about the x-axis.



39. (a) Find the **area** between $f(x) = \dfrac{1}{x}$ and the x-axis for $1 \le x \le 10$, $1 \le x \le 100$ and $1 \le x \le M$. What is the limit of the area for $1 \le x \le M$ when $M \to \infty$?

    (b) Find the **volume** swept out when the region in part (a) is revolved about the x-axis for $1 \le x \le 10$, $1 \le x \le 100$ and $1 \le x \le M$. What is the limit of the volume for $1 \le x \le M$ when $M \to \infty$?

5.2 Practice Answers

1. $\displaystyle\int_0^3 \pi \Big[\, |(3-x)-2)| \,\Big]^2 dx = \pi \int_0^3 (1-x)^2 \, dx = \pi \int_0^3 \left[1 - 2x + x^2\right] dx = \pi \left[x - x^2 + \frac{1}{3}x^3\right]_0^3 = 3\pi$

2. (a) Slicing the region vertically and rotating the slice about the x-axis results in disks, so the volume of the solid is:

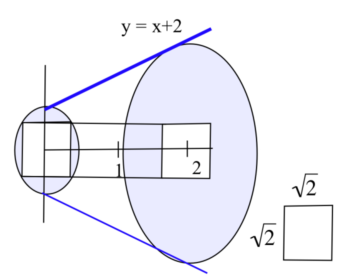
$$\int_0^2 \pi \left[x^2\right]^2 dx = \pi \int_0^2 x^4 \, dx = \pi \left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}$$

   (b) Here the slices extend from $y = x^2$ to $y = 2$ so the radius of each disk is $2 - x^2$ and the volume is:

$$\int_0^2 \pi \left[2 - x^2\right]^2 dx = \pi \int_0^2 \left[4 - 4x^2 + x^4\right] dx = \pi \left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{56\pi}{15}$$

3. $$\begin{aligned}\int_1^5 \pi\,[3 - f(x)]^2 \, dx &= \int_1^5 \pi \left[9 - 6f(x) + (f(x))^2\right] dx \\ &= \pi \left[\int_1^5 9\,dx - 6\int_1^5 f(x)\,dx + \int_1^5 [f(x)]^2 \, dx\right] = \pi\,[36 - 6 \cdot 4 + 7] = 19\pi\end{aligned}$$

4. The volume we want can be obtained by subtracting the volume of the "box" from the volume of the truncated cone generated by the rotated line segment. The volume of the truncated cone is:



$$\int_0^2 \pi\,[x+2]^2 \, dx = \pi \int_0^2 \left[x^2 + 4x + 4\right] dx = \pi \left[\frac{1}{3}x^3 + 2x^2 + 4x\right]_0^2 = \frac{56\pi}{3}$$

while the volume of the box is $\left[\sqrt{2}\right]^2 (2) = 4$ so the volume of the solid shown in the graph is $\dfrac{56\pi}{3} - 4 \approx 54.64$.

5. (a) Slicing the region vertically and rotating the slice about the line $x = 5$ results in something other than a washer, so we instead slice the region horizontally. The slice extends from $x = \dfrac{y}{2}$ (farthest from the axis of rotation) to $x = \sqrt{y}$ (closest), so the volume of the solid is:

$$\int_0^4 \left[\pi\left(5 - \frac{y}{2}\right)^2 - \pi\left(5 - \sqrt{y}\right)^2\right] dy = \frac{32\pi}{3}$$

(b) Slicing the region vertically and rotating the slice about the line $y = -5$ results in washers, so the volume is:

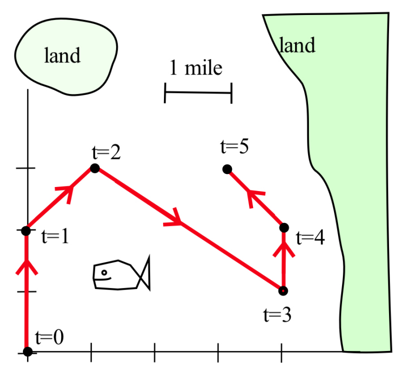
$$\int_0^2 \left[\pi\left(2x + 5\right)^2 - \pi\left(x^2 + 5\right)^2\right] dx = \frac{88\pi}{5}$$

5.3 *Arclength and Surface Area*

This section introduces two additional geometric applications of integration: finding the length of a curve and finding the area of a surface generated when you revolve a curve about a line. The general strategy remains the same: partition the problem into small pieces, approximate the solution on each small piece, add the small solutions together to form a Riemann sum and, finally, take the limit of the Riemann sum to get a definite integral.

Arclength: How Long Is a Curve?

In order to better understand an animal, biologists need to know how it moves through its environment and how far it travels. We need to know the length of the path it moves along. If we know the object's location at successive times, then we can easily calculate the distances between those locations and add them together to get a total (approximate) distance.



Example 1. In order to study the movement of whales, marine biologists implant a small transmitter on selected whales and track the location of a whale via satellite. Position data at one-hour time intervals over a five-hour period appears in the margin figure. How far did the whale swim during the first three hours?

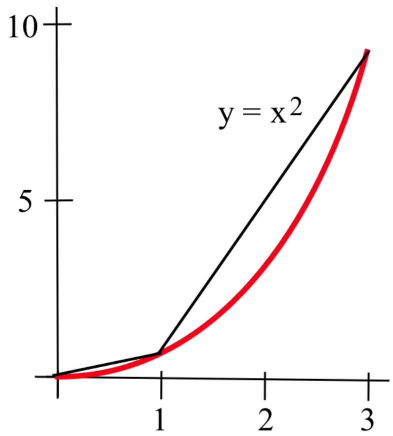
Solution. In moving from the point $(0,0)$ to the point $(0,2)$, the whale traveled *at least* 2 miles. Similarly, the whale traveled at least $\sqrt{(1-0)^2+(3-2)^2} = \sqrt{2} \approx 1.4$ miles during the second hour and at least $\sqrt{(4-1)^2+(1-3)^2} = \sqrt{13} \approx 3.6$ miles during the third hour. The scientist concluded that the whale swam at least $2+1.4+3.6 = 7$ miles during the three-hour period. ◀

Practice 1. How far did the whale swim during the entire five-hour time period?

It is unlikely that the whale swam in a straight line from location to location, so its actual swimming distance was undoubtedly more than seven miles during the first three hours. Scientists might get better distance estimates by recording the whale's position over shorter, five-minute time intervals.

Our strategy for finding the length of a curve will resemble the one the scientist used, and if the locations are given by a formula, then we can calculate the successive locations over very short intervals and get very good approximations of the total path length.

Example 2. Use the points $(0,0)$, $(1,1)$ and $(3,9)$ to approximate the length of $y = x^2$ for $0 \le x \le 3$.



Solution. The lengths of the two line segments (see margin) are:

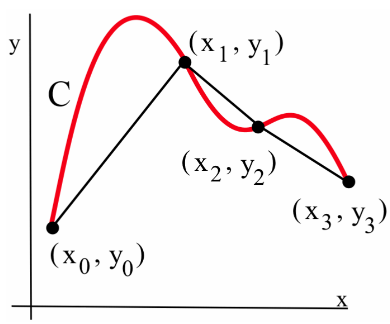
$$\sqrt{(1-0)^2+(1-0)^2} = \sqrt{1+1} = \sqrt{2} \approx 1.41$$

and:

$$\sqrt{(3-1)^2+(9-1)^2} = \sqrt{4+64} = \sqrt{68} \approx 8.25$$

so the length of the curve is approximately $1.41 + 8.25 = 9.66$. ◀

Practice 2. Get a better approximation of the length of $y = x^2$ for $0 \le x \le 3$ by using the points $(0,0)$, $(1,1)$, $(2,4)$ and $(3,9)$. Is your approximation longer or shorter than the actual length?



For a curve $\mathcal{C}$ (see margin), pick some points (x_k, y_k) along $\mathcal{C}$ and connect those points with line segments. Then the sum of the lengths of the line segments will approximate the length of $\mathcal{C}$. We can think of this as pinning a string to the curve at the selected points, and then measuring the length of the string as an approximation of the length of the curve. Of course, if we only pick a few points (as in the margin), then the total length approximation will probably be rather poor, so eventually we want lots of points (x_k, y_k) close together all along $\mathcal{C}$.

Label these points so that (x_0, y_0) is one endpoint of $\mathcal{C}$ and (x_n, y_n) is the other endpoint, and so that the subscripts increase as we move along $\mathcal{C}$. Then the distance between the successive points (x_{k-1}, y_{k-1}) and (x_k, y_k) is:

$$\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

and the total length of these line segments is simply the sum of the successive lengths:

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

This summation does not have the form $\sum g(c_k) \cdot \Delta x_k$ so it is not a Riemann sum. It is, however, algebraically equivalent to an expression very much like a Riemann sum that will lead us to a definite integral representation for the length of $\mathcal{C}$.

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, so that y is a function of x, we can factor $(\Delta x_k)^2$ from inside the radical and simplify:

$$\begin{aligned}\text{length of } C &\approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 \left[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}\right]} \\ &= \sum_{k=1}^{n} (\Delta x_k) \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k\end{aligned}$$

to get an expression that looks more like a Riemann sum. The $\dfrac{\Delta y_k}{\Delta x_k}$ inside the radical should remind you of two things: the slope of a line segment (it is, in fact, the slope of the k-th line segment in our approximation of the curve $\mathcal{C}$) and a derivative, $\frac{dy}{dx}$. If $f(x)$ is both continuous and differentiable, then the Mean Value Theorem guarantees that there is some number c_k between x_{k-1} and x_k so that:

Review Section 3.2 if you need to refresh your memory about the hypotheses and conclusions of the Mean Value Theorem.

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}$$

in which case we can write:

$$\sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k$$

This last expression *is* a Riemann sum, so it converges to a definite integral:

$$\sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k \longrightarrow \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In order to be sure that the sum converges to the integral, we need the resulting integrand to be an integrable function. If we require $f'(x)$ to be continuous on $[a, b]$ then the integrand will be a composition of continuous functions, hence continuous, and we know that a function that is continuous on a closed interval is integrable.

This definite integral provides us with a formula for the length of a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$.

Arclength Formula: $y = f(x)$ version

If	$\mathcal{C}$ is a curve given by $y = f(x)$ for $a \le x \le b$
and	$f'(x)$ exists and is continuous on $[a, b]$
then	the length L of $\mathcal{C}$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 3. Compute the length of $y = x^2$ for $0 \le x \le 3$.

Solution. Here $f(x) = x^2 \Rightarrow f'(x) = 2x$ so the length of this curve is:

$$\int_0^3 \sqrt{1 + [2x]^2}\, dx = \int_0^3 \sqrt{1 + 4x^2}\, dx$$

Unfortunately we do not (yet) have a technique to find an antiderivative of this integrand, but we can use numerical methods (such as Simpson's Rule, or a calculator or computer) to determine that the value of the integral is approximately 9.7471 (compare this with the answers from Example 2 and Practice 2). ◀

You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.

Practice 3. Compute the length of $y = x^2$ between $(1, 1)$ and $(4, 16)$.

Practice 4. Represent the length of one period of $y = \sin(x)$ as a definite integral, then find the length of this curve (using technology to approximate the value of the definite integral, if necessary).

More Arclength Formulas

Not all interesting curves are graphs of functions of the form $y = f(x)$. For a curve given by $x = g(y)$ we can mimic the previous argument (or simply swap x and y) to arrive at another arclength formula:

Arclength Formula: $x = g(y)$ version

If	$\mathcal{C}$ is a curve given by $x = g(y)$ for $c \le y \le d$
and	$g'(y)$ exists and is continuous on $[c, d]$
then	the length L of $\mathcal{C}$ is given by:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Practice 5. Compute the length of $x = \sqrt{y}$ between $(1, 1)$ and $(4, 16)$.

Review Section 2.5 to refresh your memory about parametric equations.

A curve $\mathcal{C}$ can also be described using parametric equations, where functions $x(t)$ and $y(t)$ give the coordinates of a point on the curve specified by a parameter t. We often think of t as "time," so that $(x(t), y(t))$ represents the position of a particle in the xy-plane t seconds (or minutes or hours) after time $t = 0$. In Section 2.5, we discovered that the speed of such a particle at time t is given by:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The distance traveled by the particle and the length of the curve will be equal as long as the particle does not traverse any part of the curve more than once on the interval $\alpha \le t < \beta$.

To find the distance the particle travels between times $t = \alpha$ and $t = \beta$, we could then integrate this speed function, which would also tell us the length of the curve.

Arclength Formula (Parametric Version)

If	$\mathcal{C}$ is a curve given by $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$
and	$x'(t)$ and $y'(t)$ exist and are continuous on $[\alpha, \beta]$
then	the length L of $\mathcal{C}$ is given by:

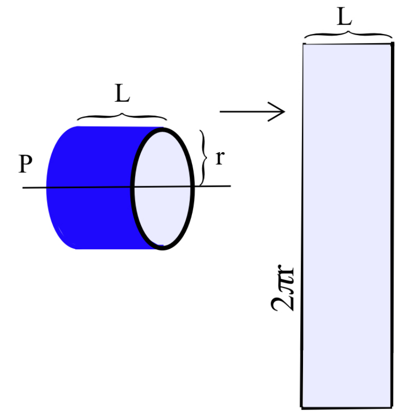
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Practice 6. Compute the length of the parametric curve given by the functions $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$.

Practice 7. Compute the length of the parametric path given by the functions $x(t) = 1 + 3t$ and $y(t) = 4t$ for $1 \le t \le 3$.

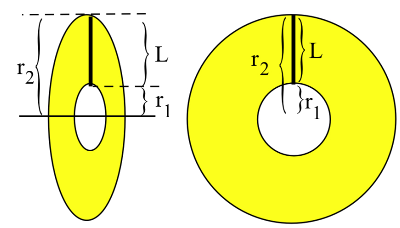
Areas of Surfaces of Revolution

In the previous section, we revolved a *region* in the xy-plane about a horizontal or vertical axis to create a *solid*, then used an integral to compute the volume of that solid. If we instead rotate a *curve* about an axis, we get a *surface*, whose **surface area** we can also compute using an integral. Just as the integral formulas for arclength came from the simple distance formula, the integral formulas for the area of a surface of revolution come from the formula for revolving a single line segment.



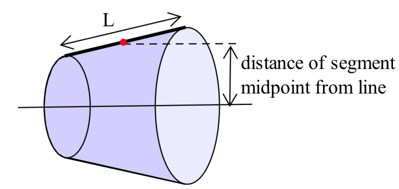
If we rotate a line segment of length L parallel to a line P (see margin) about the line P, then the resulting surface (a cylinder) can be "unrolled" and laid flat. This flattened surface is a rectangle with area $A = 2\pi \cdot r \cdot L$.

If we rotate a line segment of length L perpendicular to a line P and not intersecting P (see second margin figure) about the line P, then the resulting surface is the region between two concentric circles (an "annulus") and its area is:



$$\begin{aligned} A &= (\text{area of large circle}) - (\text{area of small circle}) \\ &= \pi\,(r_2)^2 - \pi\,(r_1)^2 = \pi\left[(r_2)^2 - (r_1)^2\right] = \pi\,(r_2 + r_1)\,(r_2 - r_1) \\ &= 2\pi\left(\frac{r_2 + r_1}{2}\right) L \end{aligned}$$

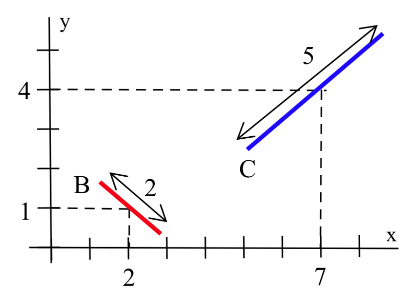
The expression $\frac{r_2+r_1}{2}$ represents the distance of the *midpoint* of the line segment L from the axis of rotation P and $2\pi\left(\frac{r_2+r_1}{2}\right)$ is the *distance* this midpoint travels when we revolve the line segment about the axis. It turns out that this pattern holds when we revolve *any* line segment of length L that does not intersect a line P about the line P (see margin):



See Problem 52 for a proof.

$$\begin{aligned} A &= (\text{distance traveled by segment midpoint}) \cdot (\text{length of line segment}) \\ &= 2\pi\,(\text{distance of segment midpoint from line } P) \cdot L \end{aligned}$$

Example 4. Compute the area of the surface generated when each line segment in the margin figure is rotated about the x-axis and the y-axis.



Solution. Line segment B has length $L = 2$ and its midpoint is at $(2, 1)$, which is 1 unit from the x-axis and 2 units from the y-axis. When B is rotated about the x-axis, the surface area is therefore:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 2 = 2\pi(1)2 = 4\pi$$

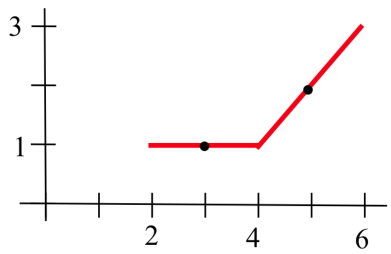
and when B is rotated about the y-axis, the surface area is:

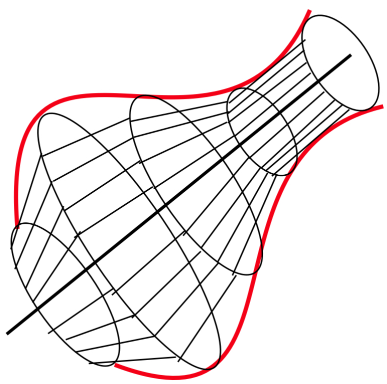
$$2\pi \cdot (\text{distance of midpoint from } y\text{-axis}) \cdot 2 = 2\pi(2)2 = 8\pi$$

Line segment $\mathcal{C}$ has length 5 and its midpoint is at $(7, 4)$. When $\mathcal{C}$ is rotated about the x-axis, the resulting surface area is:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 5 = 2\pi(4)5 = 40\pi$$

When $\mathcal{C}$ is rotated about the y-axis, the distance of the midpoint from the axis is 7, so the surface area is $2\pi(7)5 = 70\pi$. ◀





Practice 8. Find the area of the surface generated when the graph in the margin is rotated about each coordinate axis.

When we rotate a *curve* $\mathcal{C}$ (that does not intersect a line P, as in the second margin figure) about the line P, we also get a surface. To approximate the area of that surface, we can use the same strategy we used to approximate the length of a curve: select some points (x_k, y_k) along the curve, connect the points with line segments, calculate the surface area of each rotated line segment, and add together the surface areas of the rotated line segments.

The rotated line segment with endpoints (x_{k-1}, y_{k-1}) and (x_k, y_k) has midpoint:

$$(\overline{x}_k, \overline{y}_k) = \left(\frac{x_{k-1}+x_k}{2}, \frac{y_{k-1}+y_k}{2}\right)$$

and length:

$$L = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

If we rotate $\mathcal{C}$ about the x-axis, the distance from the midpoint of the k-th line segment to the x-axis is $\overline{y}_k$ so the surface area of the k-th rotated line segment will be:

$$\begin{aligned} 2\pi\left(\overline{y}_k\right) L &= 2\pi\left(\frac{y_{k-1}+y_k}{2}\right)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= 2\pi\left(\frac{y_{k-1}+y_k}{2}\right)\sqrt{1+\left[\frac{\Delta y_k}{\Delta x_k}\right]^2}\,\Delta x_k \end{aligned}$$

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, and $f'(x)$ is continuous on $[a, b]$, we can appeal to the Mean Value Theorem to find a c_k with $x_{k-1} < c_k < x_k$ and $f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$ so that our last expression becomes:

$$2\pi\left(\frac{f(x_{k-1}) + f(x_k)}{2}\right)\sqrt{1 + [f'(c_k)]^2}\,\Delta x_k$$

Adding up these approximations, we get:

$$\sum_{k=1}^{n} 2\pi\left(\frac{f(x_{k-1}) + f(x_k)}{2}\right)\sqrt{1 + [f'(c_k)]^2}\,\Delta x_k$$

This is not actually a Riemann sum, because the values x_{k-1}, x_k and c_k are not all the same. Proving that this sum converges to a definite integral requires some more advanced techniques.

which converges to a definite integral:

$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

that gives us a formula for the surface area of the revolved curve.

Example 5. Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the x-axis.

Solution. Here $f(x) = 2 + x^2 \;\Rightarrow\; f'(x) = 2x$ so, using the integral formula we just obtained:

$$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_0^3 2\pi\left(2+x^2\right)\sqrt{1+4x^2}\,dx$$

We do not (yet) know how to find an antiderivative for this integrand, but numerical approximation yields a result of 383.8. ◀

You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.

More Sufrace Area Formulas

If a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$ is instead rotated about the y-axis, then the distance from the midpoint of the k-th line segment to the axis of rotation is $\overline{x}_k$. Replacing $\overline{y}_k$ with $\overline{x}_k$ in our work on the previous page yields the formula:

$$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx$$

for the area of the surface generated by revolving $\mathcal{C}$ about the y-axis (assuming, as before, that $f'(x)$ is continuous for $a \le x \le b$).

Example 6. Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the y-axis.

Solution. Here again $f(x) = 2 + x^2 \;\Rightarrow\; f'(x) = 2x$ so, using our newest integral formula:

$$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx = \int_0^3 2\pi x\sqrt{1+4x^2}\,dx$$

We can find an antiderivative of this integrand using substitution:

$$u = 1 + 4x^2 \;\Rightarrow\; du = 8x\,dx \;\Rightarrow\; \frac{1}{8}\,du = x\,dx$$

The integral limits become $u = 1 + 4(0)^2 = 1$ and $u = 1 + 4(3)^2 = 37$, so the surface area is:

$$\int_{u=1}^{u=37} 2\pi\cdot\frac{1}{8}\sqrt{u}\,du = \frac{\pi}{4}\int_1^{37} u^{\frac{1}{2}}\,du = \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{37} = \frac{\pi}{6}\left[37\sqrt{37}-1\right]$$

or approximately 117.3. ◀

Wrap-Up

Developing formulas for the area of a surface generated by rotating a curve $x = g(y)$ for $c \le y \le d$ (or by parametric equations) present little additional difficulty. In future chapters, however, we will develop much more general—yet simpler—formulas for arclength and surface area. While the integral formulas developed in this section can be useful, more importantly their development served to illustrate yet again how relatively simple approximation formulas can lead us—via Riemann sums—to integral formulas. We will see this process again and again.

See Problems 48–51.

5.3 Problems

1. The locations (in feet, relative to an oak tree) at various times (in minutes) for a squirrel spotted in a back yard appear in the table below:

time	north	east
0	10	7
5	25	27
10	1	45
15	13	33
20	24	40
25	10	23
30	0	14

At least how far did the squirrel travel during the first 15 minutes?

2. The squirrel in the previous problem traveled at least how far during the first 30 minutes?
3. Use the partition $\{0, 1, 2\}$ to estimate the length of $y = 2^x$ between the points $(0, 1)$ and $(2, 4)$.
4. Use the partition $\{1, 2, 3, 4\}$ to estimate the length of $y = \dfrac{1}{x}$ between the points $(1, 1)$ and $\left(4, \dfrac{1}{4}\right)$.

The graphs of the functions in Problems 5–8 are line segments. Calculate each length (a) using the distance formula between two points and (b) by setting up and evaluating an appropriate arclength integral.

5. $y = 1 + 2x$ for $0 \le x \le 2$.
6. $y = 5 - x$ for $1 \le x \le 4$,
7. $x = 2 + t$, $y = 1 - 2t$ for $0 \le t \le 3$.
8. $x = -1 - 4t$, $y = 2 + t$ for $1 \le t \le 4$.
9. Calculate the length of $y = \dfrac{2}{3}x^{\frac{3}{2}}$ for $0 \le x \le 4$.
10. Calculate the length of $y = 4x^{\frac{3}{2}}$ for $1 \le x \le 9$.

Very few functions of the form $y = f(x)$ lead to integrands of the form $\sqrt{1 + [f'(x)]^2}$ that have elementary antiderivatives. In 11—14, $1 + \left[f'(x)\right]^2$ ends up being a perfect square, so you can evaluate the resulting arclength integral using antiderivatives.

11. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ for $1 \le x \le 5$.
12. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ for $1 \le x \le 9$.
13. $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}$ for $1 \le x \le 5$.
14. $y = \dfrac{x^6}{6} + \dfrac{1}{16x^4}$ for $4 \le x \le 25$.

In Problems 15–23, represent each length as a definite integral, then evaluate the integral (using technology, if necessary).

15. The length of $y = x^2$ from $(0, 0)$ to $(1, 1)$.
16. The length of $y = x^3$ from $(0, 0)$ to $(1, 1)$.
17. The length of $y = \sqrt{x}$ from $(1, 1)$ to $(9, 3)$.
18. The length of $y = \ln(x)$ from $(1, 0)$ to $(e, 1)$.
19. The length of $y = \sin(x)$ from $(0, 0)$ to $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ and from $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ to $\left(\dfrac{\pi}{2}, 1\right)$.
20. The length of the ellipse $x(t) = 3\cos(t)$, $y(t) = 4\sin(t)$ for $0 \le t \le 2\pi$.
21. The length of the ellipse $x(t) = 5\cos(t)$, $y(t) = 2\sin(t)$ for $0 \le t \le 2\pi$.
22. A robot programmed to be at location $x(t) = t\cos(t)$, $y(t) = t\sin(t)$ at time t will travel how far between $t = 0$ and $t = 2\pi$?
23. How far will the robot in the previous problem travel between $t = 10$ and $t = 20$?
24. As a tire of radius R rolls, a pebble stuck in the tread will travel a "cycloid" path, given by $x(t) = R \cdot (t - \sin(t))$, $y(t) = R \cdot (1 - \cos(t))$. As t increases from 0 to 2π, the tire makes one complete revolution and travels forward $2\pi R$ units. How far does the pebble travel?
25. Referring to the previous problem, as a tire with a 1-foot radius rolls forward 1 mile, how far does a pebble stuck in the tire tread travel?
26. Graph $y = x^n$ for $n = 1$, 3, 10 and 20. As the value of n becomes large, what happens to the graph of $y = x^n$? Estimate the value of:

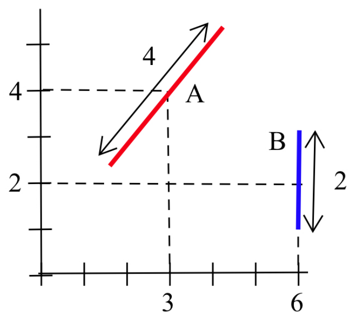
$$\lim_{n \to \infty} \int_{x=0}^{x=1} \sqrt{1 + [n \cdot x^{n-1}]^2}\, dx$$

27. Find the point on the curve $f(x) = x^2$ for $0 \le x \le 4$ that will divide the curve into two equally long pieces. Find the points that will divide the segment into three equally long pieces.

28. Find the pattern for the functions in Problems 11–14. If $y = \frac{x^n}{n} + \frac{1}{Ax^p}$, how must A and p be related to n?

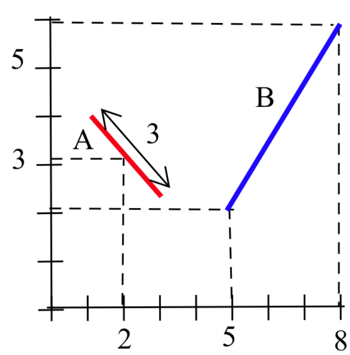
29. Use the formulas for A and p from the previous problem with $n = \frac{3}{2}$ and find a new function $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{Ax^p}$ so that $1 + \left[\frac{dy}{dx}\right]^2$ is a perfect square.

30. Find the surface area when each line segment in the figure below is rotated about the (a) x-axis and (b) y-axis.



31. Find the surface area when each line segment in the figure above is rotated about the line (a) $y = 1$ and (b) $x = -2$.

32. Find the surface area when each line segment in the figure below is rotated about the line (a) $y = 1$ and (b) $x = -2$.



33. Find the surface area when each line segment in the figure above is rotated about the (a) x-axis and (b) y-axis.

34. A line segment of length 2 with midpoint $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the y-axis? Explain your reasoning.

35. A line segment of length 2 with one end at $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the x-axis? Explain your reasoning.

In Problems 36–43, when the given curve is rotated about the given axis, represent the area of the resulting surface as a definite integral, then evaluate that integral using technology.

36. $y = x^3$ for $0 \le x \le 2$ about the y-axis

37. $y = 2x^3$ for $0 \le x \le 1$ about the y-axis

38. $y = x^2$ for $0 \le x \le 2$ about the x-axis

39. $y = 2x^2$ for $0 \le x \le 1$ about the x-axis

40. $y = \sin(x)$ for $0 \le x \le \pi$ about the x-axis

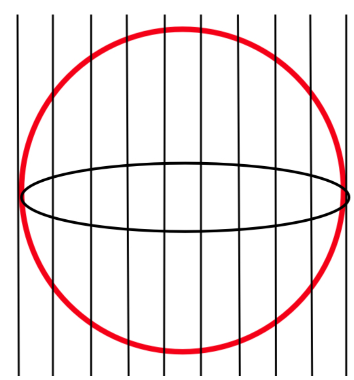
41. $y = x^3$ for $0 \le x \le 2$ about the x-axis

42. $y = \sin(x)$ for $0 \le x \le \frac{\pi}{2}$ about the y-axis

43. $y = x^2$ for $0 \le x \le 2$ about the y-axis

44. Find the area of the surface formed when the graph of $y = \sqrt{4 - x^2}$ is rotated about the x-axis:
    (a) for $0 \le x \le 1$.
    (b) for $1 \le x \le 2$.
    (c) for $2 \le x \le 3$.

45. Show that if a thin hollow sphere is sliced into pieces by equally spaced parallel cuts (see below), then each piece has the same weight. (Hint: Does each piece have the same surface area?)



46. Interpret the result of the previous problem for an orange sliced by equally spaced parallel cuts.

47. A hemispherical cake with a uniformly thick layer of frosting is sliced with equally spaced parallel cuts. Does everyone get the same amount of cake? The same amount of frosting?

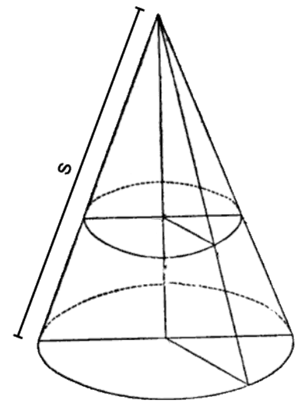
48. Devise a formula for the area of the surface generated by revolving the curve $x = g(y)$ for $c \le y \le d$ about the (a) x-axis and (b) y-axis.

49. Use the answer to the previous problem to find the area of the surface generated by revolving $x = e^y$ for $0 \le y \le 1$ about (a) the x-axis and (b) the y-axis.

50. Devise a formula for the area of the surface generated by revolving the curve given by parametric equations $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$ about the (a) x-axis and (b) y-axis.

51. Use the answer to the previous problem to find the area of the surface generated by revolving the curve given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \le t \le \frac{\pi}{2}$ about (a) the x-axis and (b) the y-axis.

52. The surface generated by revolving a line segment of length L about a line P (that does not intersect the line segment) is the **frustrum** of a cone: the surface that results from taking a larger cone of radius r_2 and removing a smaller cone of radius r_1 ("chopping off the top"). We know from geometry that the surface area of a cone is $\pi r s$ where r is the radius of the cone and s is the **slant height**:



(a) If s_1 is the slant height of the smaller cone that is removed from the bigger cone, show that:

$$s_1 + L = \frac{r_2 L}{r_2 - r_1}$$

(Hint: Use similar triangles.)

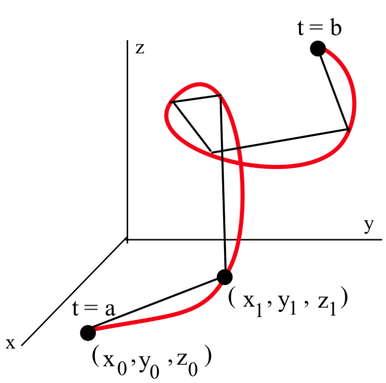
(b) Show that the surface area of the frustrum is:

$$\pi r_2 (s_1 + L) - \pi r_1 s_1$$

(c) Show that this quantity equals:

$$\pi (r_1 + r_2) L$$

(d) Show that this last quantity is the product of the distance traveled by the midpoint and the length of the line segment.



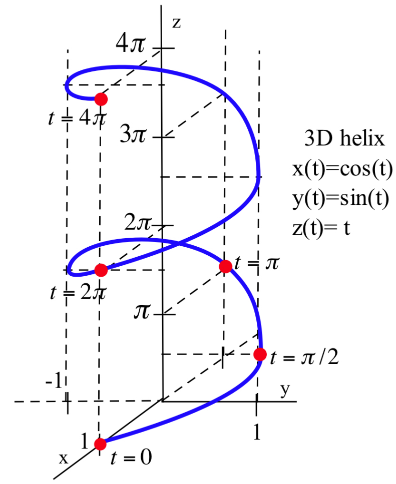
3-D Arclength

If a 3-dimensional curve $\mathcal{C}$ (see margin) is given parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$ for $\alpha \le t \le \beta$, then we can easily extend the arclength formula to three dimensions:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The remaining problems in this section use this formula to provide you with a preview of calculus in higher dimensions.

53. Find the length of the helix (see figure) given by $x = \cos(t)$, $y = \sin(t)$, $z = t$ for $0 \le t \le 4\pi$.
54. Find the length of the line segment given by $x = t$, $y = t$, $z = t$ for $0 \le t \le 1$.
55. Find the length of the curve given by $x = t$, $y = t^2$, $z = t^3$ for $0 \le t \le 1$.
56. Find the length of the "stretched helix" given by $x = \cos(t)$, $y = \sin(t)$, $z = t^2$ for $0 \le t \le 2\pi$.
57. Find the length of the curve given by $x = 3\cos(t)$, $y = 2\sin(t)$, $z = \sin(7t)$ for $0 \le t \le 2\pi$.



5.3 Practice Answers

1. At least $2 + \sqrt{2} + \sqrt{13} + 1 + \sqrt{2} \approx 9.43$ miles.
2. $L \approx \sqrt{2} + \sqrt{10} + \sqrt{26} <$ actual length
3. $\int_1^4 \sqrt{1 + [2x]^2}\, dx = \int_1^4 \sqrt{1 + 4x^2}\, dx \approx 15.34$
4. $\int_0^{2\pi} \sqrt{1 + [\cos(x)]^2}\, dx \approx 7.64$
5. Here $g(y) = \sqrt{y} = y^{\frac{1}{2}} \Rightarrow g'(y) = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$ so the arclength is:

$$\int_1^{16} \sqrt{1 + \left[\frac{1}{2\sqrt{y}}\right]^2}\, dy = \int_1^{16} \sqrt{1 + \frac{1}{4y}}\, dy \approx 15.34$$

This answer is the same as the answer to Practice 3. Should that surprise you?

6. Here $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$ so the arclength is:

$$\int_0^{2\pi} \sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\, dt = \int_0^{2\pi} \sqrt{1}\, dt = 2\pi$$

The curve in question is a circle of radius 1. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?

7. Here $x'(t) = 3$ and $y'(t) = 4$ so the arclength is:

$$\int_1^3 \sqrt{[3]^2 + [4]^2}\, dt = \int_1^3 5\, dt = 10$$

The "curve" is a line segment from $(4, 4)$ to $(10, 12)$. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?

8. The surface area of the horizontal segment revolved about x-axis is $2\pi(1)(2) = 4\pi \approx 12.57$ while the surface area of other segment revolved about the x-axis is $2\pi(2)(\sqrt{8}) \approx 35.54$, so the total surface area is approximately $12.57 + 35.54 = 48.11$ square units.

   The surface area of the horizontal segment revolved about y-axis is $2\pi(3)(2) = 12\pi \approx 37.70$ while the surface area of the other segment revolved about the y-axis is $2\pi(5)(\sqrt{8}) \approx 88.86$, so the total surface area is approximately $37.70 + 88.86 = 126.56$ square units.

5.4 More Work

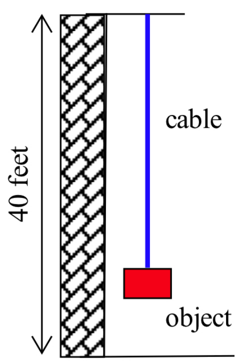
In Section 4.7 we investigated the problem of calculating the work done in lifting an object using a cable. This section continues that investigation and extends the process to handle situations in which the applied force or the distance—or both—may vary. The method we used before turns up again here. The first step is to divide the problem into small "slices" so that the force and distance vary only slightly on each slice. Then we calculate the work done for each slice, approximate the total work by adding together the work for each slice (to get a Riemann sum) and, finally, take a limit of that Riemann sum to get a definite integral representing the total work.

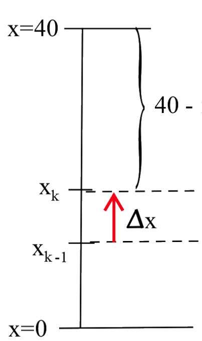
There are so many possible variations in work problems that it is vital you understand the process.

Recall that the work done on an object by a constant force is defined to be the magnitude of the force applied to the object multiplied by the distance over which the force is applied:

$$\text{work} = (\text{force}) \cdot (\text{distance})$$

A similar example appeared in Section 4.7, but it provides a good illustration of the process of dividing a problem into pieces and analyzing each piece.





Example 1. A 10-pound object is lifted 40 feet from the ground to the top of a building using a cable that weighs $\frac{1}{2}$ pound per foot (see margin figure). How much work is done?

Solution. The work done on the object is simply:

$$W = F \cdot d = (10 \text{ lbs}) \cdot (40 \text{ ft}) = 400 \text{ ft-lbs}$$

For the rope, we can partition it (see second margin figure) into n small pieces, each with length Δx. Each small piece of rope weighs:

$$\left(\frac{1}{2}\,\frac{\text{lb}}{\text{ft}}\right)(\Delta x \text{ ft}) = \frac{1}{2}\Delta x \text{ lb}$$

and the k-th slice of rope is lifted a distance of (approximately) $40 - x_k$ feet, so the work done on the k-th slice of rope is (approximately):

$$W_k = F_k \cdot d_k = \left(\frac{1}{2}\Delta x \text{ lb}\right) \cdot \Big((40 - x_k) \text{ ft}\Big) = \frac{1}{2}(40 - x_k)\,\Delta x \text{ ft-lbs}$$

and the total work done to lift the rope is therefore:

$$\sum_{k=1}^{n} \frac{1}{2}(40 - x_k)\,\Delta x \longrightarrow \int_0^{40} \frac{1}{2}(40 - x)\,dx$$

Evaluating this integral yields:

$$\frac{1}{2}\left[40x - \frac{1}{2}x^2\right]_0^{40} = \frac{1}{2}[1600 - 800] = 400 \text{ ft-lbs}$$

so the total work done to lift the object is $400 + 400 = 800$ ft-lbs. ◀

Practice 1. How much work is done lifting a 130-pound injured person to the top of a 30-foot cliff using a stretcher that weighs 10 pounds and a cable weighing 2 pounds per foot?

Work in the Metric System

All of the work problems we have considered so far measured force in pounds and distance in feet, so that work was measured in "foot-pounds." In the metric system, we often measure distance in meters (m) and force in **newton**s (N). According to Newton's second law of motion:

$$\text{force} = (\text{mass}) \cdot (\text{acceleration})$$

or, more succinctly, $F = ma$. The force in many work problems is the weight of an object, so the acceleration in question is the acceleration due to gravity, denoted by g. Near sea level on Earth, $g \approx 9.80665 \, \frac{\text{m}}{\text{sec}^2}$, although the value 9.81 is commonly used in computations. An object with a mass of 10 kg would thus have a weight of:

$$mg = (10 \text{ kg}) \cdot \left(9.81 \, \frac{\text{m}}{\text{sec}^2}\right) = 98.1 \, \frac{\text{kg} \cdot \text{m}}{\text{sec}^2} = 98.1 \, \text{N}$$

Virtually all countries other than the United States — along with U.S. scientists and engineers — use the metric system, so you need to know how to solve work problems using metric units.

Sir Isaac Newton (1643–1727) not only invented calculus, he formulated the laws of motion and universal gravitation in physics (among many other accomplishments).

Example 2. An object with a mass of 10 kg is lifted 40 m from the ground to the top of a building using a 40-meter cable with a mass of 20 kg. How much work is done?

Solution. The work done on the object is:

$$W = F \cdot d = mg \cdot d = (10 \text{ kg}) \left(9.81 \, \frac{\text{m}}{\text{sec}^2}\right) \cdot (40 \text{ m}) = 3924 \text{ N-m}$$

or 3,924 joules (a **joule**, abbreviated "J," is 1 N-m). The cable has total mass 20 kg and is 40 m long, so it has a linear density of:

$$\frac{20 \text{ kg}}{40 \text{ m}} = \frac{1}{2} \, \frac{\text{kg}}{\text{m}}$$

This unit for work is named after another English physicist, James Prescott Joule (1818–1889).

We can partition the cable into n small pieces, each with length Δx, so each small piece of cable has a mass of:

$$\left(\frac{1}{2} \, \frac{\text{kg}}{\text{m}}\right) (\Delta x \text{ m}) = \frac{1}{2} \Delta x \text{ kg}$$

and thus has a weight of:

$$F = mg = \left(\frac{1}{2} \Delta x \text{ kg}\right) \left(9.81 \, \frac{\text{m}}{\text{sec}^2}\right) = 4.905 \Delta x \, \frac{\text{kg} \cdot \text{m}}{\text{sec}^2} = 4.905 \Delta x \text{ N}$$

The k-th slice of cable is lifted a distance of approximately $40 - x_k$ m, so the work done on the k-th slice of cable is:

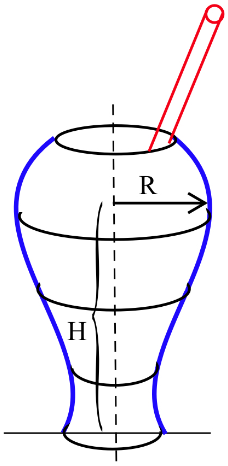
$$W_k = F_k \cdot d_k = (4.905 \Delta x \text{ N}) \cdot \Big((40 - x_k) \text{ m} \Big) = 4.905 \, (40 - x_k) \, \Delta x \text{ N-m}$$

and the total work done lifting the cable is therefore:

$$\sum_{k=1}^{n} 4.905 \, (40 - x_k) \, \Delta x \;\longrightarrow\; \int_0^{40} 4.905 \, (40 - x) \, dx = 3924 \text{ J}$$

so the total work done to lift the object is $3924 + 3924 = 7848$ J. ◀

Much of this process should look familiar. Compare the solution of Example 2 to that of Example 1 on the previous page.



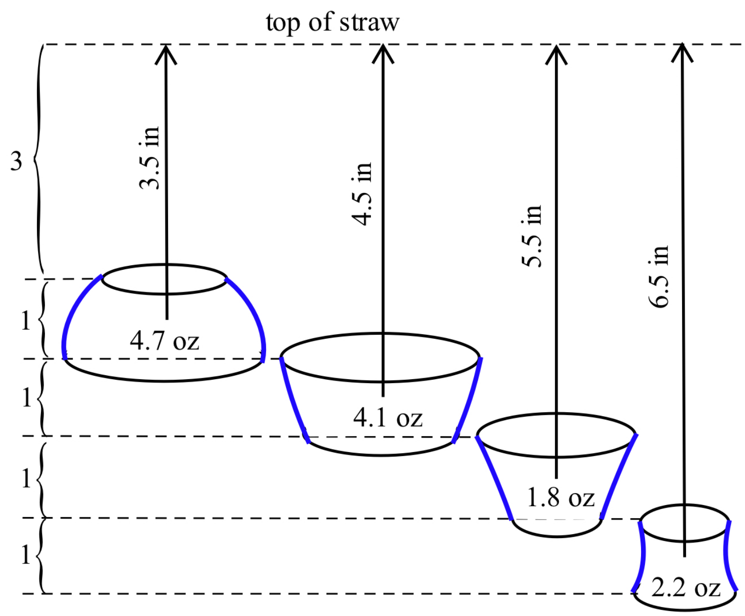
height	radius
4	1.4
3	1.6
2	1.5
1	1.0
0	1.1

Practice 2. How much work is done lifting an injured person of mass 50 kg to the top of a 30-meter cliff using a stretcher of mass 5 kg and a 30-meter cable of mass 10 kg?

Lifting Liquids

Example 3. A cola glass (see margin figure) has dimensions given in the margin table. Approximately how much work do you do when you drink a cola glass full of water by sucking it through a straw to a point 3 inches above the top edge of the glass?

Solution. The table partitions the water into 1-inch "slices":



The work needed to move each slice is approximately the weight of the slice times the distance the slice is moved. We can use the radius at the bottom of each slice to approximate the volume — and then the weight — of the slice, and a point halfway up each slice to calculate the distance the slice is moved. For the top slice:

You might wonder why the displacement is not computed by taking the distance from the bottom of the straw up to the top of the straw, but when computing work we need to use the *net* displacement.

Water's density is $62.5\frac{\text{lb}}{\text{ft}^3} = 0.5787\frac{\text{oz}}{\text{in}^3}$.

$$\text{weight} = (\text{volume})\,(\text{density}) \approx \pi\,(1.6\,\text{in})^2\,(1\,\text{in})\left(0.5787\,\frac{\text{oz}}{\text{in}^3}\right) \approx 4.7\,\text{oz}$$

and the distance this slice travels is roughly 3.5 inches, so:

$$W = F \cdot d \approx (4.7\text{ oz})\,(3.5\text{ in}) \approx 16.4\text{ oz-in}$$

For the next slice:

$$\text{weight} = (\text{volume})\,(\text{density}) \approx \pi\,(1.5\,\text{in})^2\,(1\,\text{in})\left(0.5787\,\frac{\text{oz}}{\text{in}^3}\right) \approx 4.1\,\text{oz}$$

and the distance this slice travels is roughly 4.5 inches, so:

$$W = F \cdot d \approx (4.1\text{ oz})\,(4.5\text{ in}) \approx 18.4\text{ oz-in}$$

The work for the last two slices is $(1.8\,\text{oz})\,(5.5\,\text{in}) = 9.9$ oz-in and $(2.2\,\text{oz})\,(6.5\,\text{in}) = 14.3$ oz-in. The total work is then sum of the work needed to raise each slice of water:

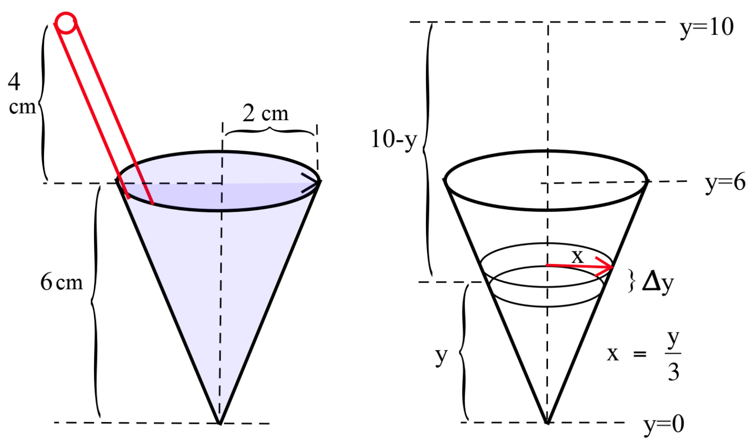
$$(16.4\,\text{oz-in}) + (18.4\,\text{oz-in}) + (9.9\,\text{oz-in}) + (14.3\,\text{oz-in}) = 59.0\,\text{oz-in}$$

or about 0.31 ft-lbs. ◀

Practice 3. Approximate the total work needed to raise the water in Example 3 by using the top radius of each slice to approximate its weight and the midpoint of each slice to approximate the distance the slice is raised.

If we knew the radius of the cola glass at *every* height, then we could improve our approximation by taking thinner and thinner slices. In fact, we could have formed a Riemann sum, taken the limit of the Riemann sum as the thickness of the slices approached 0, and obtained a definite integral. In the next Example we *do* know the radius of the container at every height.

Example 4. Find the work needed to raise the water in the cone shown below to the top of the straw.



In this example, both the force and the distance vary, and each depends on the height of the "slice" above the bottom of the cone.

Solution. We can partition the cone to get n "slices" of water. The work done raising the k-th slice is the product of the distance the slice is raised and the force needed to move the slice (the weight of the slice). For any c_k in the subinterval $[y_{k-1}, y_k]$, the slice is raised a distance of approximately $(10 - c_k)$ cm. Each slice is approximately a right circular cylinder, so its volume is:

If you want, you can choose $c_k = y_k$ like you did in Practice 3.

$$\pi\,(\text{radius})^2\,\Delta y$$

At a height y above the bottom of the cone, the radius of the cylinder is $x = \frac{y}{3}$ so at a height c_k the radius is $\frac{1}{3}c_k$; the mass of each slice is

To see this, use similar triangles in the right-hand figure above:

$$\frac{x}{y} = \frac{2}{6} \Rightarrow x = \frac{y}{3}$$

therefore:

In the metric system, a **gram** (abbreviated "g") is defined as the mass of one cubic centimeter of water, so the density of water is:

$$1\,\frac{\text{g}}{\text{cm}^3} = 1,000\,\frac{\text{kg}}{\text{m}^3}$$

$$\begin{aligned}(\text{volume})\,(\text{density}) &\approx \pi\,(\text{radius})^2\,(\Delta y)\left(1\,\frac{\text{g}}{\text{cm}^3}\right)\\ &= \pi\left(\frac{1}{3}c_k\text{ cm}\right)^2(\Delta y\text{ cm})\left(1\,\frac{\text{g}}{\text{cm}^3}\right)\\ &= \frac{\pi}{9}\,(c_k)^2\,\Delta y\text{ g}\end{aligned}$$

so the force required to raise the k-th slice is:

In the g–cm–sec version of the metric system, the standard unit of force is a **dyne** (abbreviated "dyn"), which is $1\,\frac{\text{g-cm}}{\text{sec}^2}$:

$$1\text{ N} = 100,000\text{ dyn}$$

$$F_k = m_k \cdot g \approx \left[\frac{\pi}{9}\,(c_k)^2\,\Delta y\text{ g}\right]\cdot\left[981\,\frac{\text{cm}}{\text{sec}^2}\right] = 109\pi\,(c_k)^2\,\Delta y\,\frac{\text{g-cm}}{\text{sec}^2}$$

and the work required to lift the k-th slice is:

In the g–cm–sec version of the metric system, the standard unit of work is called an **erg**, which is 1 dyn-cm:

$$1\text{ J} = 10,000,000\text{ erg}$$

$$\begin{aligned}W_k = F_k\cdot d_k &\approx \left[109\pi\,(c_k)^2\,\Delta y\text{ dyn}\right]\cdot\left[(10-c_k)\ \text{cm}\right]\\ &= 109\pi\,(c_k)^2\,(10-c_k)\,\Delta y\text{ dyn-cm}\end{aligned}$$

We can then add the work done on all n slices to get a Riemann sum:

$$W \approx \sum_{k=1}^{n} 109\pi\,(c_k)^2\,(10-c_k)\,\Delta y \longrightarrow \int_{y=0}^{y=6} 109\pi y^2(10-y)\,dy$$

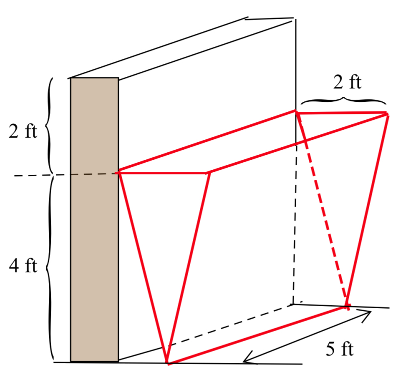
Evaluating this integral is relatively straightforward:

We integrate from $y=0$ to $y=6$ because the bottom slice of water is at a height of 0 cm and the top slice of water is at a height of 6 cm.

$$\begin{aligned}W &= 109\pi\int_0^6\left(10y^2-y^3\right)dy = 109\pi\left[\frac{10}{3}y^3-\frac{1}{4}y^4\right]_0^6\\ &= 109\pi\,[720-324] = 43164\pi\text{ erg}\end{aligned}$$

or about $135,604$ erg $= 0.0135604$ J. ◀

Practice 4. How much work is done drinking just the top 3 cm of the water in Example 4?

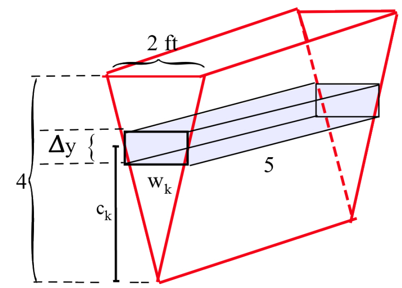


Example 5. The trough shown in the margin is filled with a liquid weighing 70 pounds per cubic foot. How much work is done pumping the liquid over the wall next to the trough?

Solution. As before, we can partition the height of the trough to get n "slices" of liquid (see margin figure at top of next page). To form a Riemann sum for the total work, we need the weight of a typical slice and the distance that slice is raised. The weight of the k-th slice is:

$$(\text{volume})\cdot(\text{density}) \approx (\text{length})\,(\text{width})\,(\text{height})\cdot\left(70\,\frac{\text{lb}}{\text{ft}^3}\right)$$

The length of each slice is 5 feet, and the height of each slice is Δy feet, but the width of each slice (w_k) varies and depends on how far the slice



is above the bottom of the trough (c_k). Using similar triangles on the edge of the trough, we can observe that:

$$\frac{w_k}{c_k} = \frac{2}{4} \Rightarrow w_k = \frac{c_k}{2}$$

so the weight of the k-th slice is therefore:

$$(5 \text{ ft}) \left(\frac{c_k}{2} \text{ ft}\right) (\Delta y \text{ ft}) \cdot \left(70 \frac{\text{lb}}{\text{ft}^3}\right) = 175 c_k \Delta y \text{ lb}$$

The k-th slice is raised from a height of c_k feet to a height of 6 feet, through a distance of $6 - c_k$ feet, so the work done on the k-th slice is:

$$W_k = F_k \cdot d_k \approx \left[175 c_k \Delta y \text{ lb}\right] \cdot \left[(6 - c_k) \text{ ft}\right] = 175 c_k (6 - c_k) \Delta y \text{ lb-ft}$$

Adding up the work done on all n slices yields a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} 175 c_k (6 - c_k) \Delta y \longrightarrow \int_0^4 175 y (6 - y) \, dy$$

We integrate from $y = 0$ to $y = 4$ because the bottom slice of liquid is at a height of 0 feet and the top slice of liquid is at a height of 4 feet.

Evaluating the integral is straightforward:

$$175 \int_0^4 \left(6y - y^2\right) dy = 175 \left[3y^2 - \frac{1}{3} y^3\right]_0^4 = \frac{14000}{3}$$

or about $4,667$ ft-lbs. ◀

You can generally handle "raise the liquid" problems by partitioning the height of the container and then focusing on a typical slice.

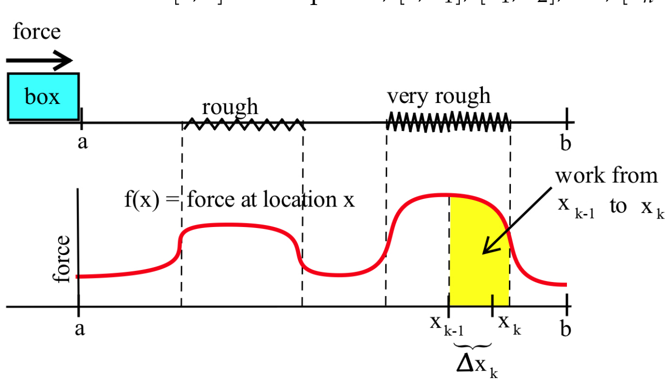
If you can calculate the weight of a typical slice and the distance it is raised, the rest of the steps are straightforward: form a Riemann sum, let it converge to a definite integral, and evaluate the integral to get the total work.

Practice 5. How much work is done pumping half of the liquid over the wall in Example 5?

Work Moving an Object Along a Straight Path

If you push a box along a flat surface (as in the figure below) that is smooth in some places and rough in others, at some places you only need to push the box lightly and in other places you have to push hard. If $f(x)$ is the amount of force needed at location x, and you want to push the box along a straight path from $x = a$ to $x = b$, then we can partition the interval $[a, b]$ into n pieces, $[a, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, b]$:

The force $f(x)$ discussed here is the minimum force required to counteract the **kinetic friction** between the box and the surface at any point. You will learn more about friction in physics and engineering classes.

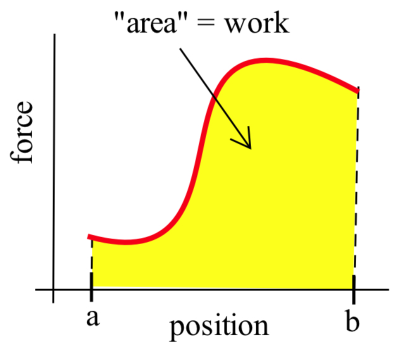


The work required to move the box through the k-th subinterval, from x_{k-1} to x_k, is approximately:

$$(\text{force}) \cdot (\text{distance}) \approx f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$$

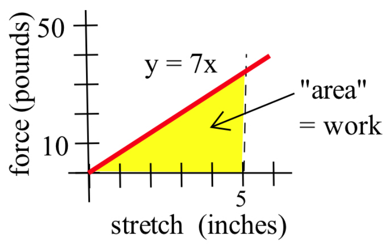
for any c_k in the subinterval $[x_{k-1}, x_k]$. The total work is the sum of the work along these n pieces, which is a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \longrightarrow \int_a^b f(x)\,dx$$



This has a simple geometric interpretation. If $f(x)$ is the force applied at position x, then the work done to move the object from position $x = a$ to position $x = b$ is the area under the graph of f between $x = a$ and $x = b$ (see margin). This formula applies in more general situations, as demonstrated in the next Example.

Example 6. If a force of $7x$ pounds is required to keep a spring stretched x inches past its natural length, how much work will be done stretching the spring from its natural length ($x = 0$) to five inches beyond its natural length ($x = 5$)?

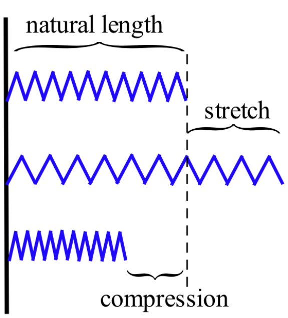


Solution. According to the formula we just developed:

$$\text{work} = \int_a^b f(x)\,dx = \int_0^5 7x\,dx = \left[\frac{7}{2}x^2\right]_0^5 = \frac{175}{2} = 87.5\,\text{in-lbs}$$

or about 7.29 ft-lbs. (See margin for a graphical interpretation.) ◀

Practice 6. How much work is done to stretch the spring in Example 6 from 5 inches past its natural length to 10 inches past its natural length?



The preceding spring example is an application of a physical principle discovered by English physicist Robert Hooke (1635–1703), a contemporary of Newton.

In fact, Hooke's Law holds for most solid objects, at least for limited forces: "Nor is it observable in these bodies only, but in all other springy bodies whatsoever, whether metal, wood, stones, baked earth, hair, horns, silk, bones, sinews, glass and the like." — Robert Hooke, *De Potentia Restitutiva, or Of Spring*

> **Hooke's Law**: The force $f(x)$ needed to keep a spring stretched (or compressed) x units beyond its natural length is proportional to the distance x: $f(x) = kx$ for some constant k.

We call the "k" in Hooke's Law a "spring constant." It varies from spring to spring (depending on the materials and dimensions of the spring — and even on the temperature of the spring), but remains constant for each spring as long as the spring is not overextended or overcompressed. Most bathroom scales use compressed springs — and Hooke's Law — to measure a person's weight.

Example 7. A spring has a natural length of 43 cm when hung from a ceiling. A mass of 40 grams stretches it to a length of 75 cm. How much work is done stretching the spring from a length of 63 cm to a length of 93 cm?

Solution. First we need to use the given information to find the value of k, the spring constant. A mass of 40 g produces a stretch of $75 - 43 = 32$ cm. Substituting $x = 32$ cm and $f(x) = 40\,\text{g} \cdot 981\,\frac{\text{cm}}{\text{sec}^2}$ into Hooke's Law $f(x) = kx$, we have:

$$40(981) = k(32) \;\Rightarrow\; k = \frac{4905}{4}$$

The length of 63 cm represents a stretch of 20 cm beyond the spring's natural length, while the length of 93 cm represents a 50-cm stretch. The work done is therefore:

$$\int_{20}^{50} \frac{4905}{4} x\,dx = \left[\frac{4905}{8}x^2\right]_{20}^{50} = 613.125\left[50^2 - 20^2\right] = 1287562.5 \text{ ergs}$$

or about 0.129 joules. ◀

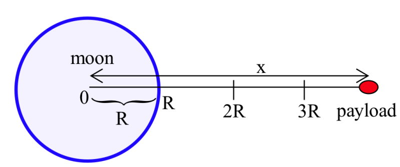
Practice 7. A spring has a natural length of 3 inches when hung from a ceiling, and a force of 2 pounds stretches it to a length of 8 inches. How much work is done stretching the spring from a length of 5 inches to a length of 10 inches?

Lifting a Payload

Calculating the work required to lift a payload from the surface of a moon (or any body with no atmosphere) can be accomplished using a similar computation. Newton's Law of Universal Gravitation says that the gravitational force between two bodies of mass M and m is:

$$F = \frac{GMm}{x^2}$$

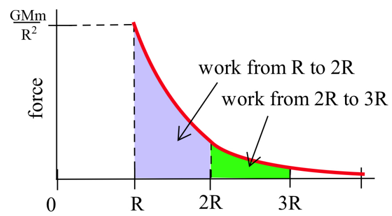
where $G \approx 6.67310^{-11}\,\text{N}\left(\frac{\text{m}}{\text{kg}}\right)^2$ is the **gravitational constant** and x is the distance between (the centers of) the two bodies.



If the moon has a radius of R m and mass M, the payload has mass m and x measures the distance (in meters) from payload to the center of the moon (so $x \geq R$), then the total amount of work done lifting the payload from the surface of the moon (an altitude of 0, where $x = R$) to an altitude of R (where $x = R + R = 2R$) is:

$$\int_{R}^{2R} \frac{GMm}{x^2}\,dx = GMm\left[\frac{-1}{x}\right]_{R}^{2R} = GMm\left[\frac{-1}{2R} + \frac{1}{R}\right] = \frac{GMm}{2R}$$

Practice 8. How much work is required to lift the payload from an altitude of R m above the surface ($x = 2R$) to an altitude of $2R$ m?



The appropriate areas under the force graph (see margin) illustrate why the work to lift the payload from $x = R$ to $x = 2R$ is much larger than the work to lift it from $x = 2R$ to $x = 3R$. In fact, the work to lift the payload from $x = 2R$ to x = $100R$ is $0.49GMmR^{-1}$, which is less than the $0.5GMmR^{-1}$ needed to lift it from $x = R$ to $x = 2R$.

The real-world problem of lifting a payload turns out to be much more challenging, because the rocket doing the lifting must also lift itself (more work) and the mass of the rocket keeps changing as it burns up fuel. Lifting a payload from a moon (or planet) with an atmosphere is even more difficult: the atmosphere produces friction, and the frictional force depends on the density of the atmosphere (which varies with height), the speed of the rocket and the shape of the rocket. Life can get complicated.

Power

Scottish engineer James Watt (1736–1819) devised horsepower to compare the output of steam engines with the power of draft horses.

In physics, **power** is defined as the rate of work done per unit of time. One traditional measurement of power, **horsepower** (abbreviated "hp"), originated with James Watt's determination in 1782 that a horse could turn a mill wheel of radius 12 feet 144 times in an hour while exerting a force of 180 pounds. Such a horse would travel:

$$144\ \frac{\text{rev}}{\text{hr}} \cdot 2\pi(12)\ \frac{\text{ft}}{\text{rev}} \cdot \frac{1}{60}\ \frac{\text{hr}}{\text{min}} = \frac{288\pi}{5}\ \frac{\text{ft}}{\text{min}}$$

and so it would produce work at a rate of:

$$(180\ \text{lb})\left(\frac{288\pi}{5}\ \frac{\text{ft}}{\text{min}}\right) = 10368\pi\ \frac{\text{ft-lb}}{\text{min}} \approx 32572\ \frac{\text{ft-lb}}{\text{min}}$$

which Watt subsequently rounded to:

$$33000\ \frac{\text{ft-lb}}{\text{min}} = 550\ \frac{\text{ft-lb}}{\text{sec}} = 1\ \text{horsepower}$$

$1\ \text{hp} \approx 746\ \text{W}$

The metric unit of power, called a **watt** (abbreviated "W") in Watt's honor, is equivalent to 1 joule per second.

Example 8. How long will it take for a 1-horsepower electric pump to pump all of the liquid in the trough from Example 5 over the wall?

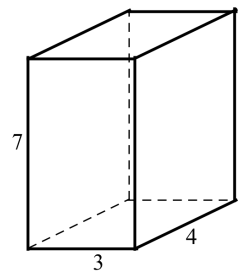
Solution. Power (P) is the rate at which work (W) is done, so:

$$P = \frac{W}{t} \Rightarrow t = \frac{W}{P} = \frac{\frac{14000}{3}\ \text{ft-lbs}}{1\ \text{hp}} = \frac{\frac{14000}{3}\ \text{ft-lbs}}{550\ \frac{\text{ft-lbs}}{\text{sec}}} = \frac{280}{33}\ \text{sec}$$

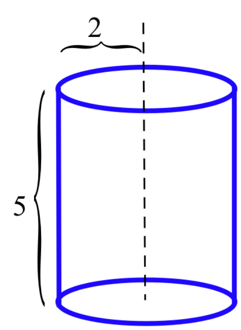
or about 8.5 seconds. ◀

5.4 Problems

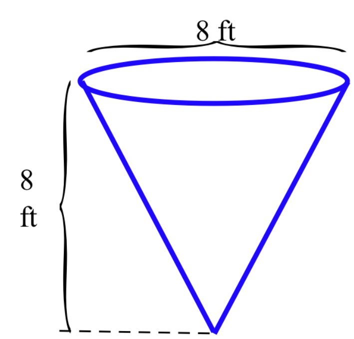
1. A tank 4 feet long, 3 feet wide and 7 feet tall (see below) is filled with water. How much work is required to pump the water out over the top edge of the tank?



2. A tank 4 feet long, 3 feet wide and 6 feet tall is filled with a oil with a density of 60 pounds per cubic foot.
   (a) How much work is needed to pump all of the oil over the top edge of the tank?
   (b) How much work is needed to pump the top 3 feet of oil over the top edge of the tank?
3. A tank 5 m long, 2 m wide and 4 m tall is filled with an oil of density 900 kg/m^3.
   (a) How much work is needed to pump all of the oil over the top edge of the tank?
   (b) How much work is needed to pump the top 10 m^3 of oil over the top edge of the tank?
   (c) How long does it take for a 200-watt pump to empty the tank?
4. A cylindrical aquarium with radius 2 feet and height 5 feet (see below) is filled with salt water (which has a density of 64 pounds per cubic foot).

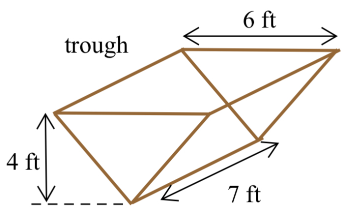


   (a) How much work is done pumping all of the water over the top edge of the aquarium?
   (b) How long does it take for a $\frac{1}{2}$-horsepower pump to empty the tank? A $\frac{1}{4}$-horsepower pump? Which pump does more work?
   (c) If the aquarium is only filled to a height of 4 feet with sea water, how much work is required to empty it?
5. A cylindrical barrel with a radius of 1 m and a height of 6 m is filled with water.
   (a) How much work is done pumping all of the water over the top edge of the barrel?
   (b) How much work is done pumping the top 1 m of water to a point 2 m above the top edge of the barrel?
   (c) How long will it take a $\frac{1}{2}$-horsepower pump to remove half of the water from the barrel?
6. The conical container shown below is filled with oats that weigh 25 pounds per ft^3.
   (a) How much work is done lifting all of the grain over the top edge of the cone?
   (b) How much work is required to lift the top 2 feet of grain over the top edge of the cone?



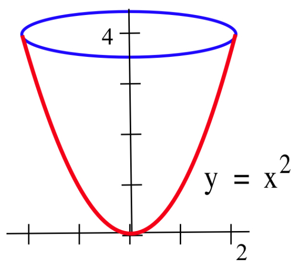
7. If you and a friend share the work equally in emptying the conical container in the previous problem, what depth of grain should the first person leave for the second person to empty?

8. A trough (see below) is filled with pig slop weighing 80 pounds per ft^3. How much work is done lifting all the slop over the top of the trough?



9. In the preceding problem, how much work is done lifting the top 14 ft^3 of slop over the top edge of the trough?

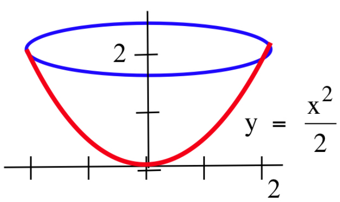
10. The parabolic container shown below (with a height of 4 m) is filled with water.



    (a) How much work is done pumping all of the water over the top edge of the tank?

    (b) How much work is done pumping all of the water to a point 3 m above the top of the tank?

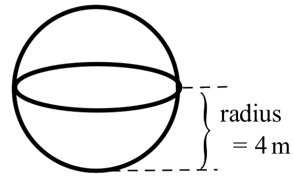
11. The parabolic container shown below (with a height of 2 m) is filled with water.



    (a) How much work is done pumping all of the water over the top edge of the tank?

    (b) How much work is done pumping all of the water to a point 3 m above the top of the tank?

12. A spherical tank with radius 4 m is full of water. How much work is done lifting all of the water to the top of the tank?



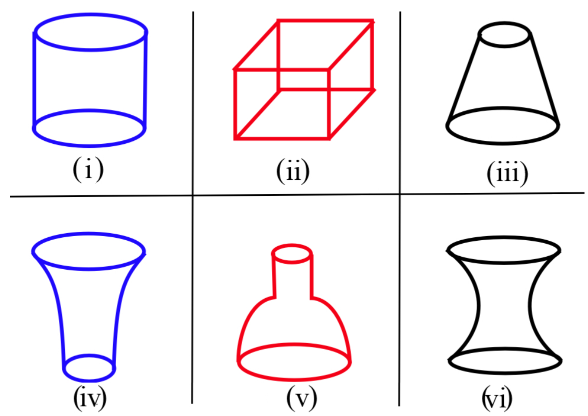
13. The spherical tank shown above is filled with water to a depth of 2 m. How much work is done lifting all of that water to the top of the tank?

14. A student said, "I've got a shortcut for these tank problems, but it doesn't always work. I figure the weight of the liquid and multiply that by the distance I have to move the 'middle point' in the water. It worked for the first five problems and then it didn't."

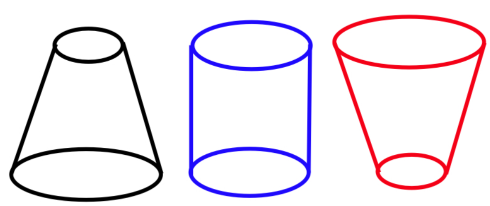
    (a) Does this "shortcut" really give the right answer for the first five problems?

    (b) How do the containers in the first five problems differ from the others?

    (c) For which of the containers shown below will the "shortcut" work?



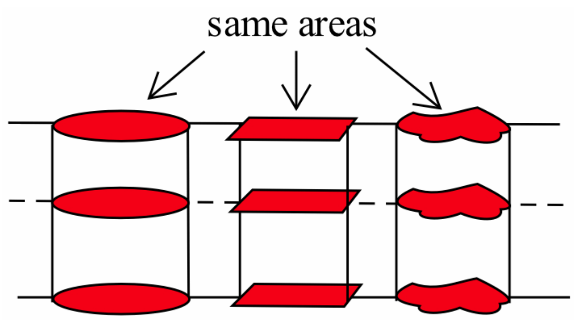
15. All of the containers shown below have the same height and hold the same volume of water.



    (a) Which requires the most work to empty? Justify your response with a detailed explanation.

    (b) Which requires the least work to empty?

16. All of the containers shown below have the same total height and at each height x above the ground they all have the same cross-sectional area.



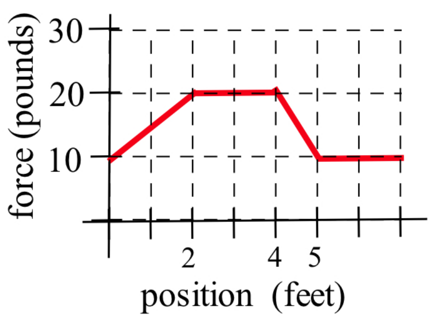
(a) Which requires the most work to empty? Justify your response with a detailed explanation.

(b) Which requires the least work to empty?

17. The figure below shows the force required to move a box along a rough surface. How much work is done pushing the box:

(a) from $x = 0$ to $x = 5$ feet?

(b) from $x = 3$ to $x = 5$ feet?



18. How much work is done pushing the box in the figure above:

(a) from $x = 3$ to $x = 7$ feet?

(b) from $x = 0$ to $x = 7$ feet?

19. A spring requires a force of $6x$ ounces to keep it stretched x inches past its natural length. How much work is done stretching the spring:

(a) from its natural length ($x = 0$) to 3 inches beyond its natural length?

(b) from its natural length to 6 inches beyond its natural length?

20. A spring requires a force of $5x$ dyn to keep it compressed x cm from its natural length. How much work is done compressing the spring:

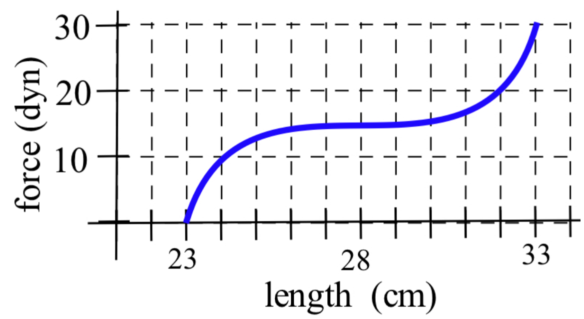
(a) 7 cm from its natural length?

(b) 10 cm from its natural length?

21. The figure below shows the force required to keep a spring that does not obey Hooke's Law stretched beyond its natural length of 23 cm. About how much work is done stretching it:

(a) from a length of 23 cm to a length 33 cm?

(b) from a length of 28 cm to a length 33 cm?



22. Approximately how much work is done stretching the defective spring in the previous problem:

(a) from a length of 23 cm to a length 26 cm?

(b) from a length of 30 cm to a length 35 cm?

23. A 3-kg object attached to a spring hung from the ceiling stretches the spring 15 cm. How much work is done stretching the spring 4 more cm?

24. A 2-lb fish stretches a spring 3 in. How much work is done stretching the spring 3 more inches?

25. A payload of mass 100 kg sits on the surface of the asteroid Ceres, a dwarf planet that is the largest object in the asteroid belt between Mars and Jupiter. Ceres has diameter 950 km and mass 896×10^{18} kg. How much work is required to lift the payload from the asteroid's surface to an altitude of (a) 10 km? (b) 100 km? (c) 500 km?

26. Calculate the amount of work required to lift *you* from the surface of the Earth's moon to an altitude of 100 km above the moon's surface. (The moon's radius is approximately 1,737.5 km and its mass is about 7.35×10^{22} kg.)

27. Calculate the amount of work required to lift *you* from the surface of the Earth's moon (see previous problem) to an altitude of:
    (a) 200 km.
    (b) 400 km.
    (c) 10,000 km.

28. An object located at the origin repels you with a force inversely proportional to your distance from the object (so that $f(x) = \frac{k}{x}$ where x is your distance from the object, measured in feet). When you are 10 feet away from the origin, the repelling force is 0.1 pound. How much work must you do to move:
    (a) from $x = 20$ to $x = 10$?
    (b) from $x = 10$ to $x = 1$?
    (c) from $x = 1$ to $x = 0.1$?

29. An object located at the origin repels you with a force inversely proportional to the square of your distance from the object (so that $f(x) = \frac{k}{x^2}$ where x is your distance from the object, measured in meters). When you are 10 m away from the origin, the repelling force is 0.1 N. How much work must you do to move:
    (a) from $x = 20$ to $x = 10$?
    (b) from $x = 10$ to $x = 1$?
    (c) from $x = 1$ to $x = 0.1$?

30. A student said "I've got a 'work along a line' shortcut that always seems to work. I figure the average force and then multiply by the total distance. Will it always work?"
    (a) Will it? Justify your answer. (Hint: What is the formula for "average force"?)
    (b) Is this a shortcut?

Work Along a Curved Path

If the location of a moving object is defined parametrically as $x = x(t)$ and $y = y(t)$ for $a \le t \le b$ (where t often represents time), and the force required to overcome friction at time t is given as $f(t)$, we can represent the work done moving along the (possibly curved) path as a definite integral. Partitioning $[a, b]$ into n subintervals of the form $[t_{k-1}, t_k]$, we can choose any c_k in $[t_{k-1}, t_k]$ and approximate the force required on $[t_{k-1}, t_k]$ by $f(c_k)$ so that the work done between $t = t_{k-1}$ and $t = t_k$ is approximately:

$$f(c_k) \cdot \sqrt{[\Delta x_k]^2 + [\Delta y_k]^2} = f(c_k) \cdot \sqrt{\left[\frac{\Delta x_k}{\Delta t_k}\right]^2 + \left[\frac{\Delta y_k}{\Delta t_k}\right]^2} \cdot \Delta t_k$$

The total work done between times $t = a$ and $t = b$ is then:

$$\sum_{k=1}^{n} f(c_k) \cdot \sqrt{\left[\frac{\Delta x_k}{\Delta t_k}\right]^2 + \left[\frac{\Delta y_k}{\Delta t_k}\right]^2} \cdot \Delta t_k \longrightarrow \int_a^b f(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

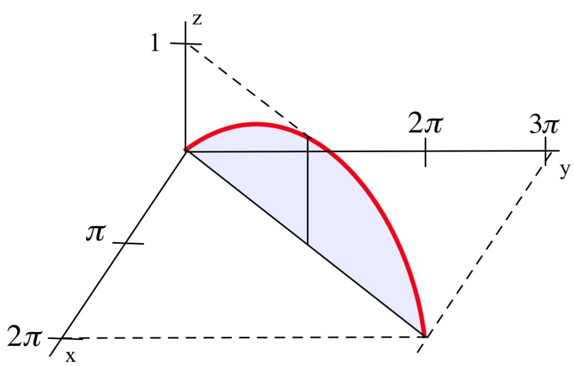
In 31–35, find the work done as an object is moved along the given parametric path (with distance measured in meters), where $f(t)$ (in newtons) is the force required at time t (in seconds). If necessary, approximate the value of the integral using technology.

31. $f(t) = t$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$

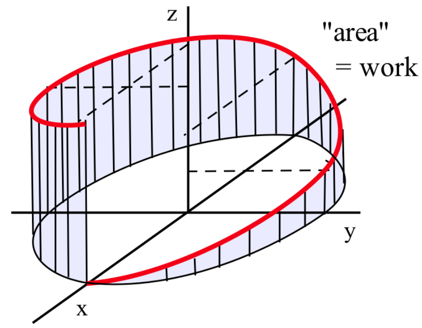
32. $f(t) = t$, $x(t) = t$, $y(t) = t^2$, $0 \le t \le 1$

33. $f(t) = t$, $x(t) = t^2$, $y(t) = t$, $0 \le t \le 1$

34. $f(t) = \sin(t)$, $x(t) = 2t$, $y(t) = 3t$, $0 \leq t \leq \pi$:



35. $f(t) = t$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \leq t \leq 2\pi$:



(Can you find a geometric way to calculate the shaded area?)

5.4 Practice Answers

1. The work done lifting the person and the stretcher is:

$$(130 \text{ lb} + 10 \text{ lb}) \cdot (30 \text{ ft}) = 4200 \text{ ft-lbs}$$

The work done lifting a small piece of cable with length Δx ft at an initial height of x feet above the ground is:

$$\left(2\ \frac{\text{lb}}{\text{ft}}\right)(\Delta x \text{ ft})\Big((30 - x) \text{ ft}\Big) = (60 - 2x)\Delta x \text{ ft-lbs}$$

so the work done lifting the cable is:

$$\sum_{k=1}^{n} (60 - 2x)\Delta x \longrightarrow \int_0^{30} (60 - 2x)\, dx = 900 \text{ ft-lbs}$$

and the total work is $4200 + 900 = 5100$ ft-lbs.

2. The work done lifting the person and the stretcher is:

$$(50 \text{ kg} + 5 \text{ kg}) \cdot \left(9.81\ \frac{\text{m}}{\text{sec}^2}\right) \cdot (30 \text{ m}) = (55 \text{ N})\,(30 \text{ m}) = 16186.5 \text{ J}$$

The work done lifting a small piece of cable with length Δx m at an initial height of x m above the ground is:

$$\left(\frac{1}{3}\ \frac{\text{kg}}{\text{m}}\right)\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)(\Delta x \text{ m})\Big((30 - x) \text{ m}\Big) = 3.27(30 - x)\Delta x \text{ J}$$

so the work done lifting the cable is:

$$\sum_{k=1}^{n} 3.27(30 - x)\Delta x \longrightarrow 3.27\int_0^{30} (30 - x)\, dx = 1471.5 \text{ J}$$

and the total work is $16186.5 + 1471.5 = 17658$ J.

3. The total work done is approximately

$$\left[\pi(1.4)^2(3.5) + \pi(1.6)^2(4.5) + \pi(1.5)^2(5.5) + \pi(1.0)^2(6.5)\right](0.5787)$$

or 67.73 oz-in $\approx$ 0.35 ft-lbs.

4. We can use the same integral as in the solution to Example 4, but instead integrate from $y = 3$ to $y = 6$:

$$W = 109\pi \int_3^6 \left(10y^2 - y^3\right) dy = 109\pi \left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_3^6$$
$$= 109\pi \left[(720 - 324) - \left(90 - \frac{81}{4}\right)\right] = 35561.25\pi \text{ erg}$$

or about $111,719$ erg $= 0.0111719$ J.

5. The total amount of liquid in the trough is $\frac{1}{2} \cdot 4 \cdot 2 \cdot 5 = 20$ ft^3, so we need to lift the top 10 ft^3 of liquid out of the trough. To find the height separating the bottom 10 ft^3 of liquid from the rest, we can recall that (from our similar-triangles computation), the width at height h is $w = \frac{h}{2}$, so the volume of liquid between height $y = 0$ and height $y = h$ is:

$$10 = \frac{1}{2} \cdot h \cdot \frac{h}{2} \cdot 5 \Rightarrow h^2 = 8 \Rightarrow h = 2\sqrt{2}$$

The work to lift the top 10 ft^3 of liquid is thus:

$$175 \int_{2\sqrt{2}}^4 \left(6y - y^2\right) dy = 175 \left[3y^2 - \frac{1}{3}y^3\right]_{2\sqrt{2}}^4$$
$$= 175 \left[\left(48 - \frac{64}{3}\right) - \left(24 - \frac{16\sqrt{2}}{3}\right)\right]$$

or about $1,786.6$ ft-lbs.

6. We can use the same integral as in the solution to Example 6, but instead integrate from $x = 5$ to $x = 10$:

$$\int_5^{10} 7x\, dx = \left[\frac{7}{2}x^2\right]_5^{10} = 350 - \frac{175}{2} = 262.5 \text{ in-lbs} = 21.875 \text{ ft-lbs}$$

7. According to Hooke's Law, 2 lb $= k \cdot (8 \text{ in} - 3 \text{ in}) \Rightarrow k = \frac{2}{5}$, so stretching the spring from $5 - 3 = 2$ in to $10 - 3 = 7$ in beyond its natural length requires:

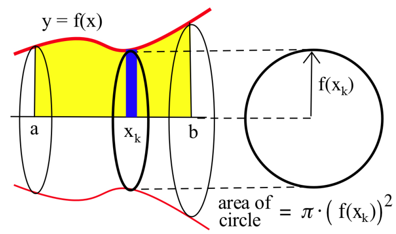
$$\int_2^7 \frac{2}{5}x\, dx = \left[\frac{1}{5}x^2\right]_2^7 = 9 \text{ in-lb} = \frac{3}{4} \text{ ft-lb}$$

8. The work required to lift the payload from $x = 2R$ to $x = 3R$ is:

$$\int_{2R}^{3R} \frac{GMm}{x^2}\, dx = GMm\left[\frac{-1}{x}\right]_{2R}^{3R} = GMm\left[\frac{-1}{3R} + \frac{1}{2R}\right] = \frac{GMm}{6R}$$

5.5 *Volumes: Tubes*

In Section 5.2, we devised the "disk" method to find the volume swept out when a region is revolved about a line. To find the volume swept out when revolving a region about the x-axis (see margin), we made cuts perpendicular to the x-axis so that each slice was (approximately) a "disk" with volume $\pi\,(\text{radius})^2 \cdot (\text{thickness})$. Adding the volumes of these slices together yielded a Riemann sum. Taking a limit as the thicknesses of the slices approached 0, we obtained a definite integral representation for the exact volume that had the form:



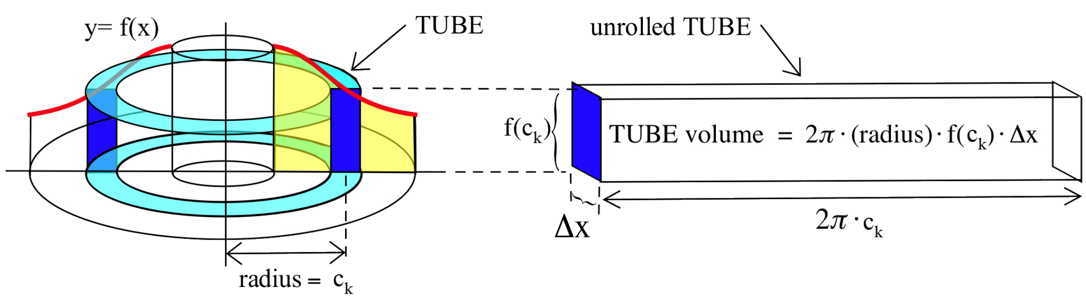
$$\int_a^b \pi\,[f(x)]^2\,dx$$

The disk method, while useful in many circumstances, can be cumbersome if we want to find the volume when a region defined by a curve of the form $y = f(x)$ is revolved about the y-axis or some other vertical line. To revolve the region about the y-axis, the disk method requires that we rewrite the original equation $y = f(x)$ as $x = g(y)$. Sometimes this is easy: if $y = 3x$ then $x = \frac{y}{3}$. But sometimes it is not easy at all: if $y = x + e^x$, then we cannot solve for x as an elementary function of y.

Refer to Examples 6(b) and 6(c) from Section 5.2 to refresh your memory.

The "Tube" Method

Partition the x-axis (as we did in the "disk" method) to cut the region into thin, almost-rectangular vertical "slices." When we revolve one of these slices about the y-axis (see below), we can approximate the volume of the resulting "tube" by cutting the "wall" of the tube and rolling it out flat:



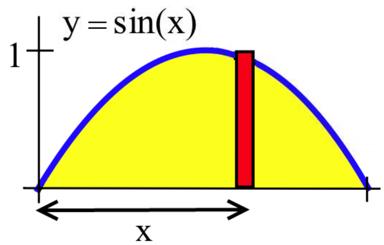
to get a thin, solid rectangular box. The volume of the tube is approximately the same as the volume of the solid box:

$$\begin{aligned} V_{\text{tube}} \approx V_{\text{box}} &= (\text{length}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2\pi \cdot [\text{radius}]) \cdot (\text{height}) \cdot (\Delta x_k) \\ &= (2\pi c_k) \Big(f(c_k) \Big) \cdot \Delta x_k \end{aligned}$$

where c_k is (as usual) any point chosen from the interval $[x_{k-1}, x_k]$.

The volume swept out when we revolve the whole region about the y-axis is (approximately) the sum of the volumes of these "tubes," which is a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} (2\pi c_k)\left(f(c_k)\right)\cdot \Delta x_k \longrightarrow \int_a^b 2\pi x \cdot f(x)\,dx$$



Example 1. Use a definite integral to represent the volume of the solid generated by rotating the region between the graph of $y = \sin(x)$ (for $0 \le x \le \pi$) and the x-axis around the y-axis.

Solution. Slicing this region vertically (see margin for a representative slice), yields slices with width Δx and height $\sin(x)$. Rotating a slice located x units away from the y-axis results in a "tube" with volume:

$$2\pi\,(\text{radius})\,(\text{height})\,(\text{thickness}) = 2\pi\,(x)\left(\sin(x)\right)\Delta x$$

where the radius of the tube (x) is the distance from the slice to the y-axis and the height of the tube is the height of the slice ($\sin(x)$). Adding the volumes of all such tubes yields a Riemann sum that converges to a definite integral:

$$\int_0^{\pi} 2\pi\,(\text{radius})\,(\text{height})\;dx = \int_0^{\pi} 2\pi x \sin(x)\,dx$$

You'll learn how to find an antiderivative for $x\sin(x)$ in Section 8.2 or, for now, you can look for this pattern in Appendix I.

We don't (yet) know how to find an antiderivative for $x\sin(x)$ but we can use technology (or a numerical method from Section 4.9) to compute the value of the integral, which turns out to be $2\pi^2 \approx 19.74$. ◀

Practice 1. Use a definite integral to compute the volume of the solid generated by rotating the region in the first quadrant bounded by $y = 4x - x^2$ about the y-axis.

If we had sliced the region in Example 1 horizontally instead of vertically, the rotated slices would have resulted in "washers"; applying the "washer" method from Section 5.2 yields the integral:

$$\int_0^1 \pi\left[(\pi - \arcsin(y))^2 - (\arcsin(y))^2\right]\,dy$$

Furthermore, the washer-method integral in this situation is more challenging to set up than the integral using the tube method, so the tube method is the most efficient choice on all counts.

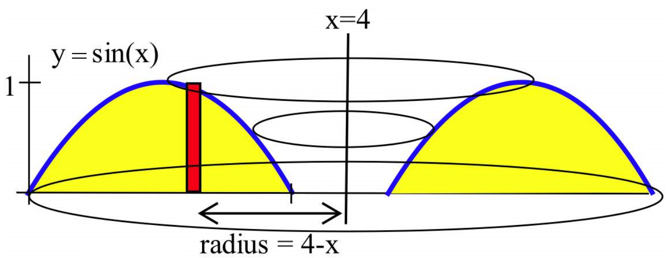
The value of this integral is also $2\pi^2$, but finding an antiderivative for this integrand will be much more challenging than finding an antiderivative for $x\sin(x)$.

Rotating About Other Axes

The "tube" method extends easily to solids generated by rotating a region about any vertical line (not just the y-axis).

Example 2. Use a definite integral to represent the volume of the solid generated by rotating the region between the graph of $y = \sin(x)$ (for $0 \le x \le \pi$) and the x-axis around the line $x = 4$.

Solution. The region is the same as the one in Example 1, but here we're rotating that region about a different vertical line:



Vertical slices again generate tubes when rotated about $x = 4$; the only difference here is that the radius for a slice located x units away from y-axis is now $4 - x$ (the distance from the axis of rotation to the slice). The volume integral becomes:

$$\int_0^{\pi} 2\pi\,(\text{radius})\,(\text{height})\;dx = \int_0^{\pi} 2\pi(4-x)\cdot\sin(x)\,dx$$

which turns out to be $2\pi(8-\pi) \approx 4.8584$. ◀

Use technology (or a table of integrals) to verify this numerical result.

Practice 2. Use a definite integral to compute the volume of the solid generated by rotating the region in the first quadrant bounded by $y = 4x - x^2$ about the line $x = -7$.

More General Regions

The "tube" method also extends easily to more general regions.

Volumes of Revolved Regions ("Tube Method")

If	the region constrained by the graphs of $y = f(x)$ and $y = g(x)$ and the interval $[a, b]$ is revolved about a vertical line $x = c$ that does not intersect the region
then	the volume of the resulting solid is:

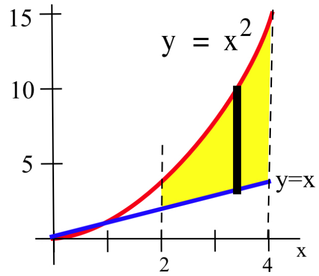
$$V = \int_a^b 2\pi \cdot |x - c| \cdot |f(x) - g(x)|\;dx$$

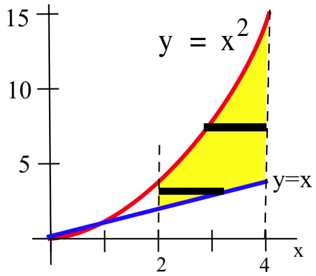
Many textbooks refer to this method as the "method of cylindrical shells" or the "shell method," but "cylindrical shells" is a mouthful (compared with "tube") and "shell method" is not precise, as shells are not necessarily cylindrical.

The absolute values appear in the general formula because the radius and the height are both distances, hence both must be positive.

You can ensure that these ingredients in your tube-method integral will be positive by always subtracting smaller values from larger values: think "right − left" for x-values and "top − bottom" for y-values.

Example 3. Compute the volume of the solid generated by rotating the region between the graphs of $y = x$ and $y = x^2$ for $2 \le x \le 4$ around the y-axis using (a) vertical slices and (b) horizontal slices.





Solution. (a) Vertical slices (see margin) result in tubes when rotated about the y-axis, and a slice x units away from the y-axis results in a tube of radius x and height $x^2 - x$, so the volume of the solid is:

$$\int_2^4 2\pi x \left[x^2 - x\right] dx = 2\pi \int_2^4 \left[x^3 - x^2\right] dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_2^4$$
$$= 2\pi \left[\left(64 - \frac{64}{3}\right) - \left(4 - \frac{8}{3}\right)\right] = \frac{248\pi}{3}$$

or about 259.7. (b) Horizontal slices result in washers when rotated about the y-axis, but we have a new problem: the lower slices (where $2 \le y \le 4$) extend from the line $x = 2$ on the left to the line $y = x$ on the right, while the upper slices (where $4 \le y \le 16$) extend from the parabola $y = x^2$ on the left to the line $x = 4$ on the right. This requires us to use *two* integrals to compute the volume:

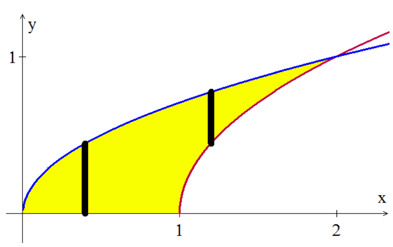
$$\int_{y=2}^{y=4} \pi \left[y^2 - 2^2\right] dy + \int_{y=4}^{y=16} \pi \left[4^2 - (\sqrt{y})^2\right] dy$$

Evaluating these integrals is straightforward, but setting them up was more time-consuming than using the tube method.

Evaluating these integrals also results in a volume of $\frac{248\pi}{3} \approx 259.7$. ◀

Practice 3. Find the volume of the solid formed by rotating the region between the graphs of $y = x$ and $y = x^2$ for $2 \le x \le 4$ around $x = 13$.

Practice 4. Compute the volume of the solid generated by rotating the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$, $y = x + 1$ and $x = 4$ around (a) the y-axis (b) the x-axis.



Example 4. Compute the volume of the solid swept out by rotating the region in the first quadrant between the graphs of $y = \sqrt{\frac{x}{2}}$ and $y = \sqrt{x-1}$ about the x-axis.

Solution. Graphing the region (see margin), it is apparent that the curves intersect where:

$$\sqrt{\frac{x}{2}} = \sqrt{x-1} \Rightarrow \frac{x}{2} = x - 1 \Rightarrow x = 2$$

Slicing the region vertically results in two cases: when $0 \le x \le 1$, the slice extends from the x-axis to the curve $y = \sqrt{\frac{x}{2}}$; when $1 \le x \le 2$, the slice extends from $y = \sqrt{x-1}$ to $y = \sqrt{\frac{x}{2}}$. Rotating the first type of slice about the x-axis results in a disk; rotating the second type of slice about the x-axis results in a washer. Using the disk method for the first interval and the washer method for the second interval, the volume of the solid is:

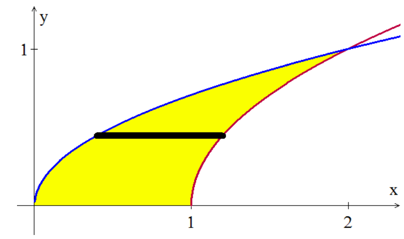
Both types of slices are perpendicular to the x-axis, so the width of each slice is of the form Δx and our integrals should involve dx.

$$\int_0^1 \pi \left[\sqrt{\frac{x}{2}}\right]^2 dx + \int_1^2 \pi \left[\left(\sqrt{\frac{x}{2}}\right)^2 - \left(\sqrt{x-1}\right)^2\right] dx$$

Evaluating these integrals is straightforward:

$$\pi \int_0^1 \frac{x}{2}\,dx + \pi \int_1^2 \left[\frac{x}{2} - (x-1)\right]\,dx = \pi\left[\frac{x^2}{4}\right]_0^1 + \pi\left[x - \frac{x^2}{4}\right]_1^2 = \frac{\pi}{2}$$

If you had instead sliced the region horizontally, you would only need one type of slice (see margin). Rotating a horizontal slice around the x-axis results in a tube. Because this slice is perpendicular to the y-axis, the thickness of the slice is of the form Δy, so the tube-method integral will include a dy and we will need to formulate the radius and "height" of the tube in terms of y. The radius of the slice is merely y, the distance between the slice and the x-axis. The "height" of the slice is its length, which is the distance between the two curves. The left-hand curve is:



$$y = \sqrt{\frac{x}{2}} \Rightarrow y^2 = \frac{x}{2} \Rightarrow x = 2y^2$$

and the right-hand curve is:

$$y = \sqrt{x-1} \Rightarrow y^2 = x - 1 \Rightarrow x = y^2 + 1$$

so the distance between the two curves is:

$$\left(y^2+1\right) - \left(2y^2\right) = 1 - y^2$$

This application of the tube method rotates a horizontal slice around a horizontal axis; in previous tube-method applications we have only rotated a vertical slice about a vertical axis. Either option results in a tube, and the general formula on page 433 can be further extended to this new situation — as we have done here — by swapping the roles of x and y,

The curves intersect where: $y^2 + 1 = 2y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$; from the graph we can see that the bottom of the region corresponds to $y = 0$ and the top of the region is at $y = 1$. Applying the tube method, the volume of the solid is:

$$\int_{y=0}^{y=1} 2\pi y \cdot \left[1 - y^2\right]\,dy = 2\pi \int_0^1 \left[y - y^3\right]\,dy = 2\pi\left[\frac{y}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

which agrees with the result above from the disk+washer method. ◀

Practice 5. Compute the volume of the solid swept out by rotating the region in the first quadrant between the graphs of $y = \sqrt{\frac{x}{2}}$ and $y = \sqrt{x-1}$ about (a) the line $x = 5$ (b) the line $y = 5$.

Which Method Is Best?

In theory, both the washer method and the tube method will work for any volume-of-revolution problem involving a horizontal or vertical axis. In practice, however, one of these methods is usually easier to use than the other — but which one is easier depends on the particular region and type of axis. As we have seen, challenges may include:

We will investigate a method for computing volumes of solids formed by rotating a region around "tilted" axes in Section 5.6.

- The necessity to split the region into two (or more) pieces, resulting in two (or more) integrals.

- The difficulty (or impossibility) of solving an equation of the form $y = f(x)$ for x or an equation of the form $x = g(y)$ for y.
- The difficulty (or impossibility) of finding an antiderivative for the resulting integrand.

With experience (and lots of practice) you will begin to develop an intuition for which method might be the best choice for a particular situation. Sketching the region along with representative horizontal and vertical slices is a vital first step.

The method that avoids the need to split the region up into more than one piece is often—but not always—the superior choice. Avoiding the need to find an inverse function for a boundary curve should also be a priority. Finally, if you need an exact value and one method results in a challenging antiderivative search, start over and try the other method.

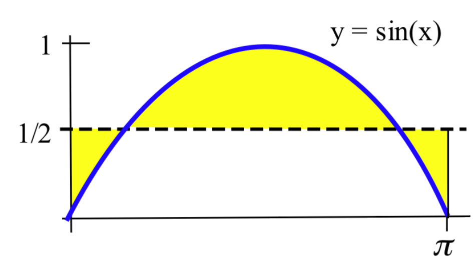
5.5 Problems

In Problems 1–6, sketch the region and calculate the volume swept out when the region is revolved about the specified vertical line.

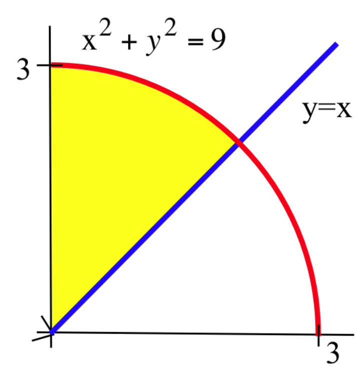
1. The region in the first quadrant between the curve $y = \sqrt{1-x^2}$ and the x-axis is rotated about the y-axis.
2. The region in the first quadrant between the curve $y = 2x - x^2$ and the x-axis is rotated about the y-axis.
3. The region in the first quadrant between between $y = 2x$ and $y = x^2$ for $0 \le x \le 3$ is rotated about the line $x = 4$.
4. The region in the first quadrant between the curve $y = \dfrac{1}{1+x^2}$, the x-axis and the line $x = 3$ is rotated about the y-axis.
5. The region between $y = \dfrac{1}{x}$, $y = \dfrac{1}{3}$ and $x = 1$ is rotated about the line $x = 5$.
6. The region between $y = x$, $y = 2x$, $x = 1$ and $x = 3$ is rotated about the line $x = 1$.

In Problems 7–11, use a definite integral to represent the volume swept out when the given region is revolved about the y-axis, then use technology to evaluate the integral.

7. The region in the first quadrant between the graphs of $y = \ln(x)$, $y = x$ and $x = 4$.
8. The region in the first quadrant between the graphs of $y = e^x$, $y = x$ and $x = 2$.
9. The region between $y = x^2$ and $y = 6 - x$ for $1 \le x \le 4$.
10. The shaded region in the figure below.



11. The shaded region in the figure below.

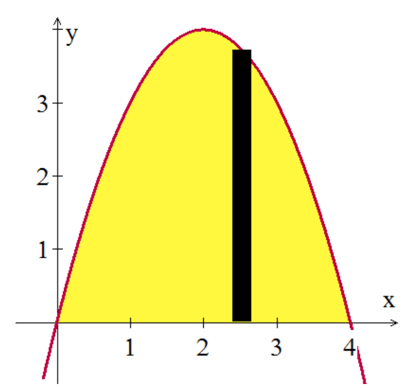


5.5 Problems

In Problems 12–30, set up an integral to calculate the volume swept out when the region between the given curves is rotated about the specified axis, using any appropriate method (disks, washers, tubes). If possible, work out an exact value of the integral; otherwise, use technology to find an approximate numerical value.

12. $y = x$, $y = x^4$, about the y-axis
13. $y = x^2$, $y = x^4$, about the y-axis
14. $y = x^2$, $y = x^4$, about the x-axis
15. $y = \sin(x^2)$, $y = 0$, $x = 0$, $x = \sqrt{\pi}$, about $x = 0$
16. $y = \cos(x^2)$, $y = 0$, $x = 0$, $x = \frac{\sqrt{\pi}}{2}$, about $x = 0$
17. $y = \dfrac{1}{\sqrt{1-x^2}}$, $y = 0$, $x = 0$, $x = \frac{1}{2}$, about $x = 0$
18. $y = \dfrac{1}{\sqrt{1-x^2}}$, $y = 0$, $x = 0$, $x = \frac{1}{2}$, about $y = 0$
19. $y = x$, $y = x^4$, about $x = 3$
20. $y = x$, $y = x^4$, about $y = 3$
21. $y = x$, $y = x^4$, about $y = -3$
22. $y = x$, $y = x^4$, about $x = -3$
23. $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = 0$, $x = 1$ about $x = 2$
24. $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = 1$, $x = \sqrt{3}$, about $x = 2$
25. $y = \dfrac{1}{1+x^2}$, $y = 1$, $x = 1$, about $x = -2$
26. $y = \dfrac{1}{1+x^2}$, $y = \frac{1}{2}$, about $x = 1$
27. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $x = 4$
28. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $x = -4$
29. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $y = 4$
30. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $y = -4$

5.5 Practice Answers

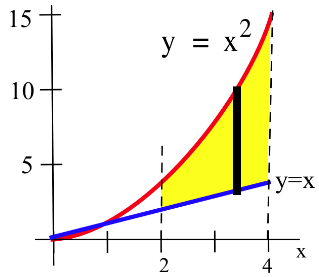


1. Graph the region (see margin) and note that the curve $y = 4x - x^2$ intersects the x-axis where $4x - x^2 = 0 \Rightarrow x(4-x) = 0 \Rightarrow x = 0$ or $x = 4$. Rotating a vertical slice around the y-axis results in a tube with radius x (the distance between the slice and the y-axis) and height $4x - x^2$ so the volume of the solid is:

$$\int_0^4 2\pi x\left(4x - x^2\right)\,dx = 2\pi\int_0^4 \left[4x^2 - x^3\right]\,dx$$
$$= 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \frac{128\pi}{3} \approx 134$$

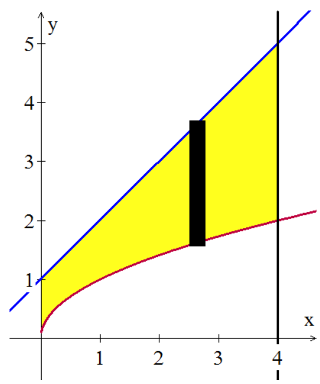
2. The region here is identical to the region in Practice 1, but we are now rotating a slice around the axis $x = -7$, so the radius of the resulting tube is $x - (-7) = x + 7$ (the distance from the slice at location x to the axis of rotation). The volume of the solid is therefore:

$$\int_0^4 2\pi(x+7)\left(4x - x^2\right)\,dx = 2\pi\int_0^4 \left[28x - 3x^2 - x^3\right]\,dx$$
$$= 2\pi\left[14x^2 - x^3 - \frac{1}{4}x^4\right]_0^4 = 192\pi \approx 603$$



3. Rotating a vertical slice (see margin figure) about the line $x = 13$ results in a tube with radius $13 - x$ and height $x^2 - x$, so the volume of the solid is:

$$\int_2^4 2\pi(13-x)(x^2-x)\,dx = 2\pi\int_2^4 \left[-13x+14x^2-x^3\right]\,dx$$
$$= 2\pi\left[-\frac{13}{2}x^2+\frac{14}{3}x^3-\frac{1}{4}x^4\right]_2^4 = 2\pi\left[\frac{392}{3}-\frac{10}{3}\right] = \frac{764\pi}{3} \approx 800$$

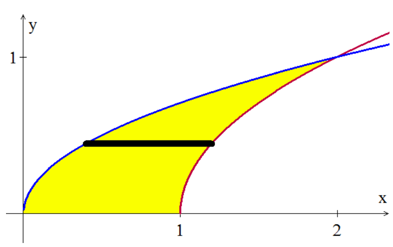


4. Graph the region (see margin) and draw a representative vertical slice. (Horizontal slices would require splitting the region into two pieces—why?) (a) Rotating the vertical slice about the y-axis results in a tube of radius x (the distance from the slice to the y-axis) and height $(x+1) - \sqrt{x}$, and the region sits between $x = 0$ and $x = 4$ so the volume of the solid is:

$$\int_0^4 2\pi x\left[x+1-x^{\frac{1}{2}}\right]\,dx = 2\pi\int_0^4 \left[x^2+x-x^{\frac{3}{2}}\right]\,dx$$
$$= 2\pi\left[\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{496\pi}{15} \approx 104$$

(b) Rotating the vertical slice around the x-axis results in a washer with big radius $x+1$ (the distance from the x-axis to the curve farthest from the x-axis) and small radius $\sqrt{x}$ (the distance from the x-axis to the closer curve) so the volume of the solid is:

$$\int_0^4 \pi\left[(x+1)^2-\left(\sqrt{x}\right)^2\right]\,dx = \pi\int_0^4 \left[x^2+x+1\right]\,dx = \frac{100\pi}{3} \approx 105$$



5. This region is the same as the one in Example 4, where it was apparent that slicing horizontally resulted in a single type of slice (compared with vertical slices, which required us to split the region into two pieces). (a) Rotating a horizontal slice around the vertical line $x = 5$ results in washers with thickness Δy (so our integral will involve dy), big radius $5 - 2y^2$ (the distance between the axis of rotation and the farthest curve) and small radius $5 - \left(y^2+1\right) = 4 - y^2$ (the distance between the axis of rotation and closest curve). Applying the washer method, the volume of the solid is:

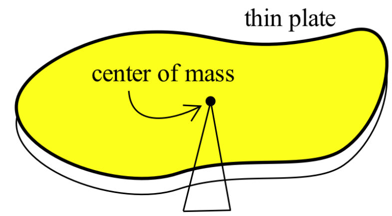
$$\int_0^1 \pi\left[\left(5-2y^2\right)^2-\left(4-y^2\right)^2\right]\,dy = \frac{14\pi}{5} \approx 8.8$$

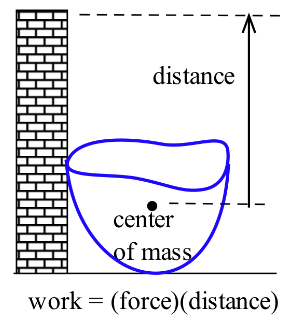
(b) Rotating a horizontal slice around the horizontal line $y = 5$ results in a tube of radius $5 - y$ (the distance between the slice and the axis of rotation) and "height" $1 - y^2$ (the length of the slice). Applying the tube method, the volume of the solid is:

$$\int_0^1 2\pi(5-y)\left(1-y^2\right)\,dy = \frac{37\pi}{6} \approx 19.4$$

5.6 *Moments and Centers of Mass*

This section develops a method for finding the center of mass of a thin, flat shape — the point at which the shape will balance without tilting (see margin). Centers of mass are important because in many applied situations an object behaves as though its entire mass is located at its center of mass. For example, the work required to pump the water in a tank to a higher point is the same as the work required to move a small object with the same mass located at the tank's center of mass to the higher point (see margin), a much easier problem (if we know the mass and the center of mass of the water). Volumes and surface areas of solids of revolution can also become easy to calculate if we know the center of mass of the region being revolved.

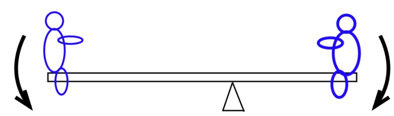




Point-Masses in One Dimension

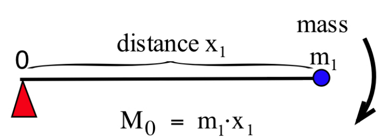
Before investigating the centers of mass of complicated regions, we consider **point-masses** (and systems of point-masses), first in one dimension and then in two dimensions.

Two people with different masses can position themselves on a seesaw so that the seesaw balances (see margin). The person on the right causes the seesaw to "want to turn" clockwise about the fulcrum, and the person on the left causes it to "want to turn" counterclockwise. If these two "tendencies" are equal, the seesaw will balance on the fulcrum. A measure of this tendency to turn about the fulcrum is called the **moment** about the fulcrum of the system, and its magnitude is the product of the mass and the distance from the mass to the fulcrum.



In this seesaw example, we need to imagine that the seesaw is constructed using a very lightweight — yet sturdy — substance, so that its mass is negligible compared with the masses of the two people.

In general, the **moment about the origin**, M_0, produced by a mass m_1 at a location x_1 is $m_1 \cdot x_1$, the product of the mass and the "signed distance" of the point-mass from the origin (see margin). For a **system** of n masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, the total mass of the system is:



$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k$$

and the moment about the origin of the system is:

$$M_0 = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot x_k$$

If the moment about the origin is positive, then the system "tends to rotate" clockwise about the origin. If the moment about the origin is negative, then the system "tends to rotate" counterclockwise about the origin. If the moment about the origin is zero, then the system does not tend to rotate in either direction about the origin: it balances on a fulcrum located at the origin.

The **moment about a point $x = p$**, M_p, produced by a mass m_1 at location $x = x_1$ is the product of the mass and the signed distance of x_1 from the point p: $m_1 \cdot (x_1 - p)$. The moment about a point $x = p$ produced by masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, is:

$$M_p = m_1(x_1 - p) + m_2(x_2 - p) + \cdots + m_n(x_n - p) = \sum_{k=1}^{n} m_k(x_k - p)$$

You have seen this "bar" notation before, in conjunction with the average value of a function. Here we can think of $\overline{x}$ as a "weighted average".

The point at which a system of point-masses balances is called the **center of mass** of the system, written $\overline{x}$ (pronounced "x-bar"). Because the system balances at $x = \overline{x}$, the moment about $\overline{x}$, $M_{\overline{x}}$, must be 0. Using this fact (and summation properties), we obtain a formula for $\overline{x}$:

We can factor $\overline{x}$ out of the second sum because it is constant.

$$\begin{aligned} 0 = M_{\overline{x}} &= \sum_{k=1}^{n} m_k \cdot (x_k - \overline{x}) = \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \left[\sum_{k=1}^{n} m_k \cdot \overline{x}\right] \\ &= \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \overline{x} \cdot \left[\sum_{k=1}^{n} m_k\right] = M_0 - \overline{x} \cdot m \end{aligned}$$

so $\overline{x} \cdot m = M_0$ and solving for $\overline{x}$ yields the following formula.

> The **center of mass of a system of point-masses** $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$ is:
>
> $$\overline{x} = \frac{M_0}{m} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k}$$

A single point-mass with mass m (the total mass of the system) located at $\overline{x}$ (the center of mass of the system) produces the same moment about any point on the line as the whole system:

$$\begin{aligned} M_p &= \sum_{k=1}^{n} m_k(x_k - p) = \left[\sum_{k=1}^{n} m_k x_k\right] - p\left[\sum_{k=1}^{n} m_k\right] = M_0 - pm \\ &= m\left(\frac{M_0}{m} - p\right) = m(\overline{x} - p) \end{aligned}$$

For many purposes, we can think of the mass of the entire system as being "concentrated at $\overline{x}$."

k	m_k	x_k
1	2	-3
2	3	4
3	1	6
4	5	-2
5	3	4

Example 1. Find the center of mass of the system consisting of the first three point-masses listed in the margin table.

Solution. $m = 2 + 3 + 1 = 6$ and $M_0 = (2)(-3) + (3)(4) + (1)(6) = 12$ so:

$$\overline{x} = \frac{M_0}{m} = \frac{12}{6} = 2$$

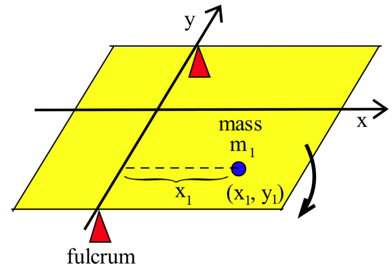
The system of three point-masses will balance on a fulcrum at $x = 2$. ◀

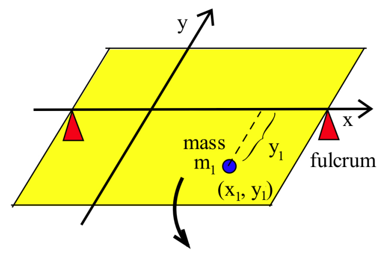
Practice 1. Find the center of mass of the system consisting of the last three point-masses listed in the margin table.

Point-Masses in Two Dimensions

The concepts of moments and centers of mass extend nicely from one dimension to a system of masses located at points in a plane. For a "knife edge" fulcrum located along the y-axis (see margin), the moment of a point-mass with mass m_1 located at the point (x_1, y_1) is the product of the mass and the signed distance of the point-mass from the y-axis: $m_1 \cdot x_1$. This "tendency to rotate about the y-axis" is called the **moment about the y-axis**, written M_y. Here, $M_y = m_1 \cdot x_1$. Similarly, a point-mass with mass m_1 located at the point (x_1, y_1) has a **moment about the x-axis** (see margin): $M_x = m_1 \cdot y_1$.

As in the seesaw example, we need to imagine that the point-masses are sitting on a thin — yet strong — plate of negligible mass compared with the point-masses.





For a **system of masses** m_k located at the points (x_k, y_k), the total mass of the system is (as before):

$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k$$

while the moment about the y-axis is:

$$M_y = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot x_k$$

and the moment about the x-axis is:

$$M_x = m_1 \cdot y_1 + m_2 \cdot y_2 + \cdots + y_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot y_k$$

At first, it may seem confusing that the formula for M_y would involve x and the formula for M_x would involve y, but keep in mind that an equation for the y-axis is $x = 0$, so we could write the moment about the y-axis as $M_{x=0}$ and the moment about the x-axis as $M_{y=0}$.

The **center of mass** of this two-dimensional system is a point $(\overline{x}, \overline{y})$ such that any line that passes through this point is a "balancing fulcrum" for the system. So we need the moment about any such line — including $x = \overline{x}$ and $y = \overline{y}$ — to be zero:

If we can find such a point, then the system will balance on a single "point-fulcrum" located at the center of mass.

$$0 = M_{x=\overline{x}} = \sum_{k=1}^{n} m_k \left(x_k - \overline{x}\right) = \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \overline{x}\left[\sum_{k=1}^{n} m_k\right] = M_y - \overline{x}m$$

so $\overline{x} = \dfrac{M_y}{m}$, and similar arithmetic shows that $\overline{y} = \dfrac{M_x}{m}$.

> The **center of mass of a system of point-masses** $m_1, m_2, \ldots, m_n$ at locations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is the point $(\overline{x}, \overline{y})$ where:
>
> $$\overline{x} = \frac{M_y}{m} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{\sum_{k=1}^{n} m_k \cdot y_k}{\sum_{k=1}^{n} m_k}$$

A single point-mass with mass m (the total mass of the system) located at $(\overline{x}, \overline{y})$ (the center of mass of the system) produces the same moment about any line as the whole system does about that line. For many purposes, we can think of the mass of the entire system being "concentrated at $(\overline{x}, \overline{y})$."

The arithmetic needed to prove this statement is similar to arithmetic we did to prove the corresponding assertion for a one-dimensional system.

k	m_k	x_k	y_k
1	2	-3	4
2	3	4	-7
3	1	6	-2
4	5	-2	1
5	3	4	-6

Example 2. Find the center of mass of the system consisting of the first three point-masses listed in the margin table.

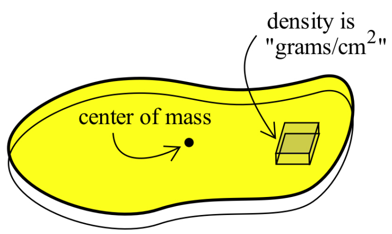
Solution. $m = 2+3+1 = 6$ and $M_y = (2)(-3)+(3)(4)+(1)(6) = 12$ while $M_x = (2)(4)+(3)(-7)+(1)(-2) = -15$ so:

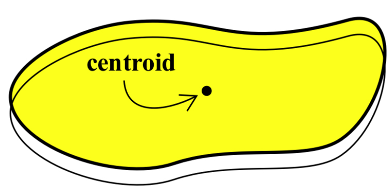
$$\overline{x} = \frac{M_y}{m} = \frac{12}{6} = 2 \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{-15}{6} = -2.5$$

The system of three point-masses will balance on any fulcrum passing through the point $(2, -2.5)$. ◀

Practice 2. Find the center of mass of the system consisting of all five point-masses listed in the margin table.

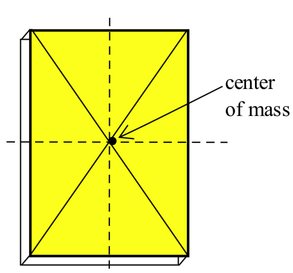
Centroid of a Region





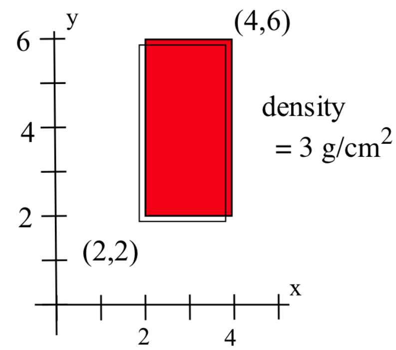
When we move from discrete point-masses to continuous regions in a plane, we move from finite sums and arithmetic to limits of Riemann sums, definite integrals and calculus. The following discussion extends ideas and calculations from point-masses to uniformly thin, flat plates (called **lamina**) that have a uniform density throughout (given as mass per area, such as "grams per cm^2"). The center of mass of one of these plates is the point $(\overline{x}, \overline{y})$ at which the plate balances without tilting. It turns out that for plates with uniform density, the center of mass $(\overline{x}, \overline{y})$ depends only on the shape (and location) of the region of the plane covered by the plate and not on the (constant) density. In these uniform-density situations, we call the center of mass the **centroid** of the region. Throughout the following discussion, you should notice that each finite sum that appeared in the discussion of point-masses has an integral counterpart for these thin plates.

Rectangles



The components of a Riemann sum typically involve areas of rectangles, so it should come as no surprise that the basic shape used to extend point-mass concepts to regions is the rectangle. The total mass of a rectangular plate is the product of the area of the plate and its (constant) density: $m = \text{mass} = (\text{area}) \cdot (\text{density})$. We will assume that the center of mass of a thin, rectangular plate is located halfway up and halfway across the rectangle, at the point where the diagonals of the rectangle cross (see margin).

The moments of the rectangle about an axis can be found by treating the rectangle as a single point-mass with mass m located at the center of mass of the rectangle.



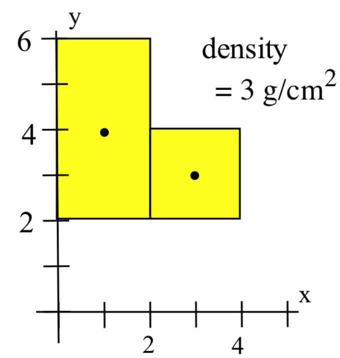
Example 3. Find the moments about the x-axis, y-axis and the line $x = 5$ of the thin, rectangular plate shown in the margin.

Solution. The density of the plate is $3\ \text{g/cm}^2$ and the area of the plate is $(2\,\text{cm})\,(4\,\text{cm}) = 8\,\text{cm}^2$ so the total mass is:

$$m = \left(8\ \text{cm}^2\right)\left(3\ \frac{\text{g}}{\text{cm}^2}\right) = 24\ \text{g}$$

The center of mass of the rectangular plate is $(\overline{x}, \overline{y}) = (3, 4)$. The moment about the x-axis is the product of the mass and the signed distance of the mass from the x-axis: $M_x = (24\,\text{g})\,(4\,\text{cm}) = 96\,\text{g-cm}$. Similarly, $M_y = (24\,\text{g})\,(3\,\text{cm}) = 72\,\text{g-cm}$. The moment about the line $x = 5$ is $M_{x=5} = (24\,\text{g})\,([5-3]\ \text{cm}) = 48\,\text{g-cm}$. ◀

To find the moments and center of mass of a plate made up of several rectangular regions, we can simply treat each of the rectangular pieces as a point-mass concentrated at its center of mass, then treat the plate as a system of discrete point-masses.

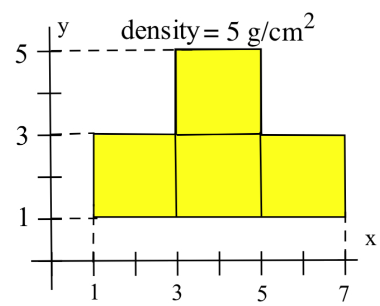


Example 4. Find the centroid of the region in the margin figure.

Solution. We can divide the plate into two rectangular plates, one with mass 24 g and center of mass $(1, 4)$, and the other with mass 12 g and center of mass $(3, 3)$. The total mass of the pair of point-masses is $m = 24 + 12 = 36$ g, and the moments about the axes are $M_x = (24\ \text{g})\,(4\ \text{cm}) + (12\ \text{g})\,(3\ \text{cm}) = 132$ g-cm and $M_y = (24\ \text{g})\,(1\ \text{cm}) + (12\ \text{g})\,(3\ \text{cm}) = 60$ g-cm. So:

$$\overline{x} = \frac{M_y}{m} = \frac{60\ \text{g-cm}}{36\ \text{g}} = \frac{5}{3}\ \text{cm} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{132\ \text{g-cm}}{36\ \text{g}} = \frac{11}{3}\ \text{cm}$$

The centroid of the plate is located at $\left(\frac{5}{3}, \frac{11}{3}\right)$. ◀



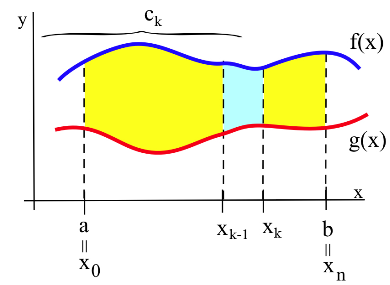
Practice 3. Find the centroid of the region in the margin figure.

To find the center of mass of a thin, non-rectangular plate, we will "slice" the plate into narrow, almost-rectangular plates and treat the collection of almost-rectangular plates as a system of point-masses located at the centers of mass of the almost-rectangles. The total mass and moments about the axes for the system of point-masses will be Riemann sums. By taking limits as the widths of the almost-rectangles approach 0, we will obtain exact values for the mass and moments as definite integrals

$\overline{x}$ *for a Region*

The Greek letter ρ (pronounced "row," as in "row your boat") is often used to represent the density of a region.

Suppose $f(x) \geq g(x)$ on the interval $[a, b]$ and $\mathcal{R}$ is a plate of uniform density ($= \rho$) sitting on the region between the graphs of $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$ (see margin figure). If we partition the interval $[a, b]$ into n subintervals of the form $[x_{k-1}, x_k]$ and choose the points c_k to be the midpoints of these subintervals, then the slice between vertical cuts at $x = x_{k-1}$ and $x = x_k$ is approximately rectangular and has mass approximately equal to:



$$\begin{aligned}(\text{area})\,(\text{density}) &= (\text{height})\,(\text{width})\,(\text{density}) \\ &\approx [f(c_k) - g(c_k)] \cdot (x_{k-1} - x_k) \cdot \rho \\ &= \rho\,[f(c_k) - g(c_k)]\,\Delta x_k\end{aligned}$$

So the mass of the whole plate is approximately

$$m = \sum_{k=1}^{n} \rho\,[f(c_k) - g(c_k)]\,\Delta x_k \longrightarrow \int_a^b \rho\,[f(x) - g(x)]\,dx = \rho \cdot A$$

where A is the area of the region $\mathcal{R}$.

The moment about the y-axis of each almost-rectangular slice is the product of the mass of the slice (m) and the distance from the centroid of the almost-rectangle to the y-axis. The x-coordinate of that centroid is located at $x = c_k$, so the distance from the centroid to the y-axis is $c_k - 0 = c_k$. The moment of the almost-rectangle about the y-axis is therefore:

$$m_k \cdot c_k = (\rho\,[f(c_k) - g(c_k)]\,\Delta x_k) \cdot c_k$$

so the moment of the entire plate about the y-axis is (approximately):

$$M_y = \sum_{k=1}^{n} \rho c_k \cdot [f(c_k) - g(c_k)]\,\Delta x_k \longrightarrow \int_a^b \rho x \cdot [f(x) - g(x)]\,dx$$

The x-coordinate of the centroid of the plate is therefore:

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x \cdot [f(x) - g(x)]\,dx}{\rho \int_a^b [f(x) - g(x)]\,dx} = \frac{\int_a^b x \cdot [f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx}$$

The density constant ρ is a factor of both M_y and m, so it cancels and has no effect on the value of $\overline{x}$. The value of $\overline{x}$ depends only on the shape and location of the region $\mathcal{R}$.

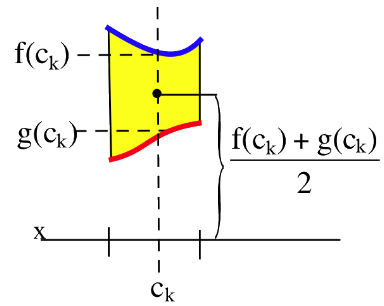
If the bottom boundary of $\mathcal{R}$ is the x-axis, then $g(x) = 0$ and the previous formulas simplify to:

$$m = \rho \int_a^b f(x)\,dx, \quad M_y = \rho \int_a^b x f(x)\,dx \quad \text{and} \quad \overline{x} = \frac{M_y}{m} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

Practice 4. Find the x-coordinate of the centroid of the region between $f(x) = x^2$, the x-axis and $x = 2$.

$\overline{y}$ for a Region

To find $\overline{y}$, the y-coordinate of the centroid of $\mathcal{R}$, we need to find M_x, the moment of $\mathcal{R}$ about the x-axis. For vertical partitions of $\mathcal{R}$ (see margin), the moment of each narrow strip about the x-axis, M_x, is the product of the strip's mass and the signed distance between the centroid of the strip and the x-axis. We've already computed the mass:



$$m_k = \rho\left[f(c_k) - g(c_k)\right]\Delta x_k$$

Because each strip is nearly rectangular, the centroid of the k-th strip is roughly halfway up the strip, at a point midway between $f(c_k)$ and $g(c_k)$, so we can average those function values to compute:

$$\overline{y}_k \approx \frac{f(c_k) + g(c_k)}{2}$$

The moment about the x-axis for this strip is thus:

$$\rho\left[f(c_k) - g(c_k)\right]\Delta x_k \cdot \left[\frac{f(c_k) + g(c_k)}{2}\right] = \frac{\rho}{2}\left[(f(c_k))^2 - (g(c_k))^2\right]\Delta x_k$$

Adding up the moments of all n strips yields:

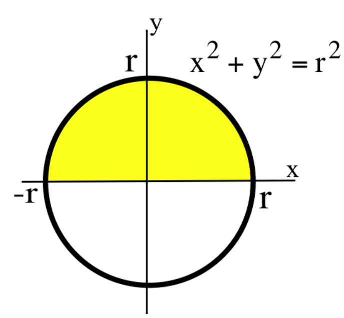
$$M_x = \sum_{k=1}^{n} \frac{\rho}{2}\left[(f(c_k))^2 - (g(c_k))^2\right]\Delta x_k \longrightarrow \int_a^b \frac{\rho}{2}\left[(f(x))^2 - (g(x))^2\right]\,dx$$

The y-coordinate of the centroid of the plate is therefore:

$$\overline{y} = \frac{M_x}{m} = \frac{\rho\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]\,dx}{\rho\int_a^b \left[f(x) - g(x)\right]\,dx} = \frac{\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]\,dx}{\int_a^b \left[f(x) - g(x)\right]\,dx}$$

If the bottom boundary of $\mathcal{R}$ is the x-axis, then $g(x) = 0$ and the previous formulas simplify to:

$$M_x = \frac{\rho}{2}\int_a^b x\left[f(x)\right]^2\,dx \quad\text{and}\quad \overline{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2\,dx}{\int_a^b f(x)\,dx}$$



Example 5. Find the y-coordinate of the centroid of the region $\mathcal{R}$ bounded below by the x-axis and above by the top half of a circle of radius r centered at the origin (see margin).

Solution. An equation for the circle is $x^2 + y^2 = r^2$ so the top half is given by $f(x) = y = \sqrt{r^2 - x^2}$, and $g(x) = 0$. The mass of the region is:

$$m = \int_{-r}^{r} \rho\sqrt{r^2 - x^2}\,dx = \rho\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \rho \cdot [\text{area of } \mathcal{R}] = \rho \cdot \frac{\pi r^2}{2}$$

The moment of $\mathcal{R}$ about the y-axis is:

$$M_y = \int_{-r}^{r} \rho x \cdot \sqrt{r^2 - x^2}\,dx = \left[-\frac{\rho}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}\right]_{x=-r}^{x=r} = 0$$

so $\overline{x} = 0$.

Could you have guessed this result merely by looking at the region?

The moment of $\mathcal{R}$ about the x-axis is:

$$M_x = \int_{-r}^{r} \frac{\rho}{2} \cdot \left[\sqrt{r^2 - x^2}\right]^2 dx = \frac{\rho}{2} \int_{-r}^{r} \left[r^2 - x^2\right] dx$$
$$= \frac{\rho}{2} \left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} = \frac{\rho}{2} \cdot \frac{4}{3}r^3 = \frac{2\rho}{3}r^3$$

Could you have guessed that centroid would be located a bit less than halfway above the bottom edge of the semicircle, merely by looking at the region?

so $\overline{y} = \dfrac{\frac{2\rho}{3}r^3}{\frac{\rho\pi}{2}r^2} = \dfrac{4}{3\pi}r \approx 0.4244r$. ◀

Practice 5. Show that the centroid of a triangular region with vertices $(0,0)$, $(0,h)$ and $(b,0)$ is located at $(\overline{x},\overline{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$.

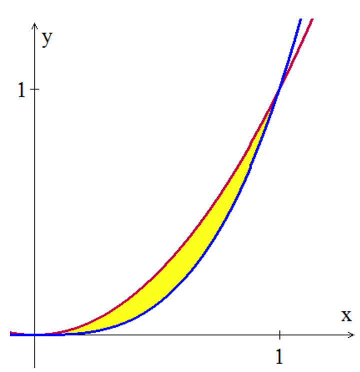
The following table summarizes and compares formulas for computing moments and centers of mass for a system of point-masses in a plane (using sums) and for a region in a plane (using integrals). The integral formulas appear in a form for calculating moments of a region $\mathcal{R}$ bounded by the graphs of two functions, $f(x)$ and $g(x)$, and two vertical lines, $x = a$ and $x = b$, where $f(x) \geq g(x)$ for $a \leq x \leq b$.

	point-masses in plane	**region $\mathcal{R}$ between f and g**
total mass:	$m = \sum_{k=1}^{n} m_k$	$m = \int_a^b \rho\,[f(x) - g(x)]\,dx = \rho \cdot \text{Area}\,(\mathcal{R})$
moment about y-axis ($x = 0$):	$M_y = \sum_{k=1}^{n} m_k \cdot x_k$	$M_y = \int_a^b \rho x \cdot [f(x) - g(x)]\,dx$
moment about x-axis ($y = 0$):	$M_x = \sum_{k=1}^{n} m_k \cdot y_k$	$M_x = \int_a^b \frac{\rho}{2}\left[(f(x))^2 - (g(x))^2\right] dx$
center of mass (ρ constant):	$\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$	$\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$

With the knowledge of Riemann sums you have developed, you should be able to set up integrals to compute masses and moments for regions bounded by curves of the form $x = g(y)$, and deal with situations where the density of a thin plate is a function of x or y.

While the integral formulas above are often useful, it is important that you understand the process used to obtain these formulas in order to compute moments and centroids of more general regions.

Example 6. Find the centroid of the region $\mathcal{R}$ bounded by the graphs of $y = x^2$ and $y = x^3$.



Solution. The curves intersect where $x^2 = x^3 \Rightarrow x^2 - x^3 = 0 \Rightarrow x^2(1 - x) = 0 \Rightarrow x = 0$ or $x = 1$. A graph (see margin) helps confirm that $x^2 \geq x^3$ on $[0,1]$. If the density of $\mathcal{R}$ is ρ then the mass of $\mathcal{R}$ is:

$$m = \int_0^1 \rho\left[x^2 - x^3\right] dx = \rho\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\rho}{12}$$

The moment of $\mathcal{R}$ about the y-axis is:

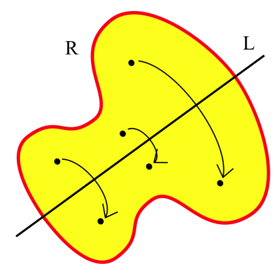
$$M_y = \rho\int_0^1 x\left[x^2 - x^3\right] dx = \rho\int_0^1 \left[x^3 - x^4\right] dx = \rho\left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{\rho}{20}$$

And the moment of $\mathcal{R}$ about the x-axis is:

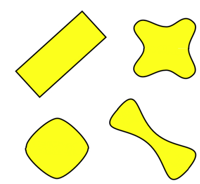
$$M_x = \frac{\rho}{2}\int_0^1 \left[\left(x^2\right)^2 - \left(x^3\right)^2\right] dx = \frac{\rho}{2}\left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \frac{\rho}{35}$$

so $\overline{x} = \dfrac{M_y}{m} = \dfrac{\frac{\rho}{20}}{\frac{\rho}{12}} = \dfrac{3}{5}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{\frac{\rho}{35}}{\frac{\rho}{12}} = \dfrac{12}{35}$. Plotting the point $\left(\frac{3}{5}, \frac{12}{35}\right) \approx (0.60, 0.34)$ along with $\mathcal{R}$ confirms that it sits inside $\mathcal{R}$ (just barely) and appears to be a reasonable candidate for the centroid. ◀

Symmetry



Symmetry is a very powerful geometric concept that can simplify many mathematical and physical problems, including the task of finding centroids of regions. For some regions, we can use symmetry alone to determine the centroid. Geometrically, a region $\mathcal{R}$ is **symmetric about a line L** if, when $\mathcal{R}$ is folded along L, each point of $\mathcal{R}$ on one side of the fold matches up with exactly one point of $\mathcal{R}$ on the other side of the fold (see margin).



Example 7. Sketch two lines of symmetry for each region shown in the margin figure.

Solution. See solution to Practice 6. ◀

A very useful fact about symmetric regions is that the centroid $(\overline{x}, \overline{y})$ of a symmetric region must lie on every line of symmetry of the region. If a region has two different lines of symmetry, then the centroid must lie on each of them, so the centroid must be located at the point where the lines of symmetry intersect.

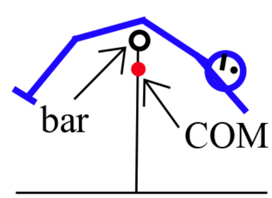
In Example 5, the half-disk was symmetric with respect to the y-axis, so we could have avoided setting up and evaluating the M_y integral by noticing that $(\overline{x}, \overline{y})$ must be located on the y-axis (the line $x = 0$) and concluding that $\overline{x} = 0$.

Practice 6. Locate the centroid of each region in Example 7.

Work

In a uniform gravitational field, the **center of gravity** of an object is located at the same point as its center of mass, and the work done to lift an object is the product of the object's weight and the distance that the center of gravity of the object is raised:

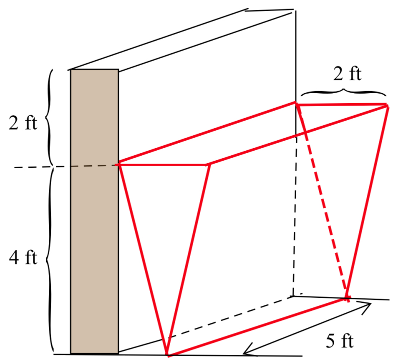
$$\text{work} = (\text{object's weight})\,(\text{distance object's center of gravity is raised})$$



In the high jump, this explains the effectiveness of the "Fosbury Flop," a technique where the jumper assumes an inverted **U** position while going over the bar (see margin): the jumper's body goes over the bar while the jumper's center of gravity goes under it, allowing the jumper to clear a higher bar with no additional upward thrust.

If you know the center of gravity of an object being lifted, some work problems become much easier.

We've already solved this problem (as Example 5 in Section 5.4) but here we try a new approach using centroids.



Example 8. The trough shown in the margin is filled with a liquid weighing 70 pounds per cubic foot. How much work is done pumping the liquid over the wall next to the trough?

Solution. This is a 3-D problem, but symmetry tells us the centroid of the liquid must be at a point 2.5 feet from either end of the trough, and 1 foot away from the wall. The vertical coordinate of the centroid will be the same as the centroid of the trough's triangular end region. Using the result of Practice 5, we can conclude that the centroid of the triangle is at a height of $\frac{2}{3} \cdot 4 = \frac{8}{3}$. The weight of the liquid is:

$$(\text{density}) \cdot (\text{volume}) = \left(70\ \frac{\text{lb}}{\text{ft}^3}\right) \cdot \frac{1}{2}\,(5\text{ ft}) \cdot (2\text{ ft}) \cdot (4\text{ ft}) = 1400\text{ lbs}$$

and the distance the center of gravity must be moved is $6 - \dfrac{8}{3} = \dfrac{10}{3}$ ft so the total work required is:

$$(1400\text{ lbs}) \cdot \left(\frac{10}{3}\text{ ft}\right) = \frac{14000}{3}\text{ ft-lbs} \approx 4666.7\text{ ft-lbs}$$

which agrees with the answer obtained in Section 5.4. ◀

Theorems of Pappus

Pappus, the last of the great Greek geometers, flourished during the first half of the fourth century.

Two theorems due to Pappus of Alexandria can make some volume and surface area calculations relatively easy.

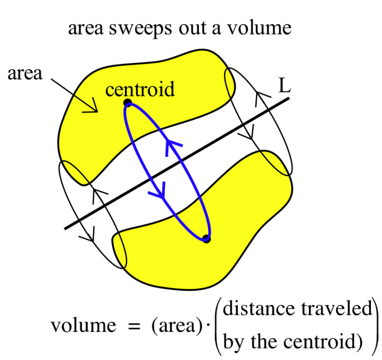
Theorem of Pappus: Volume of Revolution

If a plane region $\mathcal{R}$ with area A and centroid $(\overline{x}, \overline{y})$ is revolved around a line L in the plane that does not pass through $\mathcal{R}$

then the volume swept out by one revolution of $\mathcal{R}$ is the product of A and the distance traveled by the centroid.

Touching the boundary is OK.

The distance from the centroid to the line will be the radius of the circle swept out by the centroid, so the distance traveled by the centroid is 2π times this radius. When L is the x-axis, the volume of the solid is $A \cdot 2\pi\overline{y}$; when L is the y-axis, the volume of the solid is $A \cdot 2\pi\overline{x}$.



Example 9. Find the volume swept out when the region $\mathcal{R}$ bounded by the graphs of $y = x^2$ and $y = x^3$ is revolved around the line $x = 2$.

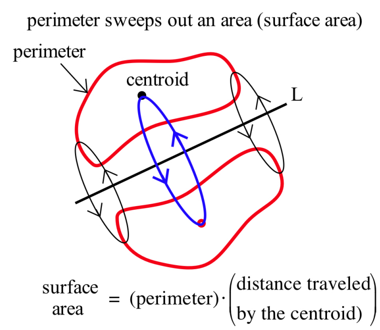
Solution. From Example 6, we know the area of $\mathcal{R}$ is $\frac{1}{12}$ and its centroid is $\left(\frac{3}{5}, \frac{12}{35}\right)$. The distance from this point to the line $x = 2$ is $2 - \frac{3}{5} = \frac{7}{5}$, so the distance traveled by the centroid is $2\pi \cdot \dfrac{7}{5} = \dfrac{14\pi}{5}$. The volume of the solid of revolution is therefore $\dfrac{1}{12} \cdot \dfrac{14\pi}{5} = \dfrac{7\pi}{30}$. ◀

Theorem of Pappus: Surface Area of Revolution

If	a plane region $\mathcal{R}$ with perimeter P and centroid $(\bar{x}, \bar{y})$ is revolved around a line L in the plane that does not pass through $\mathcal{R}$
then	the surface area swept out by one revolution of $\mathcal{R}$ is the product of P and the distance traveled by the centroid.

Touching the boundary is OK.

When L is the x-axis, the surface area of the solid is $P \cdot 2\pi\bar{y}$; when L is the y-axis, the surface area is $P \cdot 2\pi\bar{x}$.



Example 10. Find the surface area of the solid swept out when the square region $\mathcal{R}$ with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ is revolved around the line $y = 3$.

Solution. By symmetry, the centroid of the square is $(0, 0)$ and its distance from $y = 3$ is 3. The perimeter of the square is $4\sqrt{2}$, so the surface area of the solid of revolution is $4\sqrt{2} \cdot 2\pi \cdot 3 = 24\pi\sqrt{3}$. ◀

5.6 Problems

1. (a) Find the total mass and the center of mass for a system consisting of the three point-masses in the table below left.
   (b) Where should you locate a new object with mass 8 so the new system has its center of mass at $x = 5$?
   (c) What mass should you put at $x = 10$ so the original system plus the new mass has its center of mass at $x = 6$?

m_k	2	5	5
x_k	4	2	6

m_k	5	3	2	6
x_k	1	7	5	5

2. (a) Find the total mass and the center of mass for a system consisting of the four point-masses in the table above right.
   (b) Where should you locate a new object with mass 10 so the new system has its center of mass at $x = 6$?
   (c) What mass should you put at $x = 14$ so the original system plus the new mass has its center of mass at $x = 6$?

3. (a) Find the total mass and the center of mass for a system consisting of the three point-masses in the table below.
   (b) Where should you locate a new object with mass 10 so the new system has its center of mass at $(5, 2)$?

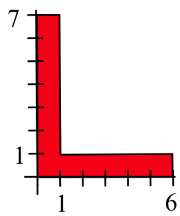
m_k	2	5	5
x_k	4	2	6
y_k	3	4	2

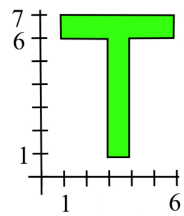
4. (a) Find the total mass and the center of mass for a system consisting of the four point-masses in the table below.
   (b) Where should you locate a new object with mass 12 so the new system has its center of mass at $(3, 5)$?

m_k	5	3	2	6
x_k	1	7	5	5
y_k	4	7	0	8

In Problems 5–10, divide the plate shown into rectangles and semicircles, calculate the mass, moments and centers of mass of each piece, then find the center of mass of the plate. Assume the density of the plate is $\rho = 1$. Plot the location of the center of mass for each shape. (Refer to Example 5 for centroids of semicircular regions.)

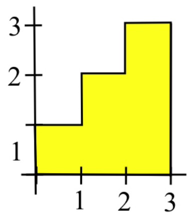
5. Use the figure below left.

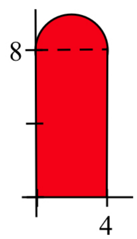




6. Use the figure above right.

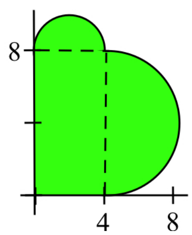
7. Use the figure below left.

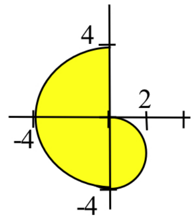




8. Use the figure above right.

9. Use the figure below left.





10. Use the figure above right.

In Problems 11–26, sketch the region bounded by the the given curves and find the centroid of each region (use technology to evaluate integrals, if necessary). Plot the location of the centroid on your sketch of the region.

11. $y = x$, the x-axis, $x = 3$

12. $y = x^2$, the x-axis, $x = -2$, $x = 2$

13. $y = x^2$, $y = 4$

14. $y = \sin(x)$, the x-axis, the y-axis, $x = \pi$

15. $y = 4 - x^2$ and the x-axis for $-2 \le x \le 2$

16. $y = x^2$, $y = x$

17. $y = 9 - x$, $y = 3$, $x = 0$, $x = 3$

18. $y = \sqrt{1 - x^2}$, the x-axis, $x = 0$, $x = 1$

19. $y = \sqrt{x}$, the x-axis, $x = 9$

20. $y = \ln(x)$, the x-axis, $x = e$

21. $y = e^x$, $y = e$ and the y-axis

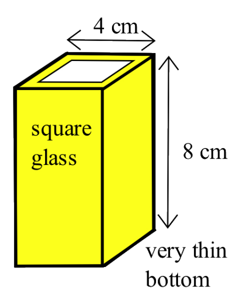
22. $y = x^2$ and $y = 2x$

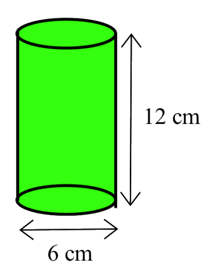
23. An empty box in the shape of a cube measuring 1 foot on each side weighs 10 pounds. By symmetry, we know its center of mass is 6 inches above its bottom. When the box is full of a liquid with density 60 lb/ft^3, the center of mass of the box-liquid system is again (due to symmetry) 6 inches above the bottom of the box.

    (a) Find the height of the center of mass of the box-liquid system as a function of h, the height of water in the box.

    (b) To what height should you fill the box so that the box-liquid system has the lowest center of gravity (and the greatest stability)?

24. The empty glass shown below left has a mass of 100 g when empty. Find the height of the center of mass of the glass-water system as a function of the height of water in the glass.



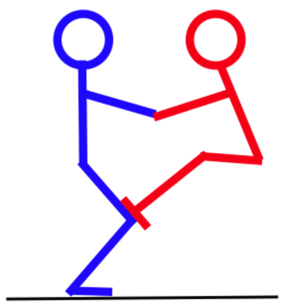


25. The empty soda can shown above right has a mass of 15 g when empty and 400 g when full of soda. Find the height of the center of mass of the can-soda system as a function of the height of the soda in the can.

26. Give a practical set of directions someone could actually use to find the height of the center of gravity of their body with their arms at their sides. How will the height of the center of gravity change if they lift their arms?

27. Try the following experiment. Stand straight with your back and heels against a wall. Slowly raise one leg, keeping it straight, in front of you. What happened? Why?

28. Why can't two dancers stand in the position shown below?



29. If a shape has exactly two lines of symmetry, the lines can meet at right angles. Must they meet at right angles?

30. Sketch regions with exactly two lines of symmetry, exactly three lines of symmetry, and exactly four lines of symmetry.

31. A rectangular box is filled to a depth of 4 feet with 300 pounds of water. How much work is done pumping the water to a point 10 feet above the bottom of the box?

32. A cylinder is filled to a depth of 2 feet with 40 pounds of water. How much work is done pumping the water to a point 7 feet above the bottom of the cylinder?

33. A sphere of radius 2 m is filled with water. How much work is done pumping the water to a point 3 m above the top of the sphere?

34. A sphere of radius 2 feet is filled with water. How much work is done pumping the water to a point 5 feet above the top of the sphere?

35. The center of a square region with sides of length 2 cm is located at the point $(3, 4)$. Find the volume swept out when the square region is rotated:
   (a) about the x-axis.
   (b) about the y-axis.
   (c) about the line $y = 6$
   (d) about the line $x = 6$
   (e) about the line $2x + 3y = 6$

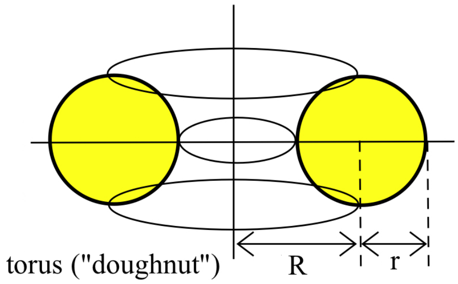
36. The lower left corner of a rectangular region with an 8-inch base and a 4-inch height is located at the point $(3, 5)$. Find the volume swept out when the rectangular region is rotated:
   (a) about the x-axis.
   (b) about the y-axis.
   (c) about the line $y = x + 5$

37. The center of a square region with sides of length 2 cm is located at the point $(3, 4)$. Find the surface area swept out when the square region is rotated:
   (a) about the x-axis.
   (b) about the y-axis.
   (c) about the line $y = 6$
   (d) about the line $x = 6$
   (e) about the line $2x + 3y = 6$

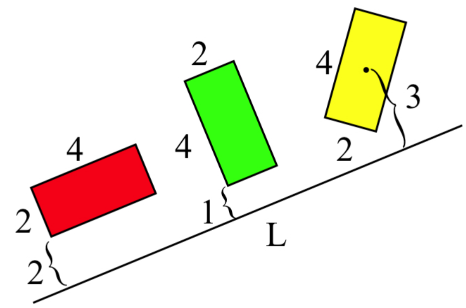
38. The lower left corner of a rectangular region with an 8-inch base and a 4-inch height is located at the point $(3, 5)$. Find the surface area swept out when the rectangular region is rotated:
   (a) about the x-axis.
   (b) about the y-axis.
   (c) about the line $y = x + 5$

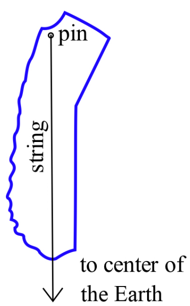
39. Find the volume and surface area swept out when the region inside the circle $(x - 3)^2 + (y - 5)^2 = 4$ is rotated:
   (a) about the x-axis.
   (b) about the y-axis.
   (c) about the line $y = 9$
   (d) about the line $x = 6$
   (e) about the line $2x + 3y = 6$

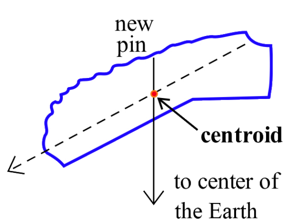
40. Find the volume and surface area swept out when the center of a circle with radius r and center $(R, 0)$ is rotated about the y-axis (see below).



41. Find the volumes and surface areas swept out when the rectangles shown below are rotated about the line L. (Measurements are in feet.)







Physically Approximating Centroids of Regions

You can approximate the location of a centroid of a region experimentally, even if the region—such as a state or country—is not described by a formula.

Cut the shape out of a piece of some uniformly thick material, such as paper or cardboard, and pin an edge to a wall. The shape will pivot about the pin until its center of mass is directly below the pin (see margin) so the center of mass of the shape must lie directly below the pin, on the line connecting the pin with the center of mass of Earth. Repeat the process using a different point near the edge of the shape to find a different line. The center of mass also lies on the new line, so you can conclude that the centroid of the shape is located where the two lines intersect (see margin). It is a good idea to pick a third point near the edge and plot a third line to check that this third line also passes through the point of intersection of the first two lines.

You can experimentally approximate the "population center" of a region by attaching masses proportional to the populations of the cities and then repeating the "pin" process with this weighted model. The point on the new model where the lines intersect is the approximate "population center" of the region.

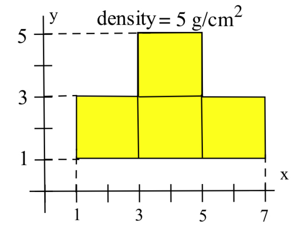
42. Determine the centroid of your state.

43. Which state would result in the easiest centroid problem? The most difficult centroid problem?

5.6 Practice Answers

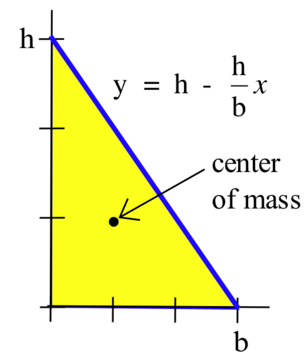
1. $m = 1+5+3 = 9$; $M_0 = (1)(6)+(5)(-2)+(3)(4) = 8$; $\overline{x} = \dfrac{M_0}{m} = \dfrac{8}{9}$; the three point-masses will balance on a fulcrum located at $\overline{x} = \frac{8}{9}$.

2. $m = 2+3+1+5+3 = 14$
$M_y = (2)(-3)+(3)(4)+(1)(6)+(5)(-2)+(3)(4) = 14$
$M_x = (2)(4)+(3)(-7)+(1)(-2)+(5)(1)+(3)(-6) = -28$
$\overline{x} = \dfrac{M_y}{m} = \dfrac{14}{14} = 1$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{-28}{14} = -2$
The five point-masses balance at the point $(1,-2)$.



3. There are several ways to break the region into "easy" pieces—one way is to consider the four 2 cm-by-2 cm squares. The center of mass of each square is located at the center of the square (at $(2,2)$, $(4,2)$, $(6,2)$ and $(4,4)$), and each square has mass $\left(4 \text{ cm}^2\right)\left(5\,\frac{\text{g}}{\text{cm}^2}\right) = 20$ g so: $m = 4\,(20\,\text{g}) = 80\,\text{g}$, $M_y = 2(20)+4(20)+6(20)+4(20) = 320\,\text{g-cm}$ and $M_x = 2(20)+2(20)+2(20)+4(20) = 200$ g-cm. Therefore $\overline{x} = \dfrac{M_y}{m} = \dfrac{320 \text{ g-cm}}{80 \text{ g}} = 4$ cm and $\overline{y} = \dfrac{M_x}{m} = \dfrac{200 \text{ g-cm}}{80 \text{ g}} = 2.5$ cm so the center of mass is located at $(4, 2.5)$.

4. For simplicity, let $\rho = 1$. Then the mass is $m = \displaystyle\int_0^2 x^2\,dx = \frac{8}{3}$ while $M_y = \displaystyle\int_0^2 x\cdot x^2\,dx = \int_0^2 x^3\,dx = 4$ so $\overline{x} = \dfrac{4}{\frac{8}{3}} = \dfrac{3}{2} = 1.5$.



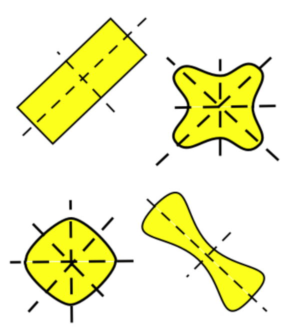
5. The triangular region appears in the margin. Here $f(x) = h - \dfrac{h}{b}x$ for $0 \le x \le b$ and $g(x) = 0$. The "mass" is just the area of the triangle, so $m = \frac{1}{2}\cdot b\cdot h$ while:

$$M_y = \int_0^b x\left[h - \frac{h}{b}x\right]\,dx = \int_0^b \left[hx - \frac{h}{b}x^2\right]\,dx = \left[\frac{h}{2}x^2 - \frac{h}{3b}x^3\right]_0^b = \frac{b^2h}{6}$$

and:

$$M_x = \int_0^b \frac{1}{2}\left[h - \frac{h}{b}x\right]^2\,dx = \left[\frac{1}{6}\left(-\frac{b}{h}\right)\left(h - \frac{h}{b}x\right)^3\right]_0^b = 0 + \frac{b}{6h}\cdot h^3 = \frac{bh^2}{6}$$

So $(\overline{x},\overline{y}) = \left(\dfrac{\frac{b^2h}{6}}{\frac{bh}{2}}, \dfrac{\frac{bh^2}{6}}{\frac{bh}{2}}\right) = \left(\dfrac{b}{3}, \dfrac{h}{3}\right)$.



6. The centroid of each region is located at the point where the lines of symmetry intersect (see margin figure).

5.7 *Improper Integrals*

In Section 5.4, we computed the work required to lift a payload of mass m from the surface of a moon of mass M and radius R to a height H above the surface of the moon:

$$\int_R^{R+H} \frac{GMm}{x^2}\,dx = \left[-\frac{GMm}{x}\right]_R^{R+H} = \frac{GMm}{R} - \frac{GMm}{R+H}$$

Notice that as the height H grows very large, the second term in this answer becomes very small and the total work approaches $\dfrac{GMm}{R}$. We can write:

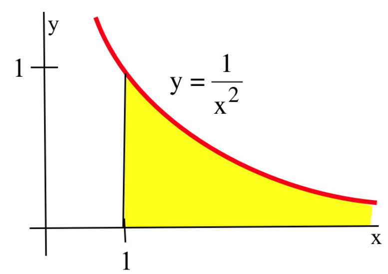
$$\lim_{H\to\infty}\left[\frac{GMm}{R} - \frac{GMm}{R+H}\right] = \frac{GMm}{R}$$

Here we're taking a limit of an expression that arose as the value of a definite integral, so we can also write:

$$\frac{GMm}{R} = \lim_{H\to\infty}\left[\frac{GMm}{R} - \frac{GMm}{R+H}\right] = \lim_{H\to\infty}\int_R^{R+H}\frac{GMm}{x^2}\,dx$$

We could write this last integral, at least informally, as:

$$\int_R^{\infty}\frac{GMm}{x^2}\,dx$$



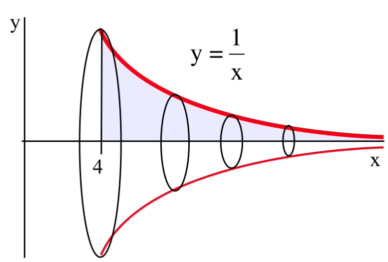
We call this new type of integral an **improper integral** because the interval of integration is infinite, violating an assumption we made when originally developing the definite integral $\int_a^b f(x)\,dx$ using Riemann sums that the length of the interval of integration, $[a, b]$, was finite.

Example 1. Represent the area of the infinite region between $f(x) = \dfrac{1}{x^2}$ and the x-axis for $x \geq 1$ (see margin) as an improper integral.

Solution. We can represent the area of region (which has infinite length) as:

$$\int_1^{\infty}\frac{1}{x^2}\,dx$$

We don't yet know whether this area is finite or infinite. ◀



Practice 1. Represent the volume swept out when the infinite region between $f(x) = \dfrac{1}{x}$ and the x-axis for $x \geq 4$ is revolved about the x-axis (see margin) using an improper integral.

General Strategy for Improper Integrals

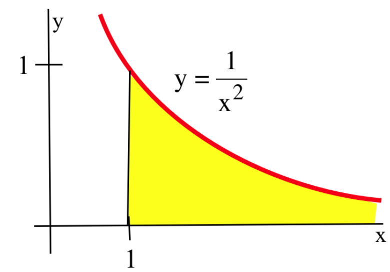
In the lifting-a-payload example above, we defined our first improper integral as the limit of a "proper" integral over a finite interval as the length of the interval became larger and larger.

Our general approach to evaluate improper integrals over infinitely long intervals — as well as another type of improper integral introduced later in this section — will mimic this strategy: Shrink the interval of integration so you have a (proper) definite integral you can evaluate, then let the interval grow to approach the desired interval of integration. The value of the improper integral will be the limiting value of the (proper) definite integrals as the intervals grow to the interval you want, provided that this limit exists.

Infinitely Long Intervals of Integration

To evaluate an improper integral on an infinitely long interval:

- replace the infinitely long interval with a finite interval
- evaluate the integral on the finite interval
- let the finite interval grow longer and longer, approaching the original infinitely long interval



Example 2. Evaluate $\int_1^{\infty} \frac{1}{x^2}\,dx$ (see margin).

Solution. The interval $[1, \infty)$ is infinitely long, but we can evaluate the integral on finite intervals such as $[1, 2]$, $[1, 10]$, $[1, 1000]$ and, more generally, $[1, M]$ where M is some massive positive number:

$$\int_1^{2} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\int_1^{10} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{10} = \left[-\frac{1}{10}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{10} = \frac{9}{10}$$

$$\int_1^{1000} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{1000} = \left[-\frac{1}{1000}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{1000} = \frac{999}{1000}$$

and, more generally, $\int_1^{M} \frac{1}{x^2}\,dx = 1 - \frac{1}{M}$ so:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{M\to\infty} \int_1^{M} \frac{1}{x^2}\,dx = \lim_{M\to\infty} \left[1 - \frac{1}{M}\right] = 1$$

The value of the improper integral is 1. ◀

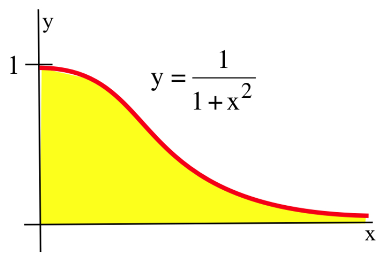
We say that the improper integral $\int_1^{\infty} \frac{1}{x^2}\,dx$ in the Example 2 "is **convergent**" and that it "**converges to** 1."

Furthermore, from Example 1, we know that this improper integral represents the area of an infinitely long region. We now have an example — which you may find highly counterintuitive — of a region with infinite length but finite area.

Not all improper integrals converge, however.

Example 3. Evaluate each improper integral. (See margin for graphical interpretations of these integrals as areas of unbounded regions.)

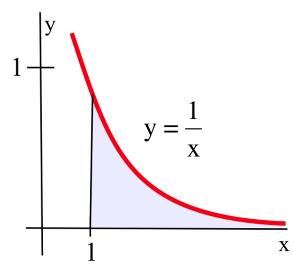
$$\text{(a)} \int_0^\infty \frac{1}{1+x^2}\,dx \qquad \text{(b)} \int_1^\infty \frac{1}{x}\,dx \qquad \text{(c)} \int_0^\infty \cos(x)\,dx$$



Solution. (a) Replacing the upper limit of the improper integral with a massive positive number M:

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_0^M \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \Big[\arctan(x)\Big]_0^M$$
$$= \lim_{M\to\infty} \Big[\arctan(M) - 0\Big] = \frac{\pi}{2}$$

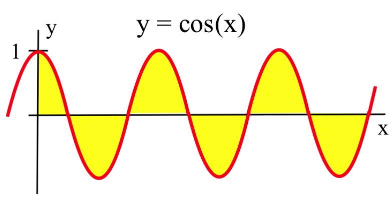
so the improper integral is convergent and converges to $\frac{\pi}{2}$.



(b) Replacing the upper limit of the improper integral with a massive positive number M:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{x}\,dx = \lim_{M\to\infty} \Big[\ln(x)\Big]_1^M = \lim_{M\to\infty} \ln(M) = \infty$$

Because this limit diverges, we say the improper integral **is divergent** or that it **diverges**.



(c) Once again replacing ∞ with M in the upper limit of the integral:

$$\lim_{M\to\infty} \int_0^M \cos(x)\,dx = \lim_{M\to\infty} \Big[\sin(x)\Big]_0^M = \lim_{M\to\infty} \sin(M)$$

As M grows without bound, the values of $\sin(M)$ oscillate between -1 and 1, never approaching a single value, so the limit does not exist; we say that this improper integral diverges. ◀

Practice 2. Evaluate: (a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ (b) $\displaystyle\int_0^\infty \sin(x)\,dx$

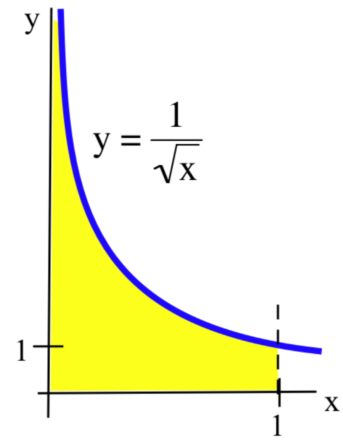
Definition: For any integrable function $f(x)$ defined for all $x \geq a$ and any integrable function $g(x)$ defined for all $x \leq b$:

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty} \int_a^M f(x)\,dx$$
$$\int_{-\infty}^b g(x)\,dx = \lim_{N\to-\infty} \int_N^b g(x)\,dx$$

If the limit in question exists and is finite, we say that the corresponding improper integral **converges** or **is convergent** and define the value of the improper integral to be the value of the limit. If the limit in question does not exist, we say that the corresponding improper integral **diverges** or **is divergent**.

Functions Undefined at an Endpoint of the Interval of Integration

Consider the graph of $\frac{1}{\sqrt{x}}$ on the interval $(0,1]$ (see margin) and compare this region to the graph from Example 2. It appears we can generate the new region by reflecting the old region across $y = x$ and adding a rectangle (of area 1) at the bottom, so we might reasonably assume that the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is a finite number. This integral is over a finite interval, $[0,1]$, but we have a new problem: the integrand is undefined at $x = 0$, one of the endpoints of the interval of integration. This violates another assumption we made when developing the definition of a definite integral as a limit of Riemann sums.



If the function you want to integrate is unbounded at one of the endpoints of an interval of finite length, as in this situation, you can shrink the interval of integration so that the function is bounded at both endpoints of the new, smaller interval, then evaluate the integral over the smaller interval, and finally let the smaller interval grow to approach the original interval.

Example 4. Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution. The function $\dfrac{1}{\sqrt{x}}$ is not defined at $x = 0$, the lower endpoint of integration, but the function is bounded on intervals such as $[0.36,1]$, $[0.09,1]$ and, more generally, on the interval $[c,1]$ for any $c > 0$:

$$\int_{0.36}^{1} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{0.36}^{1} = 2\sqrt{1} - 2\sqrt{0.36} = 2 - 1.2 = 0.8$$
$$\int_{0.09}^{1} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{0.09}^{1} = 2\sqrt{1} - 2\sqrt{0.09} = 2 - 0.6 = 1.4$$

and, in general:

$$\int_{c}^{1} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{c}^{1} = 2\sqrt{1} - 2\sqrt{c} = 2 - 2\sqrt{c}$$

so, taking the limit as c decreases toward 0:

$$\lim_{c\to 0^+} \int_{c}^{1} \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \left[2 - 2\sqrt{c}\right] = 2$$

which is what you should have expected based on the graph. ◀

A region of area 1 plus a rectangle of area 1 should have an area of $1 + 1 = 2$.

> **Definition**: For any function $f(x)$ defined and continuous on $(a,b]$ and any function $g(x)$ defined and continuous on $[a,b)$:
>
> $$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$
> $$\int_a^b g(x)\,dx = \lim_{c\to b^-} \int_a^c g(x)\,dx$$

If the limit exists, we say the integral **converges** and define the value of the integral to be the value of the limit. If the limit does not exist, we say that the integral **diverges**.

Practice 3. Show that (a) $\displaystyle\int_1^{10} \frac{1}{\sqrt{10-x}}\,dx = 6$ and (b) $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges.

If an integrand is unbounded at one or more points *inside* the interval of integration, you can split the original improper integral into two or more improper integrals over subintervals where the integrand is unbounded at only one endpoint of each subinterval.

See Problems 22–26 for practice with integrals of this type.

Testing for Convergence: The P-Test and the Comparison Test

Sometimes we care only whether or not an improper integral converges. We now consider two methods for testing the convergence of an improper integral. Neither method gives you the actual value of the integral, but each enables you to determine whether or not certain improper integrals converge. The **Comparison Test for Integrals** enables you to determine the convergence (or divergence) of certain integrals by comparing them with other (easier) integrals. The **P-Test** involves special cases often used with the Comparison Test for Integrals.

P-Test for integrals: For any $a > 0$, the improper integral $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \le 1$.

Proof. It is easiest to consider three cases rather than two: $p = 1$, $p > 1$ and $p < 1$. If $p = 1$ then:

$$\int_a^\infty \frac{1}{x^p}\,dx = \int_a^\infty \frac{1}{x}\,dx = \lim_{M\to\infty} \int_a^M \frac{1}{x}\,dx = \lim_{M\to\infty} \Big[\ln(|x|)\Big]_a^M$$
$$= \lim_{M\to\infty} \Big[\ln(M) - \ln(a)\Big] = \infty$$

so the improper integral diverges. For the other two cases, $p \ne 1$, so:

$$\lim_{M\to\infty} \int_a^M x^{-p}\,dx = \lim_{M\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^M = \lim_{M\to\infty} \left[\frac{M^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right]$$

If $p > 1$, then $1 - p < 0$ so $\displaystyle\lim_{M\to\infty} \frac{M^{1-p}}{1-p} = 0$ and:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{M\to\infty} \left[\frac{M^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \frac{a^{1-p}}{p-1}$$

which is a finite number, so the improper integral converges. If $p < 1$, then $1 - p > 0$ so $\displaystyle\lim_{M\to\infty} \frac{M^{1-p}}{1-p} = \infty$ and:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{M\to\infty} \left[\frac{M^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \infty$$

so the improper integral diverges. □

Example 5. Determine the convergence or divergence of each integral.

$$\text{(a)} \int_5^\infty \frac{1}{x^2}\,dx \qquad \text{(b)} \int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \text{(c)} \int_1^8 \frac{1}{\sqrt[3]{x}}\,dx$$

Solution. (a) The integral matches the form required by the P-Test with $p = 2 > 1$, so the improper integral converges. The P-Test does *not* tell us the value of the integral.

(b) The integral matches the form required by the P-Test with $p = \frac{1}{2} < 1$, so the improper integral diverges.

(c) This is not an improper integral, so the P-Test does not apply, but:

$$\int_1^8 \frac{1}{\sqrt[3]{x}}\,dx = \int_1^8 x^{-\frac{1}{3}}\,dx = \left[\frac{3}{2}x^{\frac{2}{3}}\right]_1^8 = \frac{3}{2}\left[8^{\frac{2}{3}} - 1^{\frac{2}{3}}\right] = \frac{3}{2}[4-1] = \frac{9}{2}$$

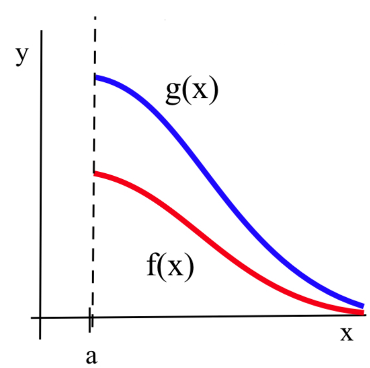
so the value of the integral is 4.5. ◀

The following **Comparison Test** enables us to determine the convergence or divergence of an improper integral of a positive function by comparing this function with functions whose improper integrals we already know converge or diverge.

Comparison Test for Integrals of Positive Functions: Suppose $f(x)$ and $g(x)$ are defined and integrable for all $x \geq a$ with $0 \leq f(x) \leq g(x)$. Then:

- $\int_a^\infty g(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converges.
- $\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ diverges.

The proof involves a straightforward application of the definition of an improper integral and various facts about limits, but the graph in the margin provides a geometrically intuitive way of understanding why these results must hold. If $\int_a^\infty g(x)\,dx$ converges, then the area under the graph of $g(x)$ is finite, so the (smaller) area under the graph of $f(x)$ must also be finite, and $\int_a^\infty f(x)\,dx$ must converge as well. If $\int_a^\infty f(x)\,dx$ diverges, then the area under the graph of $f(x)$ is infinite, so the (bigger) area under the graph of $g(x)$ must also be infinite, and $\int_a^\infty g(x)\,dx$ must also diverge.



Just as important as understanding what this Comparison Test *does* tell us is realizing what the Comparison Test does *not* tell us. If $\int_a^\infty g(x)\,dx$ diverges, or if $\int_a^\infty f(x)\,dx$ converges, the Comparison Test *tells us absolutely nothing* about the convergence or divergence of the other integral. Geometrically, if $\int_a^\infty g(x)\,dx$ diverges, then the area under the graph of $g(x)$ is infinite, but the (smaller) area under the graph of $f(x)$ could be either finite or infinite, so we can't conclude anything about the convergence or divergence of $\int_a^\infty f(x)\,dx$. Likewise, if $\int_a^\infty f(x)\,dx$ converges, then the area under the graph of $f(x)$ is finite, but the (bigger) area under the graph of $g(x)$ could be either finite or infinite, so we can't conclude anything about the convergence or divergence of $\int_a^\infty g(x)\,dx$.

Example 6. Determine whether each of these integrals is convergent or divergent by comparing it with an appropriate integral that you already know converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{7}{x^3+5}\,dx$ (b) $\displaystyle\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ (c) $\displaystyle\int_6^\infty \frac{9}{\sqrt{x-5}}\,dx$

Solution. (a) We know that $5 > 0$ and $x \ge 1$ so:

$$x^3+5 > x^3 \;\Rightarrow\; 0 < \frac{1}{x^3+5} < \frac{1}{x^3} \;\Rightarrow\; 0 < \frac{7}{x^3+5} < \frac{7}{x^3}$$

We also know, by the P-Test with $p = 3 > 1$, that $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ converges, so $\displaystyle\int_1^\infty \frac{7}{x^3}\,dx = 7\cdot\int_1^\infty \frac{1}{x^3}\,dx$ also converges. By the Comparison Test, the smaller integral $\displaystyle\int_1^\infty \frac{7}{x^3+5}\,dx$ must converge as well.

The numerator of the original integrand is constant and the dominant term in the denominator of that integrand is x^3, so it should make sense to compare the original integrand with $\frac{1}{x^3}$.

(b) We know that $-1 \le \sin(x) \le 1$, so:

$$2 \le 3+\sin(x) \le 4 \;\Rightarrow\; 0 < \frac{3+\sin(x)}{x^2} \le \frac{4}{x^2} = 4\cdot\frac{1}{x^2}$$

By the P-Test with $p = 2 > 1$, $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, so $\displaystyle\int_1^\infty \frac{4}{x^2}\,dx = 4\cdot\int_1^\infty \frac{1}{x^2}\,dx$ also converges. By the Comparison Test, the smaller integral $\displaystyle\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ must converge as well.

The numerator of the original integrand fluctuates between 2 and 4 while the dominant (and only) term in its denominator is x^2, so it should make sense to compare the original integrand with $\frac{1}{x^2}$.

(c) We know that $\sqrt{u}$ is an increasing function, so:

$$x-5 < x \;\Rightarrow\; \sqrt{x-5} < \sqrt{x} \;\Rightarrow\; \frac{1}{\sqrt{x-5}} > \frac{1}{\sqrt{x}}$$

By the P-Test with $p = \frac{1}{2} < 1$, $\displaystyle\int_6^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, so the bigger integral $\displaystyle\int_6^\infty \frac{1}{\sqrt{x-5}}\,dx$ must also diverge. ◀

5.7 Problems

In Problems 1–26, evaluate each improper integral, or show why it diverges.

1. $\int_{10}^{\infty} \frac{1}{x^3}\,dx$

2. $\int_{e}^{\infty} \frac{5}{x \cdot [\ln(x)]^2}\,dx$

3. $\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2}\,dx$

4. $\int_{1}^{\infty} \frac{2}{e^x}\,dx$

5. $\int_{e}^{\infty} \frac{5}{x \cdot \ln(x)}\,dx$

6. $\int_{0}^{\infty} \frac{x}{1+x^2}\,dx$

7. $\int_{3}^{\infty} \frac{1}{x-2}\,dx$

8. $\int_{3}^{\infty} \frac{1}{(x-2)^2}\,dx$

9. $\int_{3}^{\infty} \frac{1}{(x-2)^3}\,dx$

10. $\int_{3}^{\infty} \frac{1}{x+2}\,dx$

11. $\int_{3}^{\infty} \frac{1}{(x+2)^2}\,dx$

12. $\int_{3}^{\infty} \frac{1}{(x+2)^3}\,dx$

13. $\int_{0}^{4} \frac{1}{\sqrt{x}}\,dx$

14. $\int_{0}^{8} \frac{1}{\sqrt[3]{x}}\,dx$

15. $\int_{0}^{16} \frac{1}{\sqrt[4]{x}}\,dx$

16. $\int_{0}^{2} \frac{1}{\sqrt{2-x}}\,dx$

17. $\int_{0}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$

18. $\int_{0}^{2} \frac{3x^2}{\sqrt{8-x^3}}\,dx$

19. $\int_{-2}^{\infty} \sin(x)\,dx$

20. $\int_{\pi}^{\infty} \sin(x)\,dx$

21. $\int_{0}^{\frac{\pi}{2}} \tan(x)\,dx$

22. $\int_{0}^{3} \frac{1}{x-2}\,dx$

23. $\int_{0}^{\pi} \tan(x)\,dx$

24. $\int_{3}^{\infty} \frac{1}{x\sqrt{x}}\,dx$

25. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

26. $\int_{-\infty}^{0} \frac{1}{x^2}\,dx$

In Problems 27–44, determine whether each improper integral converges or diverges, but do not evaluate the integral.

27. $\int_{1}^{\infty} \frac{1}{x^5}\,dx$

28. $\int_{2}^{\infty} \frac{1}{\sqrt[5]{x}}\,dx$

29. $\int_{3}^{\infty} \frac{1}{\sqrt[5]{x^6}}\,dx$

30. $\int_{4}^{\infty} \frac{1}{\sqrt[5]{x^4}}\,dx$

31. $\int_{5}^{\infty} \frac{1}{\sqrt{x\sqrt[3]{x}}}\,dx$

32. $\int_{6}^{\infty} x^{-\frac{4}{7}}\,dx$

33. $\int_{7}^{\infty} x^{-\frac{7}{4}}\,dx$

34. $\int_{8}^{\infty} \frac{1}{1+x^2}\,dx$

35. $\int_{3}^{\infty} \frac{1}{x^2+5}\,dx$

36. $\int_{4}^{\infty} \frac{7}{x^2+5}\,dx$

37. $\int_{5}^{\infty} \frac{1}{x^3+x}\,dx$

38. $\int_{6}^{\infty} \frac{1}{x-2}\,dx$

39. $\int_{e}^{\infty} \frac{7}{x+\ln(x)}\,dx$

40. $\int_{2}^{\infty} \frac{1}{x^2-1}\,dx$

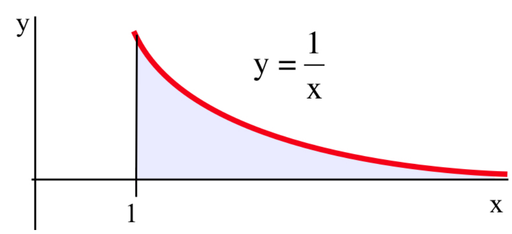
41. $\int_{\pi}^{\infty} \frac{1+\cos(x)}{x^2}\,dx$

42. $\int_{0}^{\infty} \frac{x^4}{x^6+1}\,dx$

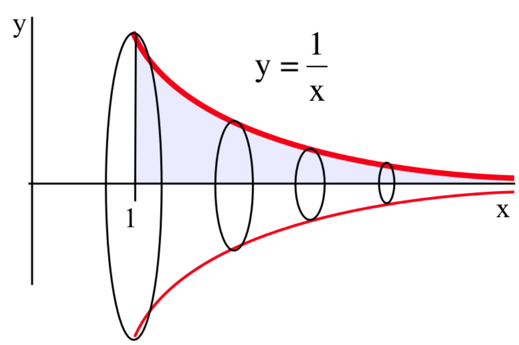
43. $\int_{0}^{\infty} \frac{x^4}{x^5+1}\,dx$

44. $\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$

45. Example 3(b) showed that $\int_{1}^{M} \frac{1}{x}\,dx$ grew arbitrarily large as M grew arbitrarily large, so no finite amount of paint would cover the region bounded by the x-axis and the graph of $f(x) = \frac{1}{x}$ for $x > 1$:

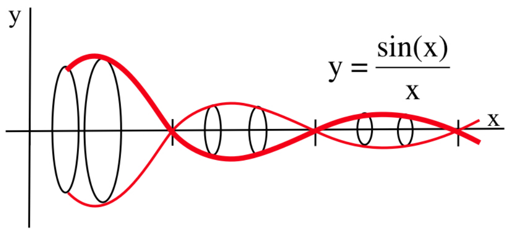


Show that the volume of the solid obtained when the region graphed above is revolved about the x-axis:

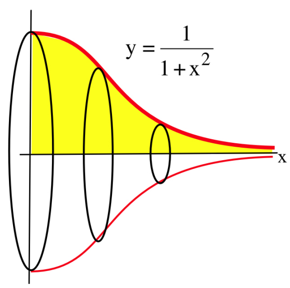


is finite, so the 3-dimensional trumpet-shaped region can be filled with a finite amount of paint. Does this present a contradiction?

46. Determine whether or not the volume of the solid obtained by revolving the region between the x-axis and the graph of $f(x) = \dfrac{\sin(x)}{x}$ for for $x \geq 1$ (see below) about the x-axis is finite.

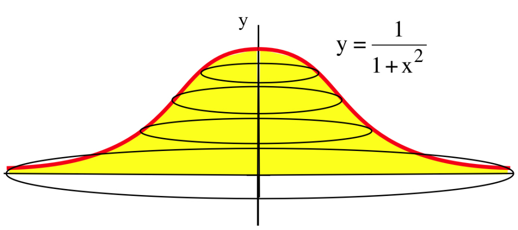


47. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = \dfrac{1}{x^2+1}$ (see below) is revolved about the x-axis.



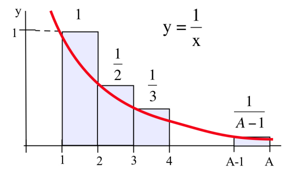
48. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = e^{-x}$ is revolved about the x-axis.

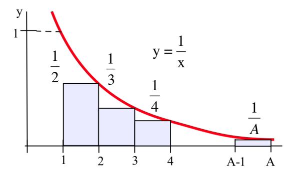
49. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = \dfrac{1}{x^2+1}$ (see below) is revolved about the y-axis.



50. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = e^{-x}$ is revolved about the y-axis.

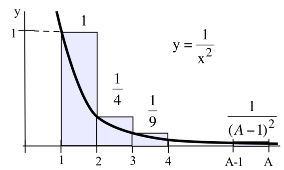
51. Use the figure below left to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x}\,dx$ or $\displaystyle\sum_{k=1}^{A-1} \frac{1}{k}$.

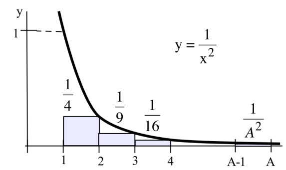




52. Use the figure above right to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x}\,dx$ or $\displaystyle\sum_{k=2}^{A} \frac{1}{k}$.

53. Use the figure below left to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x^2}\,dx$ or $\displaystyle\sum_{k=1}^{A-1} \frac{1}{k^2}$.





54. Use the figure above right to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x^2}\,dx$ or $\displaystyle\sum_{k=2}^{A} \frac{1}{k^2}$.

The **Laplace transform** of a function $f(t)$ is defined using an improper integral involving a parameter s:

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

Laplace transforms are often used to solve differential equations.

55. Compute the Laplace transform of the constant function $f(t) = 1$.

56. Compute the Laplace transform of $f(t) = e^{4t}$.

57. Define a function $g(t)$ by:

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

Compute the Laplace transform of $g(t)$.

58. Define a function $h(t)$ by:

$$h(t) = \begin{cases} 1 & \text{if } t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

Compute the Laplace transform of $h(t)$.

59. Devise a "Q-Test" to determine whether $\displaystyle\int_0^b \frac{1}{x^q}\,dx$ converges or diverges for any number $b > 0$.

60. Use the result of the previous problem to test the convergence of $\displaystyle\int_0^e \frac{1}{\sqrt[3]{x}}\,dx$ and $\displaystyle\int_0^\pi \frac{1}{x\cdot\sqrt[3]{x}}\,dx$.

5.7 Practice Answers

1. $\displaystyle\int_4^\infty \pi\cdot\left(\frac{1}{x}\right)^2 dx = \pi\int_4^\infty \frac{1}{x^2}\,dx$

2. (a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx = \lim_{M\to\infty}\int_1^M x^{-3}\,dx = \lim_{M\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^M = \lim_{M\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{M^2}+\frac{1}{2}\right] = \frac{1}{2}$

   (b) Replacing ∞ with M in the upper limit of the integral:

$$\begin{aligned}\int_0^\infty \sin(x)\,dx &= \lim_{M\to\infty}\int_0^M \sin(x)\,dx = \lim_{M\to\infty}\Big[-\cos(x)\Big]_0^M \\ &= \lim_{M\to\infty}\Big[-\cos(M)+1\Big] = \lim_{M\to\infty}[1-\cos(M)\end{aligned}$$

   This limit does not exist (the values of $1-\cos(M)$ oscillate between 0 and 2 and never approach any fixed number) so the improper integral diverges.

3. (a) The integral is improper at its upper limit, where $x = 10$, so:

$$\begin{aligned}\int_1^{10} \frac{1}{\sqrt{10-x}}\,dx &= \lim_{c\to 10^-}\int_1^c (10-x)^{-\frac{1}{2}}\,dx = \lim_{c\to 10^-}\left[-2\sqrt{10-x}\right]_1^c \\ &= \lim_{c\to 10^-}\left[-2\sqrt{10-c}+2\sqrt{9}\right] = 0+2\cdot 3 = 6\end{aligned}$$

   (b) The integral is improper at its lower limit, where $x = 0$, so:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+}\Big[\ln(|x|)\Big]_c^1 = \lim_{c\to 0^+}[\ln(1)-\ln(c)] = \infty$$

   so the integral diverges.

5.8 *Additional Applications*

This section introduces two additional applications of integrals that once again illustrate the process of going from an applied problem to a Riemann sum and on to a definite integral. A third application does not follow this process: it uses the idea of "area" to model an election and to qualitatively understand why certain election outcomes occur.

The main point of this section is to demonstrate the power of definite integrals to solve a wide variety of applied problems. Each of these new applications is treated more briefly than those in the previous sections. These are far from the only applications that could be included here. By now, however, you should have developed enough of an understanding of the Riemann-sum process so that when you encounter other applications (in physics, engineering, biology, statistics, probability, economics, computer graphics . . .) you will be able to use that process to set up an integral to compute or approximate a desired quantity.

Fluid Pressures and Forces

In physics, **pressure** is defined as force per unit of area. The **hydrostatic pressure** on an object immersed in a fluid (such as water) is the product of the density of that fluid and the depth of the object:

$$\text{pressure} = (\text{density})(\text{depth})$$

Fluids exert pressure in all possible directions, and the forces due to this pressure act on solid objects in a direction perpendicular to the object.

The total **hydrostatic force** applied against an immersed object is the sum of the hydrostatic forces against each part of the object.

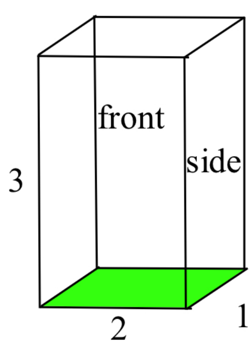
If an entire object is at the same depth, we can determine the total hydrostatic force against that (necessarily flat) object simply by multiplying the density of the fluid times the depth of the object times the object's area. If the unit of density is "pounds per cubic foot" and depth is measured in "feet," then the unit of pressure is "pounds per square foot," a measure of force per unit of area. If pressure, with the units "pounds per square foot," is multiplied by an area with units "square feet," the result is a force, measured in "pounds."

In the metric system, the standard unit of force is a **pascal** (abbreviated "Pa"):

$$1\,\text{Pa} = 1\,\frac{\text{N}}{\text{m}}$$

and named after Blaise Pascal (1623–1662), a French mathematician, physicist, inventor, writer and philosopher.

Example 1. Find the total hydrostatic force against the bottom of the freshwater aquarium shown in the margin.

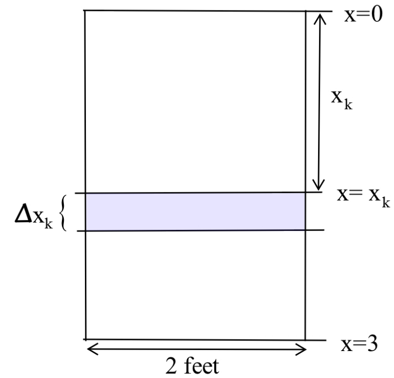


Solution. Water's desity is $62.5\,\frac{\text{lb}}{\text{ft}^3}$, so the total hydrostatic force is:

$$(\text{density}) \cdot (\text{depth}) \cdot (\text{area}) = \left(62.5\,\frac{\text{lb}}{\text{ft}^3}\right) \cdot (3\,\text{ft}) \cdot \left(2\,\text{ft}^2\right)$$

or 375 lbs. Finding the total hydrostatic force against the *front* of the aquarium is a very different problem, because different parts of that front face are located at different depths and subject to different pressures. ◀

To compute the force against the *front* of the aquarium, we can partition it into n thin horizontal slices (see margin) and focus on one of them. Because the slice is very thin, every part of the k-th slice is at (almost) the same depth, so every part of that slice is subject to (almost) the same pressure. We can approximate the total hydrostatic force against the slice at the depth x_k as:



$$(\text{density}) \cdot (\text{depth}) \cdot (\text{area}) = \left(62.5\,\frac{\text{lb}}{\text{ft}^3}\right) \cdot (x_k\,\text{ft}) \cdot (2\,\text{ft})\,(\Delta x_k\,\text{ft})$$

or $125x_k \cdot \Delta x_k$ lbs. The total hydrostatic force against the front is the sum of the forces against each slice:

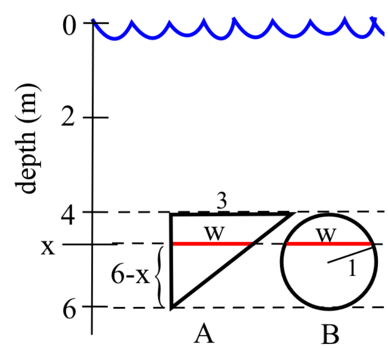
$$\text{total hydrostatic force} \approx \sum_{k=0}^{n} 125x_k \cdot \Delta x_k$$

which is a Riemann sum. The limit of this Riemann sum as the slices get thinner ($\Delta x_k \to 0$) is a definite integral:

$$\sum_{k=0}^{n} 125x_k \cdot \Delta x_k \;\longrightarrow\; \int_{x=0}^{x=3} 125x\,dx = \Big[62.5x^2\Big]_{x=0}^{x=3} = 562.5\,\text{lbs}$$

Practice 1. Find the total hydrostatic force against one side of the aquarium and the total force against the entire aquarium.

Example 2. Find the total hydrostatic force against viewing windows A and B in the freshwater aquarium shown in the margin.



Solution. For window A, using similar triangles, the width w of a slice at depth x m satisfies:

$$\frac{w}{6-x} = \frac{3}{2} \;\Rightarrow\; w = \frac{3}{2}(6-x) = 9 - \frac{3}{2}x$$

so the area of a slice of height Δx_k m at depth x_k m is $\left(9 - \frac{3}{2}x_k\right)\Delta x_k\,\text{m}^2$. The density of water is $1000\,\frac{\text{kg}}{\text{m}^3}$. Multiplying this density by area (with units m^2) would give kg per m, but pressure is measured in N per m, so we need to multiply by the acceleration due to gravity, $g \approx 9.81\,\frac{\text{m}}{\text{sec}^2}$. The hydrostatic force applied to the k-th slice is thus:

$$1000(9.81)x_k\left(9 - \frac{3}{2}x_k\right)\Delta x_k$$

and the total hydrostatic force applied to the window is therefore:

$$\int_{x=4}^{x=6} 9810\left[9x - \frac{3}{2}x^2\right]dx = 9810\left[\frac{9}{2}x^2 - \frac{1}{2}x^3\right]_4^6$$
$$= 9810\,[(162-108)-(72-32)] = 137340\,\text{N}$$

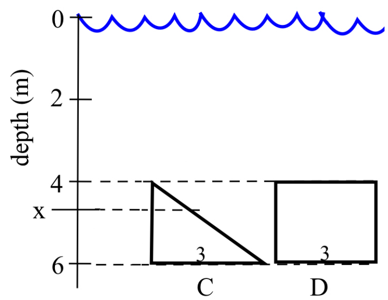
For window B, applying the Pythagorean Theorem yields:

$$(5-x)^2 + \left(\frac{w}{2}\right)^2 = 1 \Rightarrow w = 2\sqrt{1-(5-x)^2}$$

The total hydrostatic force is thus:

$$\int_{x=4}^{x=6} 1000(9.81)x \cdot 2\sqrt{1-(5-x)^2}\,dx$$

which (using technology) is approximately 154,095 N. ◀



Practice 2. Find the total hydrostatic force against viewing windows C and D of the freshwater aquarium shown in the margin.

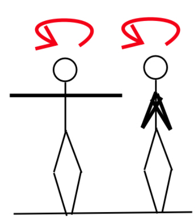
Because the total force at even moderate depths is so large, underwater windows are made of thick glass or plastic and strongly secured to their frames. Similarly, the bottom of a dam is much thicker than the top in order to withstand the greater force against the bottom.

Kinetic Energy

Physicists define the **kinetic energy** (energy of motion) of an object with mass m and velocity v to be:

$$\text{KE} = \frac{1}{2}m \cdot v^2$$

The greater the mass of an object or the faster it is moves, the greater its kinetic energy. If every part of the object has the same velocity, computing its kinetic energy becomes relatively easy.

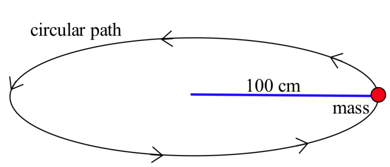


Sometimes, however, different parts of an object move with different velocities. For example, if an ice skater is spinning with an angular velocity of 2 revolutions per second, her arms travel further in one second (have a greater *linear* velocity) when they are extended than when drawn in close to her body (see margin). So the ice skater, spinning at 2 revolutions per second, has greater kinetic energy when her arms are extended. Similarly, the tip of a rotating propeller (or the barrel of a swinging baseball bat) has a greater linear velocity than other parts of the propeller (or the bat's handle).

If the units of mass are kg and the units of velocity are m/sec^2, then:

$$\text{KE} = \frac{1}{2}(m\,\text{kg}) \cdot \left(v\,\frac{\text{m}}{\text{sec}}\right)^2 = \frac{1}{2}mv^2\,\text{kg} \cdot \text{m} \cdot \frac{\text{m}}{\text{sec}^2}$$

so the units of kinetic energy are N-m, or Joules, the same as work. Similarly, if the units of mass are g and the units of velocity are cm/sec^2, then the units of kinetic energy are dyn-cm, or ergs.



Example 3. A point-mass of 1 gram at the end of a (massless) 100-cm string rotates at a rate of 2 revolutions per second (see margin).

(a) Find the kinetic energy of the point-mass.

(b) Find its kinetic energy if the string is 200 cm long.

Solution. (a) In one second, the mass travels twice around a circle with radius 100 cm so it travels $2 \cdot (2\pi \cdot 100) = 400\pi$ cm. Its velocity is thus $v = 400\pi$ cm/sec, and:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(1\text{ g}) \cdot \left(400\pi\ \frac{\text{cm}}{\text{sec}}\right)^2 = 80000\pi^2 \text{ ergs}$$

or about 0.079 J.

(b) If the string is 200 cm long, then the velocity is $2 \cdot (2\pi \cdot 400) = 800\pi$ cm/sec and:

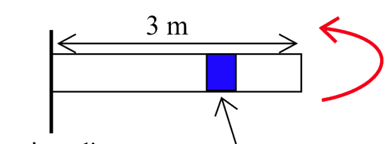
$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(1\text{ g}) \cdot \left(800\pi\ \frac{\text{cm}}{\text{sec}}\right)^2 = 320000\pi^2 \text{ ergs}$$

or about 0.316 J. ◀

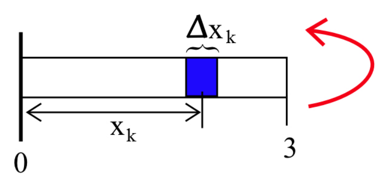
When the length of the string doubles, the velocity doubles and the kinetic energy quadruples.

Practice 3. A 1-gram point-mass at the end of a 2-meter (massless) string rotates at a rate of 4 revolutions per second. Find the kinetic energy of the point mass.

If different parts of a rotating object are different distances from the axis of rotation, then those parts have different linear velocities, and it becomes more difficult to calculate the total kinetic energy of the object. By now the method should seem very familiar: partition the object into small pieces, approximate the kinetic energy of each piece, and add the kinetic energies of the small pieces (a Riemann sum) to approximate the total kinetic energy of the object. The limit of the Riemann sum as the pieces get smaller is a definite integral.



Example 4. The density of a narrow bar (see margin) is 5 grams per meter of length. Find the kinetic energy of the 3-meter-long bar when it rotates at a rate of 2 revolutions per second.



Solution. Partition the bar (see margin) into n pieces so that the mass of the k-th piece is:

$$m_k \approx (\text{length}) \cdot (\text{density}) = (\Delta x_k \text{ m})\left(5\ \frac{\text{g}}{\text{m}}\right) = 5 \cdot \Delta x_k \text{ g}$$

During one second, the k-th piece, located at a distance of x_k m from the pivot line, will make two revolutions, traveling approximately:

$$2\left(2\pi\left[\text{radius}\right]\right) = 4\pi\left[100x_k \text{ cm}\right] = 400\pi x_k \text{ cm}$$

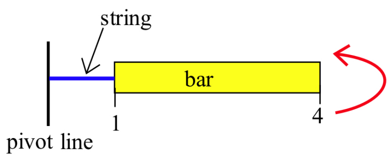
so $v_k \approx 400\pi x_k$ cm/sec. The kinetic energy of the k-th piece is:

$$\frac{1}{2}m_k \cdot v_k^2 \approx \frac{1}{2}(5\Delta x_k \text{ g})\left(400\pi x_k\ \frac{\text{cm}}{\text{sec}}\right)^2 = 400000\pi^2 x_k^2 \text{ ergs}$$

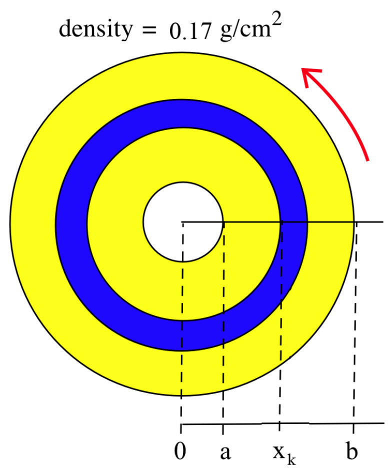
and the total kinetic energy of the rotating bar is therefore:

$$\sum_{k=1}^{n} 400000\pi^2 x_k^2 \cdot \Delta x_k \longrightarrow \int_{x=0}^{x=3} 400000\pi^2 x^2 \, dx = 400000\pi^2 \left[\frac{1}{3}x^3\right]_0^3$$

which equals $3600000\pi^2$ ergs, or about 3.55 J. ◀



Practice 4. Find the kinetic energy of the bar in the previous Example if it rotates at 2 revolutions per second at the end of a 100-centimeter (massless) string (see margin).



The "slices" that give rise to the Riemann sum in this problem are—unlike most examples we have seen previously—*not* rectangles. We also use here the notion that if Δx is small, then $(\Delta x)^2$ is *very* small, so we can essentially ignore it in our approximation of area.

Example 5. Find the kinetic energy of the thin, flat object with density 0.17 g/cm^2 shown in the margin when it rotates at 45 revolutions per minute.

Solution. We can partition the object along one radial line and form n annular "slices" each Δx cm wide. Then the "slice" between x_k and $x_k + \Delta x$ is a thin annulus (a disk with a smaller disk removed from its center) with area:

$$\begin{aligned} \pi\left(x_k + \Delta x\right)^2 - \pi\left(x_k\right)^2 &= \pi\left[x_k^2 + 2x_k\Delta x + (\Delta x)^2 - x_k^2\right] \\ &= 2\pi x_k \Delta x + \pi\left(\Delta x\right)^2 \approx 2\pi x_k \Delta x \end{aligned}$$

and mass $(0.17)2\pi x_k \Delta x$. During one revolution, a point on this slice travels approximately $2\pi x_k$ cm and 45 rev/min is equivalent to $\frac{3}{4}$ rev/sec, so the linear velocity of the point is $2\pi x_k \cdot \frac{3}{4} = \frac{3}{2}\pi x_k$ cm/sec. The kinetic energy of this slice is therefore:

$$\left((0.17)2\pi x_k \Delta x\right)\left(\frac{3}{2}\pi x_k\right)^2 = \frac{9}{2}(0.17)\pi^2 x_k^3 \Delta x$$

so the total kinetic energy of the object is:

$$\sum_{k=1}^{n} \frac{9}{2}(0.17)\pi^2 x_k^3 \cdot \Delta x \longrightarrow \int_a^b \frac{9}{2}(0.17)\pi^2 x^3 \, dx$$

Evaluating this integral yields:

$$\frac{9}{2}(0.17)\pi^2\left[\frac{1}{4}x^4\right]_a^b = \frac{9}{8}(0.17)\pi^2\left[b^4 - a^4\right]$$

Because b is raised to the fourth power, a small increase in the value of b (if $b > 1$) leads to a large increase in the object's kinetic energy.

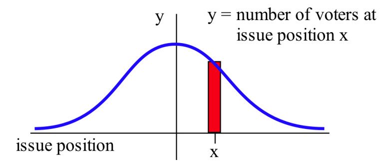
In the not-so-distant past your grandparents (and perhaps even your parents) used such objects to listen to music—and each one only held two songs!

If $a = 0.75$ in ≈ 1.905 cm and $b = 3.75$ in ≈ 9.525 cm, the total mass of the object is 42 g and its total kinetic energy is about 15,512 ergs. ◀

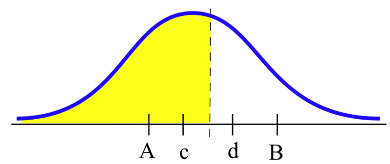
Areas and Elections

The previous applications in this chapter have used definite integrals to determine areas, volumes, pressures and energies precisely. But exactness and numerical precision are not the same as "understanding," and sometimes we can gain insight and understanding simply by determining which of two areas or integrals is larger. One situation of this type involves models of elections.

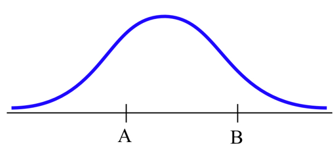
Suppose the voters of a state have been surveyed about their positions on a single issue, with their responses recorded on a quantitative scale. The distribution of voters who place themselves at each position on this issue appears in the margin. Suppose also that each voter casts his or her vote for the candidate whose position on this issue is closest to his or her position.

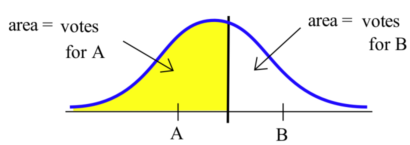


If two candidates have taken the positions labeled A and B, then a voter at position c votes for the candidate at A because A is closer to c than B is to c. Similarly, a voter at position d votes for the candidate at B. The total votes for the candidate at A in this election is represented by the shaded area under the curve, and the candidate with the larger number of votes — the larger area — wins the election. In this illustration, the candidate at A wins.



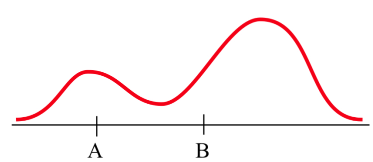
Example 6. The distribution of voters on an issue appears below left. If these voters decide between candidates on the basis of that single issue, which candidate will win the election?



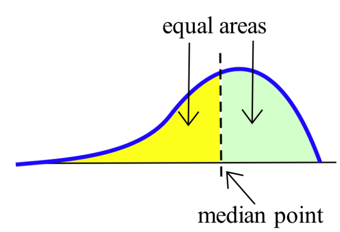


Solution. The figure above right illustrates that A corresponds to a larger area (more votes) than B: A will win. ◀

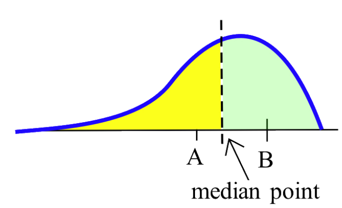
Practice 5. In an election between candidates with positions A and B in the margin figure, who will win?

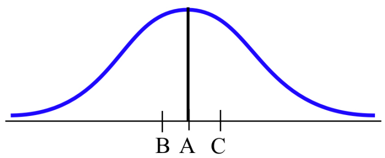


If voters behave as described and if the election is between two candidates, then we can give the candidates some advice. The best position for a candidate is at the "median point," the location that divides the voters into two equal-sized (equal-area) groups so that half of the voters are on one side of the median point and half are on the other side (see margin). A candidate at the median point gets more votes than a candidate at any other point. (Why?)



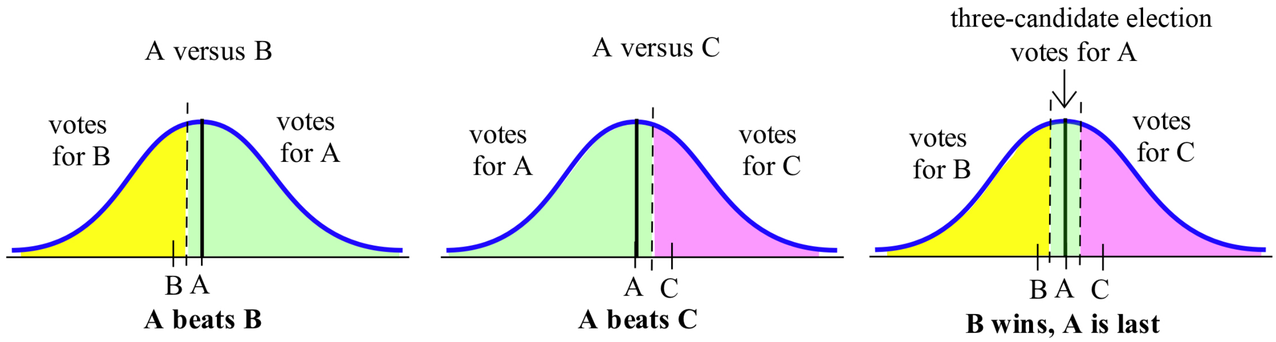
If two candidates have positions on opposite sides of the median point (see margin), then a candidate can get more votes by moving a bit toward the median point. This "move toward the middle ground"





commonly occurs in elections as candidates attempt to sell themselves as "moderates" and their opponents as "extremists."

If more than two candidates are running in an election, the situation changes dramatically. A candidate at the median position, the unbeatable place in a two-candidate election can even get the fewest votes. If the margin figure represents the distribution of voters on the single issue in the election, then candidate A would beat B in an election just between A and B (below left) and A would beat C in an election just between A and C (below center). But in an election among all three candidates, A would get the fewest votes (below right).



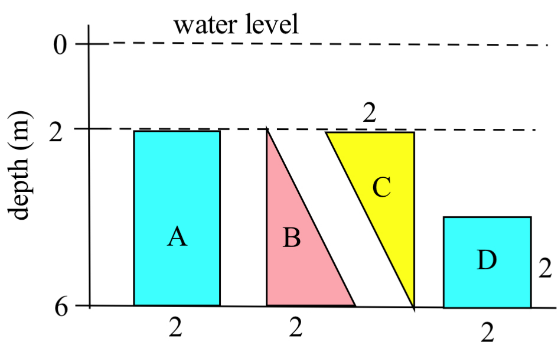
This type of situation really does occur. It leads to the political saying about a primary election with many candidates and a general election between the final nominees from two parties: "extremists can win primaries, but moderates are elected to office."

The previous discussion of elections and areas is greatly oversimplified. Most elections involve several issues of different importance to different voters, and the views of the voters are seldom completely known before the election. Many candidates take "fuzzy" positions on issues. And it is not even certain that real voters vote for the candidate with the "closest" position: perhaps they don't vote at all unless some candidate is "close enough" to their position. But this very simple model of elections can still help us understand how and why some things happen in elections. It is also a starting place for building more sophisticated models to help understand more complicated election situations and to test assumptions about how voters really do make voting decisions.

5.8 Problems

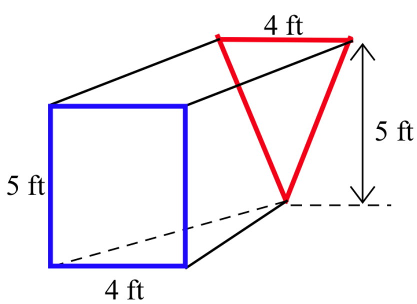
In Problems 1–5, use ρ for the density of the fluid in the given container.

1. Calculate the force against windows A and B in the figure below.

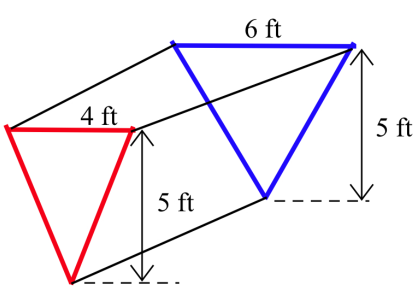


2. Calculate the force against windows C and D in the figure above.

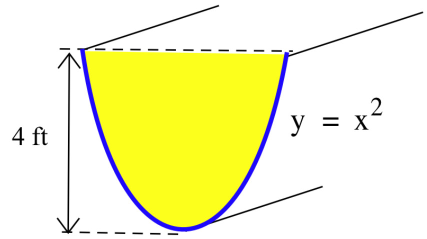
3. Calculate the total force against each end of the tank shown below. How does the total force against the ends of the tank change if the length of the tank is doubled?



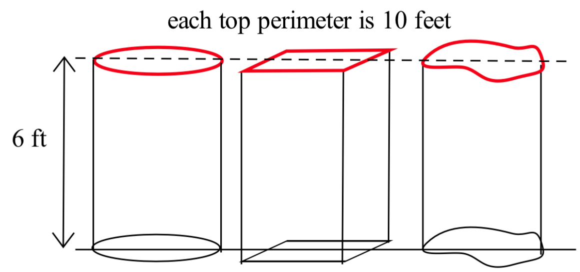
4. Calculate the total force against each end of the tank shown below.



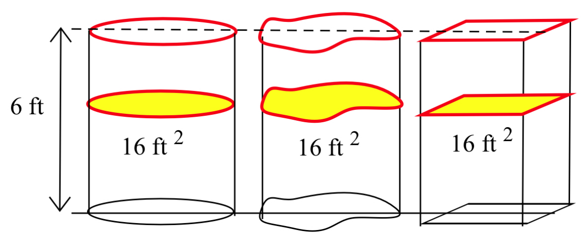
5. Calculate the total force against the end of the tank shown below.



6. The three tanks shown below are all 6 feet tall and the top perimeter of each tank is 10 feet. Which tank has the greatest total force against its sides?



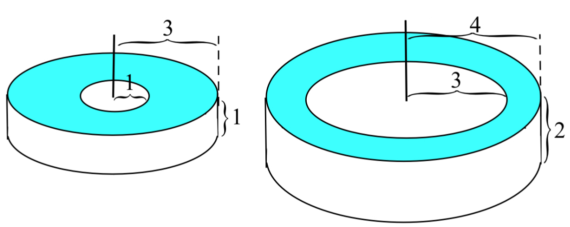
7. The three tanks shown below are all 6 feet tall and the cross-sectional area of each tank is 16 ft^2. Which tank has the greatest total force against its sides?



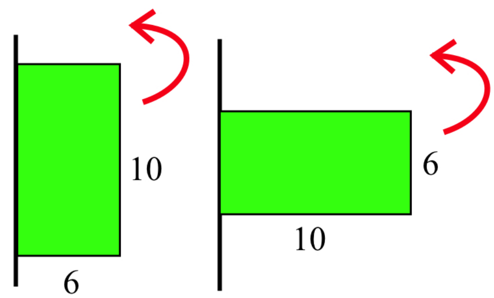
8. Calculate the total force against the bottom 2 feet of the sides of a tank with a square 40-foot by 40-foot base that is filled with water (a) to a depth of 30 feet. (b) to a depth of 35 feet.

9. Calculate the total force against the bottom 2 feet of the side of a cylindrical tank with a radius of 20 feet that is filled with water (a) to a depth of 30 feet. (b) to a depth of 35 feet.

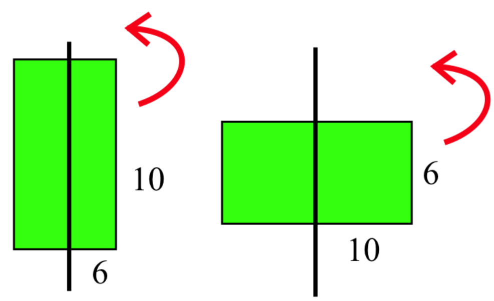
10. Calculate the total force against the side and bottom of a cylindrical aluminum soda can with diameter 6 cm and height 12 cm if it is filled with 385 g of soda. (Assume the can has been opened so carbonization is not a factor.)
11. Find the kinetic energy of a 20-gram object rotating at 3 revolutions per second at the end of (a) a 15-cm (massless) string and (b) a 20-cm string.
12. Each centimeter of a metal bar has a mass of 3 grams. Calculate the kinetic energy of the 50-centimeter bar if it is rotating at a rate of 2 revolutions per second about one of its ends.
13. Each centimeter of a metal bar has a mass of 3 grams. Calculate the kinetic energy of the 50-centimeter bar if it is rotating at a rate of 2 revolutions per second at the end of a 10-cm cable.
14. Calculate the kinetic energy of a 20-gram meter stick if it is rotating at a rate of 1 revolution per second about one of its ends.
15. Calculate the kinetic energy of a 20-gram meter stick if it is rotating at a rate of 1 revolution per second about its center point.
16. A flat, circular plate is made from material that has a density of 2 grams per cubic centimeter. The plate is 5 centimeters thick, has a radius of 30 centimeters and is rotating about its center at a rate of 2 revolutions per second. (a) Calculate its kinetic energy. (b) Find the radius of a plate that would have twice the kinetic energy of the first plate, assuming the density, thickness and rotation rate are the same.
17. Each "washer" in the figure below is made from material with density of 1 gram per cm^3, and each is rotating about its center at a rate of 3 revolutions per second. Calculate the kinetic energy of each washer (dimensions are in cm).



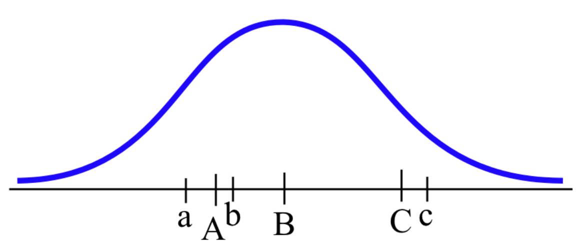
18. The rectangular plate shown below is 1 cm thick, 10 cm long and 6 cm wide and is made of a material with a density of 3 grams per cm^3. Calculate the kinetic energy of of the plate if it is rotating at a rate of 2 revolutions per second (a) about its 10-cm side and (b) about its 6-cm side.



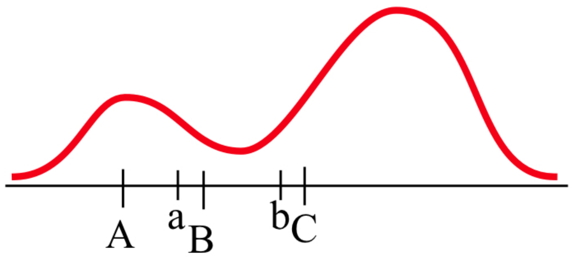
19. Calculate the kinetic energy of the plate in Problem 18 if it is rotating at a rate of 2 revolutions per second about a vertical line through the center of the plate, as shown below left.



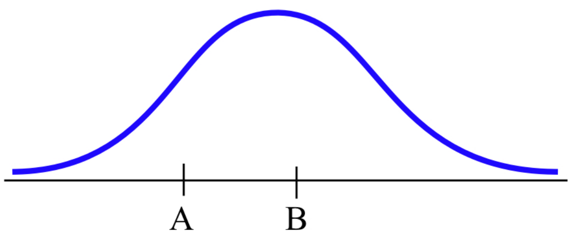
20. Calculate the kinetic energy of the plate in Problem 18 if it is rotating at a rate of 2 revolutions per second about a vertical line through the center of the plate, as shown above right.
21. For the voter distribution shown below, which candidates would the voters at positions a, b and c vote for?



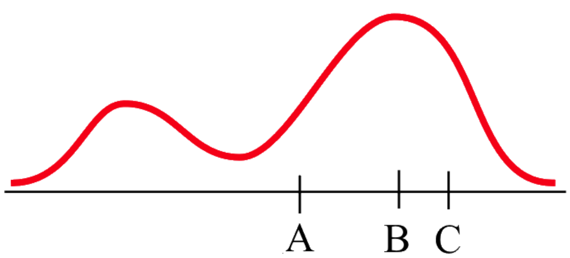
22. For the voter distribution shown below, which candidates would the voters at positions a, b and c vote for?



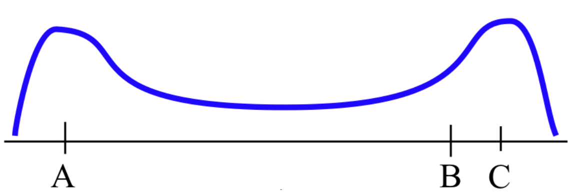
23. Shade the region representing votes for candidate A in the distribution shown below. Which candidate wins?



24. Shade the region representing votes for candidate A in the distribution shown below. Which candidate wins?



25. Refer to the voter distribution shown below.

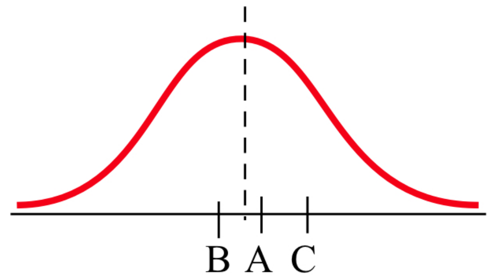


(a) Which candidate wins?

(b) If candidate B withdraws before the election, which candidate will win?

(c) If candidate B stays in the election but C withdraws, then who wins?

26. Refer to the voter distribution shown below.

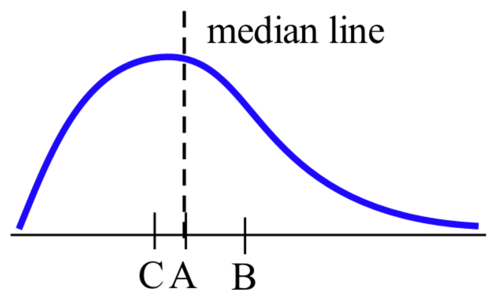


(a) Which candidate wins?

(b) If candidate B withdraws before the election, which candidate will win?

(c) If candidate B stays in the election but C withdraws, then who wins?

27. Refer to the voter distribution shown below.

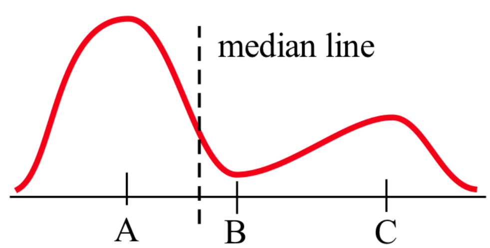


(a) If the election is between A and B, who wins?

(b) If the election is between A and C, who wins?

(c) If the election is among A, B and C, who wins?

28. Refer to the voter distribution shown below.



(a) If the election is between A and B, who wins?

(b) If the election is between A and C, who wins?

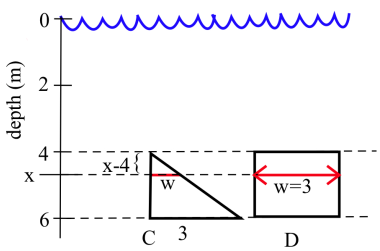
(c) If the election is among A, B and C, who wins?

29. Sketch a distribution for a two-issue election.

5.8 Practice Answers

1. The reasoning for a side of the aquarium is exactly the same as for the front, except a side is 1 foot long instead of 2, so the force is half of that against the front: 281.25 lbs. The total force against all sides (and the bottom) is:

$$2(281.25) + 2(562.5) + 375 = 2062.5 \text{ lbs}$$



2. For window C, using similar triangles (see margin), the width w of a slice at depth x m satisfies:

$$\frac{w}{x-4} = \frac{3}{2} \Rightarrow w = \frac{3}{2}(x-4) = \frac{3}{2}x - 6$$

so the area of a slice of height Δx_k m at depth x_k m is $\left(\frac{3}{2}x_k - 6\right) \Delta x_k \text{ m}^2$. The density of water is $1000 \frac{\text{kg}}{\text{m}^3}$ and $g \approx 9.81 \frac{\text{m}}{\text{sec}^2}$, so the hydrostatic force applied to the k-th slice is:

$$1000(9.81)x_k \left(\frac{3}{2}x_k - 6\right) \Delta x_k$$

and the total hydrostatic force applied to the window is therefore:

$$\int_{x=4}^{x=6} 9810 \left[\frac{3}{2}x^2 - 6x\right] dx = 9810 \left[\frac{1}{2}x^3 - 3x^2\right]_4^6$$
$$= 9810 \Big[(108 - 108) - (32 - 48)\Big] = 156960 \text{ N}$$

Windows A (from Example 2) and C (with a horizontal flip) fit together to form window D, so it is encouraging that the sum of the total hydrostatic forces against A and C is $137340 + 156960 = 294300$ N, the total hydrostatic force against window D.

For window D, the width is 3 at all depths, so the total hydrostatic force against the window is:

$$\int_{x=4}^{x=6} 1000(9.81)x \cdot 3\, dx = 14715x^2 \Big|_4^6 = 294300 \text{ N}$$

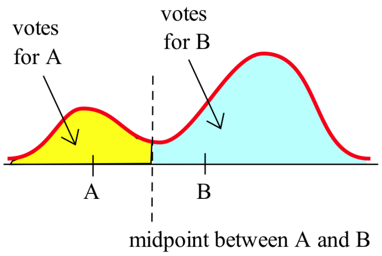
3. The object travels $2\pi\,(2 \text{ m}) = 4\pi$ m during one revolution, so during the 1 second it takes to make 4 revolutions, the object travels 16π m; its velocity is thus $v = 1600\pi \frac{\text{cm}}{\text{sec}}$ and its kinetic energy is:

$$\frac{1}{2}m \cdot v^2 = \frac{1}{2}(1 \text{ g}) \left(1600\pi \frac{\text{cm}}{\text{sec}}\right)^2 = 1280000\pi^2 \text{ ergs} \approx 12633094 \text{ ergs}$$

4. Everything remains the same as in Example 4, except for the endpoints of integration:

$$\int_{x=1}^{x=4} 400000\pi^2 x^2\, dx = 400000\pi^2 \left[\frac{1}{3}x^3\right]_1^4 = 8400000\pi^2 \text{ ergs}$$

which is approximately 82,904,677 ergs, or 8.29 J.



5. The shaded regions in the margin figure show the total votes for each candidate: B wins.

6
Differential Equations

This chapter introduces you to differential equations, a major field in applied and theoretical mathematics that provides useful tools for engineers, scientists and others studying changing phenomena.

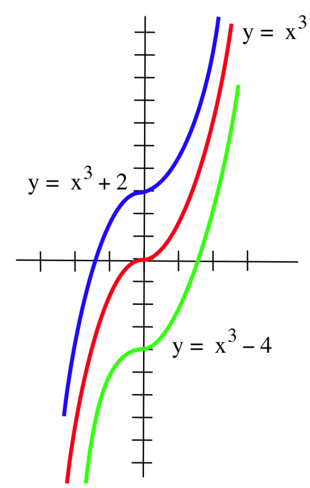
Physical laws of motion, heat and electricity can be expressed using differential equations. The growth of a population, the changing gene frequencies in that population, and the spread of a disease can be described by differential equations. Economic and social models use differential equations, and the earliest examples of "chaos" came from differential equations used for modeling atmospheric behavior. Some scientists even assert that the main purpose of a calculus course should be to teach people to understand and solve differential equations.

The purpose of this chapter is to introduce some basic ideas, vocabulary and techniques for differential equations and to explore additional applied problems that can be solved using calculus. Applications in this chapter include exponential population growth, calculating how long a population takes to double in size, radioactive decay and its use for dating ancient objects and detecting fraud, describing the motion of an object, and chemical mixtures and rates of reaction.

This chapter merely provides an introduction. In the near future you may very well take one or more classes solely devoted to solving differential equations.

6.1 *Introduction to Differential Equations*

Algebraic equations involve constants and variables, and solutions of algebraic equations typically involve numbers. For example, $x = 3$ and $x = -2$ are solutions of the algebraic equation $x^2 = x + 6$. **Differential equations** contain derivatives (or differentials) of functions and solutions of differential equations are functions. The differential equation $y' = 3x^2$ has infinitely many solutions, and two of those solutions are the functions $y = x^3 + 2$ and $y = x^3 - 4$ (see margin).



You have already solved lots of differential equations: every time you found an antiderivative of a function $f(x)$, you solved the differential equation $y' = f(x)$ to get a solution y. You have also used differential equations in applications. Areas, volumes, work and motion problems all involved integration and finding antiderivatives, so they all involved solving a differential equation. The differential equation $y' = f(x)$, however, is just the beginning. Other applications generate differential equations that may involve higher-order derivatives of y (such as y'') and functions of y as well as x.

Checking Solutions of Differential Equations

Whether a differential equation is easy or difficult to solve, it is important to be able to check that a possible solution actually satisfies the differential equation. A possible solution of an algebraic equation can be checked by putting the solution into the equation to see if it results in true statment: $x = 3$ is a solution of $5x + 1 = 16$ because $5(3) + 1 = 16$ is true; $x = 4$ is not a solution, because $5(4) + 1 \neq 16$.

Similarly, a solution of a differential equation can be checked by substituting the function (and its appropriate derivatives) into the original equation to see if the result is true: $y = x^2$ is a solution of $xy' = 2y$ because $y = x^2 \Rightarrow y' = 2x$ and $x \cdot 2x = 2 \cdot x^2$ is a true statement for all values of x.

Example 1. Check that (a) $y = x^2 + 5$ is a solution of $y'' + y = x^2 + 7$ and (b) $y = x + \frac{5}{x}$ is a solution of $y' + \frac{y}{x} = 2$.

Solution. (a) $y = x^2 + 5 \Rightarrow y' = 2x \Rightarrow y'' = 2$. Substituting these functions for y and y'' into the left side of the differential equation $y'' + y = x^2 + 7$ yields:

$$y'' + y = (2) + (x^2 + 5) = x^2 + 7$$

so $y = x^2 + 5$ is a solution of the differential equation.

(b) $y = x + \frac{5}{x} \Rightarrow y' = 1 - \frac{5}{x^2}$. Substituting these functions for y and y' into the left side of the differential equation $y' + \frac{y}{x} = 2$, we have:

$$y' + \frac{y}{x} = \left[1 - \frac{5}{x^2}\right] + \frac{1}{x}\left[x + \frac{5}{x}\right] = 1 - \frac{5}{x^2} + 1 + \frac{5}{x^2} = 2$$

which matches the right side of the original differential equation. ◀

Practice 1. Check that (a) $y = 2x + 6$ is a solution of $y - 3y' = 2x$ and (b) $y = e^{3x}$, $y = 5e^{3x}$ and $y = Ae^{3x}$ (where A is any constant) are all solutions of $y'' - 2y' - 3y = 0$.

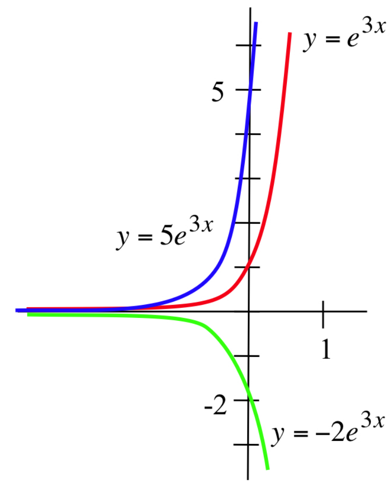
A solution of a differential equation with the **initial condition** $y(x_0) = y_0$ is a function that satisfies the differential equation as well as the initial condition. To check the solution of an **initial value problem** (or **IVP**), we must check that a solution function satisfies both the equation and the initial condition.

Example 2. Which of the given functions is a solution of the initial value problem $y' = 3y$, $y(0) = 5$?

(a) $y = e^{3x}$ (b) $y = 5e^{3x}$ (c) $y = -2e^{3x}$

Solution. All three functions satisfy the differential equation, but only one of them satisfies the initial condition that $y(0) = 5$. If $y = e^{3x}$, then $y(0) = e^{3(0)} = 1 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition (see margin). If $y = 5e^{3x}$, then $y(0) = 5e^{3(0)} = 5$ so $y = 5e^{3x}$ does satisfy the initial condition. If $y = -2e^{3x}$, then $y(0) = -2e^{3(0)} = -2 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition. ◀

You should check that they do.



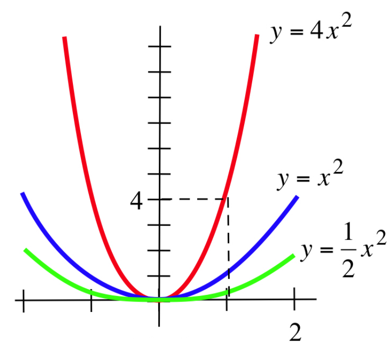
Practice 2. Which function is a solution of the initial value problem $y'' + 9y = 0$, $y(0) = 2$?

(a) $y = \sin(3x)$ (b) $y = 2\sin(3x)$ (c) $y = 2\cos(3x)$

Finding the Value of the Constant

Differential equations usually have many solutions, typically a whole "family" of them, with each solution in the family satisfying a different initial condition. To find which solution of a differential equation also satisfies a given initial condition of the form $y(x_0) = y_0$, we replace x and y in an equation describing the solution family with the values x_0 and y_0, then algebraically solve for the value of an unknown constant.

Example 3. For every value of C, the function $y = Cx^2$ is a solution of $xy' = 2y$ (see margin). Find the value of C so that $y(5) = 50$.



Solution. Substituting the initial condition $x = 5$ and $y = 50$ into the solution $y = Cx^2$:

$$50 = C(5^2) \Rightarrow C = \frac{50}{25} = 2$$

so the function $y = 2x^2$ satisfies both the differential equation and the initial condition. ◀

Practice 3. For every value of C, the function $y = e^{2x} + C$ is a solution of $y' = 2e^{2x}$. Find the value of C so that $y(0) = 7$.

Types of Differential Equations

In Chapter 14, we will begin studying **partial derivatives** of functions of more than one variable, which can appear in differential equations called **partial differential equations** (or **PDE**s). Because of this, we will call differential equations involving ordinary derivatives, such as $\frac{dy}{dx}$, $\frac{dy}{dt}$ or $\frac{d^2y}{dt^2}$, **ordinary differential equations** (or **ODE**s).

6.1 Problems

In Problems 1–10, check that the function y is a solution of the given differential equation.

1. $y' + 3y = 6$; $y = e^{-3x} + 2$
2. $y' - 2y = 8$; $y = e^{2x} - 4$
3. $y'' - y' + y = x^2$; $y = x^2 + 2x$
4. $3y'' + y' + y = x^2 - 4x$; $y = x^2 - 6x$
5. $xy' - 3y = x^2$; $y = 7x^3 - x^2$
6. $xy'' - y' = 3$; $y = x^2 - 3x + 5$
7. $y' + y = e^x$; $y = \frac{1}{2}e^x + 2e^{-x}$
8. $y'' + 25y = 0$; $y = \sin(5x) + 2\cos(5x)$
9. $y' = -\dfrac{x}{y}$; $y = \sqrt{7 - x^2}$
10. $y' = x - y$; $y = x - 1 + 2e^{-x}$

In Problems 11–20, check that the function y is a solution of the given initial value problem.

11. $y' = 6x^2 - 3$, $y(1) = 2$; $y = 2x^3 - 3x + 3$
12. $y' = 6x + 4$, $y(2) = 3$; $y = 3x^2 + 4x - 17$
13. $y' = 2\cos(2x)$, $y(0) = 1$; $y = \sin(2x) + 1$
14. $y' = 1 + 6\sin(2x)$, $y(0) = 2$; $y = x - 3\cos(2x) + 5$
15. $y' = 5y$, $y(0) = 7$; $y = 7e^{5x}$
16. $y' = -2y$, $y(0) = 3$; $y = 3e^{-2x}$
17. $xy' = -y$, $y(1) = -4$; $y = -\dfrac{4}{x}$
18. $y \cdot y' = -x$, $y(0) = 3$; $y = \sqrt{9 - x^2}$
19. $y' = \dfrac{5}{x}$, $y(e) = 3$; $y = 5\ln(x) - 2$
20. $y' + y = e^x$, $y(0) = 5$; $y = \frac{1}{2}e^x + \frac{9}{2}e^{-x}$

In Problems 21–30, find the value of the constant C so a function from the given family of solutions satisfies the given initial value problem.

21. $y' = 2x$, $y(3) = 7$; $y = x^2 + C$
22. $y' = 3x^2 - 5$, $y(1) = 2$; $y = x^3 - 5x + C$
23. $y' = 3y$, $y(0) = 5$; $y = Ce^{3x}$
24. $y' = -2y$, $y(0) = 3$; $y = Ce^{-2x}$
25. $y' = 6\cos(3x)$, $y(0) = 4$; $y = 2\sin(3x) + C$
26. $y' = 3 - 2\sin(2x)$, $y(0) = 1$; $y = 3x + \cos(2x) + C$
27. $y' = \dfrac{1}{x}$, $y(e) = 2$; $y = \ln(x) + C$
28. $y' = \dfrac{1}{x^2}$, $y(1) = 3$; $y = -\frac{1}{x} + C$
29. $y' = -\dfrac{y}{x}$, $y(2) = 10$; $y = -\frac{C}{x}$
30. $y' = -\frac{x}{y}$, $y(3) = 4$; $y = \sqrt{C - x^2}$

In Problems 31–40, find the find the function y that satisfies the given initial value problem.

31. $y' = 4x^2 - x$, $y(1) = 7$
32. $y' = x - e^x$, $y(0) = 3$
33. $y' = \dfrac{3}{x}$, $y(1) = 2$
34. $xy' = 1$, $y(e) = 7$
35. $y' = 6e^{2x}$, $y(0) = 1$
36. $y' = 36(3x - 2)^2$, $y(1) = 8$
37. $y' = x \cdot \sin(x^2)$, $y(0) = 3$
38. $y' = \dfrac{6}{x^2}$, $y(1) = 2$
39. $xy' = 6x^3 - 10x^2$, $y(2) = 5$
40. $x^2y' = 6x^3 - 1$, $y(1) = 10$
41. Show that if $y = f(x)$ and $y = g(x)$ are both solutions to $y' + 5y = 0$, then $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$ and $y = A \cdot f(x) + B \cdot g(x)$ are solutions for any constants A and B.
42. Show that if $f(x)$ and $g(x)$ are both solutions to $y'' + 2y' - 3y = 0$, then so are $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$ and $y = A \cdot f(x) + B \cdot g(x)$ for any constants A and B.
43. Show that $y = \sin(x) + x$ and $y = \cos(x) + x$ are both solutions of $y'' + y = x$. Are $y = 3[\sin(x) + x]$ and $y = [\sin(x) + x] + [\cos(x) + x]$ solutions of $y'' + y = x$?
44. Show that $y = e^{3x} - 2$ and $y = 5e^{3x} - 2$ are both solutions of $y' - 3y = 6$. Are $y = 7\left[e^{3x} - 2\right]$ and $y = \left[e^{3x} - 2\right] + \left[5e^{3x} - 2\right]$ also solutions?

45. The ODE $\frac{dy}{dt} = A - By$ (where A and B are positive constants) describes the concentration y of glucose in a person's blood at time t. Check that $y = \frac{A}{B} - C \cdot e^{-Bt}$ is a solution of the ODE for any value of the constant C.

46. The ODE $\frac{dy}{dt} = Ay$ (where A is a positive constant) is used to model "exponential" growth and decay. Check that $y = C \cdot e^{At}$ is a solution of the differential equation for any value of the constant C.

47. The ODE $L \cdot \frac{dI}{dt} + RI = E$ (where L, R and E are positive constants) describes the current $I(t)$ in an electrical circuit. Show that $I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ is a solution of the ODE.

48. The ODE $m \cdot y'' + C \cdot y = 0$ (where C is a positive constant) describes the position y of an object hung from a spring as it moves up and down. Show that $y = A \cdot \sin(\omega t) + B \cdot \cos(\omega t)$ with $\omega = \sqrt{\frac{C}{m}}$ is a solution of the ODE for all values of the constants A and B.

6.1 Practice Answers

1. (a) $y = 2x + 6 \Rightarrow y' = 2$; $y - 3y' = (2x+6) - 3(2) = 2x$ (OK)

   (b) $y = e^{3x} \Rightarrow y' = 3e^{3x} \Rightarrow y'' = 9e^{3x}$ so:

   $$y'' - 2y' - 3y = 9e^{3x} - 2(3e^{3x}) - 3(e^{3x}) = 0 \quad \text{(OK)}$$

   $y = 5e^{3x} \Rightarrow y' = 15e^{3x} \Rightarrow y'' = 45e^{3x}$ so:

   $$y'' - 2y' - 3y = 45e^{3x} - 2(15e^{3x}) - 3(5e^{3x}) = 0 \quad \text{(OK)}$$

   $y = Ae^{3x} \Rightarrow y' = 3Ae^{3x} \Rightarrow y'' = 9Ae^{3x}$ so:

   $$y'' - 2y' - 3y = 9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 0 \quad \text{(OK)}$$

2. We want $y'' + 9y = 0$ and $y(0) = 2$.

   (a) $y = \sin(3x) \Rightarrow y' = 3\cos(3x) \Rightarrow y'' = -9\sin(3x)$ so we have $y'' + 9y = -9\sin(3x) + 9 \cdot \sin(3x) = 0$ (OK) but checking the initial condition: $y(0) = \sin(0) = 0 \neq 2$.

   (b) $y = 2\sin(3x) \Rightarrow y' = 6\cos(3x) \Rightarrow y'' = -18\sin(3x)$ so we have $y'' + 9y = -18\sin(3x) + 9 \cdot 2\sin(3x) = 0$ (OK) but checking the initial condition: $y(0) = 2\sin(0) = 0 \neq 2$

   (c) $y = 2\cos(3x) \Rightarrow y' = -6\sin(3x) \Rightarrow y'' = -18\cos(3x)$ so $y'' + 9y = -18\cos(3x) + 9 \cdot 2\cos(3x) = 0$ (OK) and checking the initial condition: $y(0) = 2\cos(0) = 2$ (OK).

   Only $y = 2\cos(3x)$ satisfies both the ODE and the initial condition.

3. $y = e^{2x} + C \Rightarrow y' = 2e^{2x}$ (OK) so, plugging in the initial values:

   $$7 = y(0) = e^{2 \cdot 0} + C \Rightarrow 7 = 1 + C \Rightarrow C = 6$$

   so $y = e^{2x} + 6$.

6.2 *The Differential Equation $y' = f(x)$*

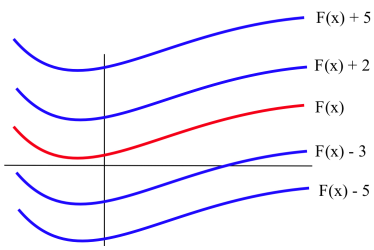
This section introduces some basic concepts and vocabulary of the study of ODEs as they apply to the familiar problem, $y' = f(x)$: the notion of a general solution of an ODE, the (possibly unique) solution to an IVP and the direction field of an ODE.

We'll use these concepts in later sections as we examine more complicated differential equations and their applications.

Solving $y' = f(x)$

The solution of the ODE $y' = f(x)$ is the collection of all antiderivatives of f: $y = \int f(x)\,dx$. If $y = F(x)$ is one antiderivative of f, then we have essentially found *all* antiderivatives of f because any antiderivative of f has the form $F(x) + C$, for some value of the constant C. If F is one particular antiderivative of f, the collection of functions $F(x) + C$ is called the **general solution** of $y' = f(x)$. The general solution consists of a **one-parameter family** of functions (the parameter here is C).

This is due to Corollary 2 to the Mean Value Theorem in Section 3.2.



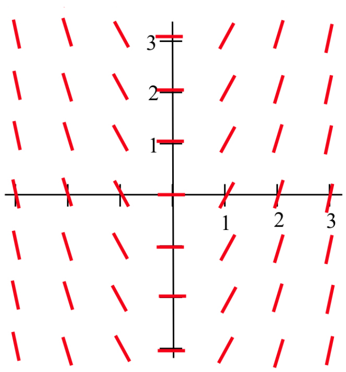
Example 1. Find the general solution of the ODE $y' = 2x + e^{3x}$.

Solution. $y = \int \left[2x + e^{3x}\right] dx = x^2 + \frac{1}{3}e^{3x} + C$. ◀

Practice 1. Find general solutions for $y' = x + \dfrac{3}{x+2}$ and $y' = \dfrac{6}{x^2+1}$.

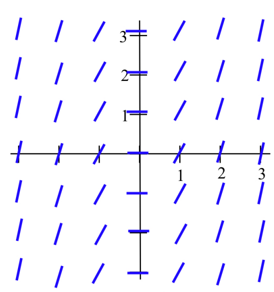
Direction Fields

Geometrically, a derivative tells us the slope of the tangent line to a curve, so we can interpret the ODE $y' = f(x)$ as a geometric condition: at each point (a, b) on the graph of the solution function y, the slope of the tangent line is $f(a)$. The ODE $y' = 2x$ says that at each point (a, b) on the graph of y, the slope of the line tangent to the graph is $2a$: if the point $(5, 3)$ is on the graph of y, then the slope of the tangent line there is $2 \cdot 5 = 10$. We can present this information graphically as a **direction field**: a collection of short line segments through some sample points in the plane so that the slope of the segment through (a, b) is $f(a)$. A direction field for $y' = 2x$ appears in the margin: at a point (a, b), the slope is $2a$. Direction fields for $y' = 3x^2$ and $y' = \cos(x)$ appear below:

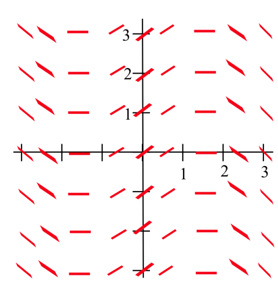


y' = 2x

What are the solutions to $y' = 2x$? Can you "see" those solution curves in the direction field graphed above?



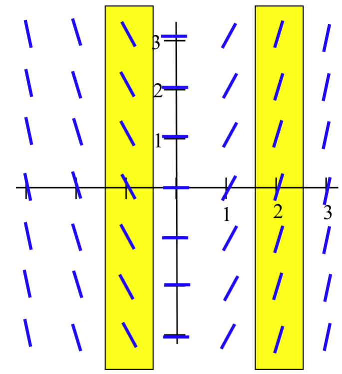
y' = 3x^2



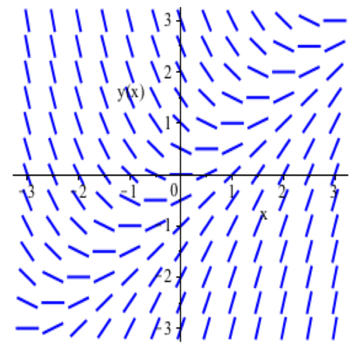
y' = cos(x)

What are the solutions to $y' = 3x^2$? To $y' = \cos(x)$? Can you "see" those solution curves in the direction fields?

For any ODE of the form $y' = f(x)$, the values of y' depend only on x, so along any vertical line (where x is fixed) all the small line segments have the same y', hence the same slope, and they are parallel (see margin). If y' depends on both x and y, then the slopes of the line segments will depend on both x and y, and the slopes of the small line segments along a vertical line are not all the same. The second margin figure shows a direction field for $y' = x - y$, where y' is a function of both x and y. A direction field of an ODE $y' = g(x, y)$ is a collection of short line segments with slope $g(a, b)$ at the point (a, b).



Direction field for y' = f(x)

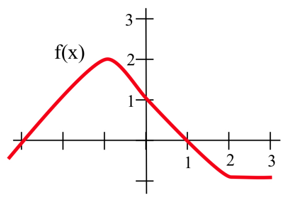


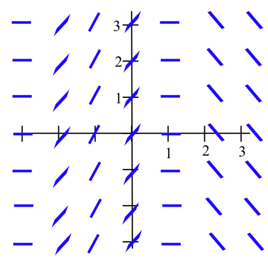
Direction field for y' = x - y

Practice 2. Construct direction fields for (a) $y' = x + 1$ and (b) $y' = x + y$ by sketching a short line segment with slope y' at each point (a, b) with integer coordinates from -3 to 3.

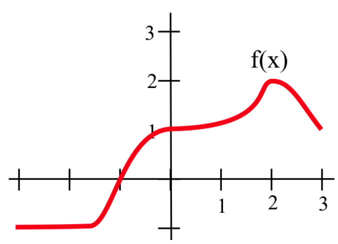
As you discovered in the previous Practice problem, direction fields are usually tedious to plot by hand, but computers (and some calculators) can plot them quickly. If you only have a graph of the function f in the differential equation $y' = f(x)$, you can construct an approximate direction field using the information you have about f from its graph.

Example 2. Construct a direction field for the differential equation $y' = f(x)$ for the f given graphically below left.



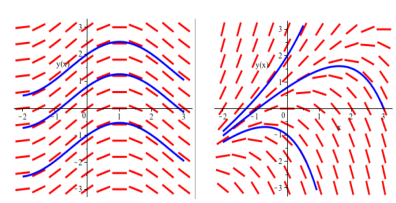


Solution. If $x = 0$ then $y' = f(0) = 1$, so at every point on the vertical line where $x = 0$ (the y-axis) the line segments of the direction field have slope $y' = 1$ (above right). Similarly, if $x = 1$ then $y' = f(1) = 0$, so the line segments of the direction field have slope $y' = 0$ at every point on the vertical line where $x = 1$. The small line segments along any vertical line are parallel. ◀



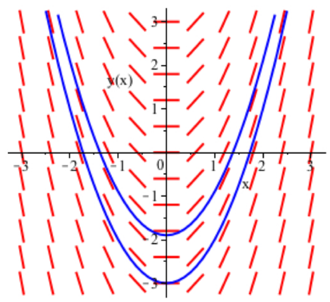
Practice 3. Construct a direction field for the differential equation $y' = f(x)$ for the f given graphically in the margin.

Once you have a direction field for an ODE, you can sketch curves that have the appropriate tangent-line slopes so you can "see" the shapes of the solution curves even if you do not have formulas for them. (See margin.) These shapes can be useful for estimating which initial conditions lead to straight-line solutions or periodic solutions or solutions with other properties, and they can help us understand the behavior of machines and organisms in applied problems.



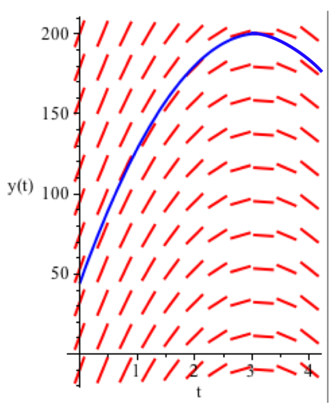
Initial Value Problems

An initial condition $y(x_0) = y_0$ specifies that the solution y of the differential equation should go through the point (x_0, y_0) in the plane. To solve a differential equation with an initial condition, you typically use integration to find the general solution (a family of solutions containing an arbitrary constant) and then you use algebra to find the one value for the constant so the solution satisfies the initial condition.



Example 3. Solve the differential equation $y' = 2x$ with the initial condition $y(2) = 1$.

Solution. The general solution is $y = \int 2x\,dx = x^2 + C$. Substituting $x_0 = 2$ and $y_0 = 1$ into the general solution: $1 = (2)^2 + C \Rightarrow C = -3$. So the solution we want is $y = x^2 - 3$. (A quick check verifies that $\left[x^2 - 3\right]' = 2x$ and $2^2 - 3 = 1$.) The margin shows a direction field for $y' = 2x$ and the solution curve that goes through $(2, 1)$, $y = x^2 - 3$, along with the solution of the ODE that satisfies $y(1) = -1$. ◀



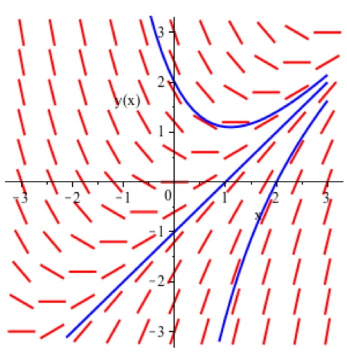
Example 4. If you toss a ball upward with an initial velocity of 100 feet per second, its height y (in feet) at time t (in seconds) satisfies the differential equation $y' = 100 - 32t$. Sketch the direction field for y (for $0 \le t \le 4$) and then sketch the solution that satisfies the condition that the ball is 200 feet high after 3 seconds.

Solution. $y = \int [100 - 32t]\,dt = 100t - 16t^2 + C$; if $y(3) = 200$:

$$200 = 100(3) - 16(3)^2 + C \;\Rightarrow\; 200 = 156 + C \;\Rightarrow C = 44$$

The function we want is $y = 100t - 16t^2 + 44$. The direction field and the solution satisfying $y(3) = 200$ appear in the margin. ◀

Practice 4. Find the solution of $y' = 9x^2 - 6\sin(2x) + e^x$ that goes through the point $(0, 6)$.



Example 5. A direction field for $y' = x - y$ appears in the margin. Sketch the three solutions of the ODE $y' = x - y$ that satisfy the initial conditions $y(0) = 2$, $y(0) = -1$ and $y(1) = -2$.

Solution. See margin figure. ◀

Existence and Uniqueness

When solving IVPs, three questions often present themselves:

- Does a solution to the IVP exist?
- Is that solution unique?
- On what interval(s) is that solution valid?

We typically hope the answer the first two questions is "yes," but (as we will see in the next section) that will not always be true.

Surprisingly, for reasonably "nice" IVPs, we will eventually be able to answer these questions without actually finding the solution.

For IVPs of the form $y' = f(x)$, $y(x_0) = y_0$, where $f(x)$ is continuous on some interval (a, b) with $a < x_0 < b$, the answer to the first two questions is "yes" and the answer to third question is "on the interval (a, b), and possibly on some larger interval." In this situation, define:

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

The Fundamental Theorem of Calculus tells us that $y'(x) = f(x)$, so this $y(x)$ solves the ODE, and:

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\,dt = y_0 + 0 = y_0$$

so it also satisfies the initial value condition. If $\tilde{y}(x)$ is any other solution to the IVP, then $\tilde{y}'(x) = f(x) = y'(x)$ so $\tilde{y}(x) = y(x) + C$, but we also know that $\tilde{y}(x_0) = y_0 = y(x_0)$:

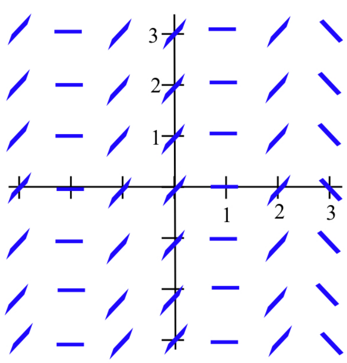
$$\tilde{y}(x_0) = y(x_0) + C \;\Rightarrow\; y_0 = y_0 + C \;\Rightarrow\; C = 0 \;\Rightarrow \tilde{y}(x) = y(x)$$

which tells us that $y(x)$ is the *only* solution. Finally, $f(x)$ is continuous on (a, b), hence integrable on (a, b), so the integral definition of $y(x)$ is defined (and solves the IVP) on (a, b).

6.2 Problems

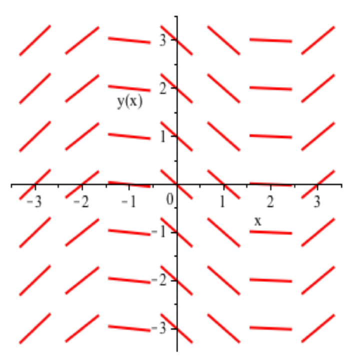
In Problems 1–6, use the given direction field to sketch the solutions of the underlying ODE that satisfy the given initial conditions.

1. See figure below; $y(0) = 1$, $y(1) = -2$ and $y(1) = 3$.



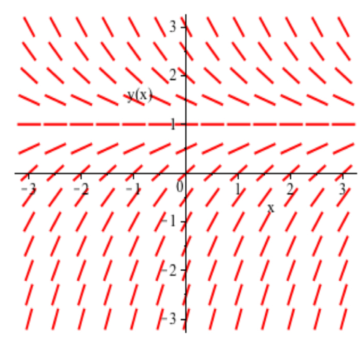
2. See figure above; $y(0) = 2$, $y(1) = -1$ and $y(1) = -2$.

3. See figure below; $y(-2) = 1$, $y(0) = 1$ and $y(2) = 1$.



4. See figure above; $y(-2) = -1$, $y(0) = -1$, $y(2) = -1$.

5. See figure below; $y(0) = -2$, $y(0) = 0$ and $y(0) = 2$.



6. See figure above; $y(2) = -2$, $y(2) = 0$ and $y(2) = 2$.

7. How do the three solutions in Problem 5 behave for large values of x?
8. How do the three solutions in Problem 6 behave for large values of x?

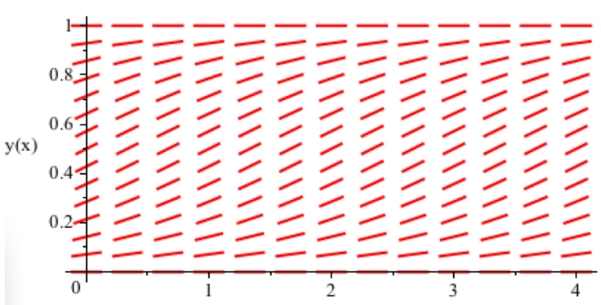
In Problems 9–14, (a) sketch a direction field for the given ODE and (b) without solving the ODE, sketch solutions that go through the points $(0, 1)$ and $(2, 0)$.

9. $y' = 2x$
10. $y' = 2 - x$
11. $y' = 2 + \sin(x)$
12. $y' = e^x$
13. $y' = 2x + y$
14. $y' = 2x - y$

In Problems 15–20, (a) find the family of functions that solve the given ODE, (b) find the member of the family that satisfies the given IVP and (c) report the interval on which the solution to the IVP is valid.

15. $y' = 2x - 3, \; y(1) = 4$
16. $y' = 1 - 2x, \; y(2) = -3$
17. $y' = e^x + \cos(x), \; y(0) = 7$
18. $y' = \sin(2x) - \cos(x), \; y(0) = -5$
19. $y' = \dfrac{6}{2x+1} + \sqrt{x}, \; y(1) = 4$
20. $y' = \dfrac{e^x}{1+e^x}, \; y(0) = 0$

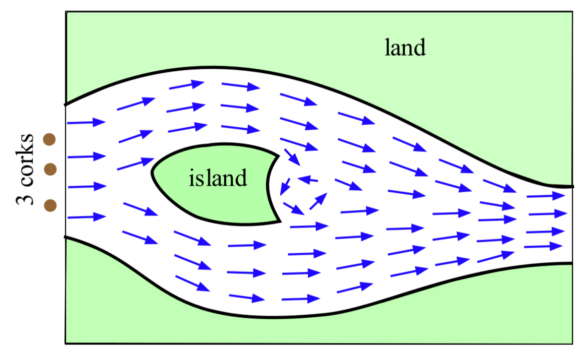
Problems 21–22 concern a direction field (shown below) that comes from an ODE called the **logistic equation**, $y' = y(1 - y)$, used to model the growth of a population in an environment with renewable but limited resources. (It is also used to describe the spread of a rumor or disease through a population.)



21. Sketch the solution that satisfies the initial condition $y(0) = 0.1$. What letter of the alphabet does this solution resemble?
22. Sketch several solutions that have different initial values for $y(0)$. What appears to happen to all of these solutions after a "long time" (for large values of x)?

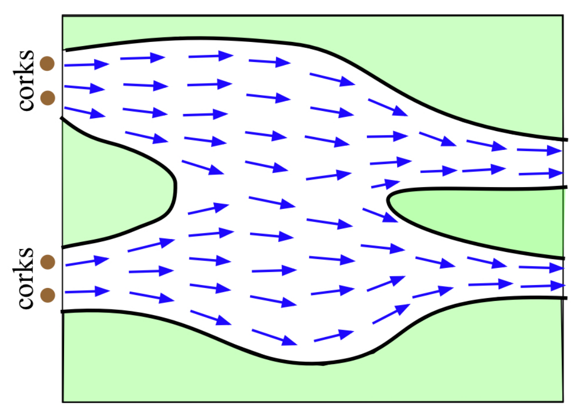
In Problems 23–24, the given figures show the direction of surface flow at different locations along a river. Sketch the paths small corks will follow if they are put into the river at the dots in each figure. (Because they indicate both the magnitude and the direction of flow, each diagram is called a **vector field**.) Notice that corks that start close to each other can drift far apart, and corks that start far apart can drift close together.

23. See figure below.



Surface flow along a river

24. See figure below.



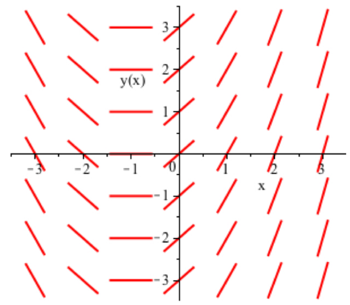
Surface flow along a river

6.2 Practice Answers

1. $y' = x + \dfrac{3}{x+2} \;\Rightarrow\; y = \displaystyle\int \left[x + \frac{3}{x+2}\right] dx = \frac{1}{2}x^2 + 3\ln\left(|x+2|\right) + C$

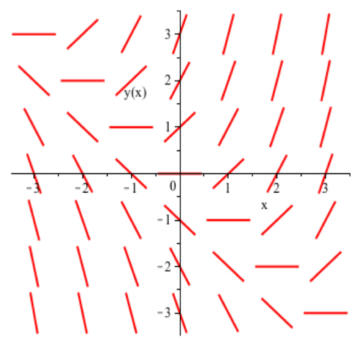
   $y' = \dfrac{6}{x^2+1} \;\Rightarrow\; y = \displaystyle\int \frac{6}{x^2+1}\, dx = 6\arctan(x) + C$

2. (a) If $y' = x + 1$ then the table below lists values of y' for integer values of x and y from -3 to 3 (notice that y' does not depend on the value of y); the margin figure shows the direction field.



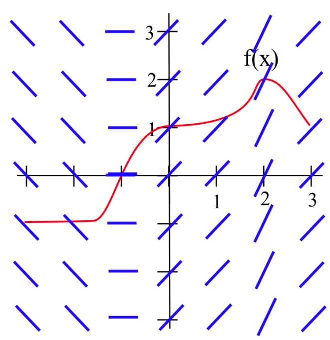
	$x=-3$	-2	-1	0	1	2	3
$y=3$	-2	-1	0	1	2	3	4
2	-2	-1	0	1	2	3	4
1	-2	-1	0	1	2	3	4
0	-2	-1	0	1	2	3	4
-1	-2	-1	0	1	2	3	4
-2	-2	-1	0	1	2	3	4
-3	-2	-1	0	1	2	3	4

   (b) If $y' = x + y$ then the table below lists values of y' for integer values of x and y from -3 to 3 (notice that y' here *does* depend on both x and y); the margin figure shows the direction field.



	$x=-3$	-2	-1	0	1	2	3
$y=3$	0	1	2	3	4	5	6
2	-1	0	1	2	3	4	5
1	-2	-1	0	1	2	3	4
0	-3	-2	-1	0	1	2	3
-1	-4	-3	-2	-1	0	1	2
-2	-5	-4	-3	-2	-1	0	1
-3	-6	-5	-4	-3	-2	-1	0

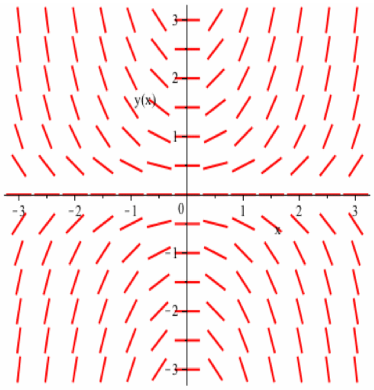
3. An approximate direction field for the ODE $y' = f(x)$ appears in the margin. (The function shown is $f(x)$, not a solution to the ODE.)



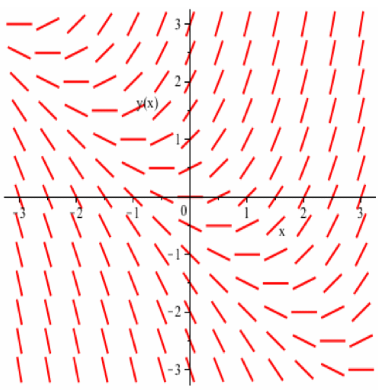
4. $y = \displaystyle\int \left[9x^2 - 6\sin(2x) + e^x\right] dx = 3x^3 + 3\cos(2x) + e^x + C$ so:

   $$6 = y(0) = 3\cdot 0^3 + 3\cos(2\cdot 0) + e^0 + C = 0 + 3 + 1 + C \;\Rightarrow\; C = 2$$

   and therefore $y = 3x^3 + 3\cos(2x) + e^x + 2$.



Direction field for y' = xy



Direction field for y' = x+y

6.3 *Separable Equations*

So far, we have only learned how to solve ODEs of the form $y' = f(x)$ (which you already knew how to solve), where y' depended only on x and the slopes of the line segments of a corresponding direction field did not depend on the y-coordinate of the location of the line segment. In many situations, however, y' depends on both x and y, for example, $y' = xy$ (see first margin figure for a direction field) or $y' = x + y$ (see second margin figure for a direction field).

In Example 2 of Section 2.9, we considered the problem of finding a tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. You can solve this problem geometrically. Or you can solve the equation for y to get $y = \sqrt{25 - x^2}$ and then compute $\frac{dy}{dx}$. Or you can implicitly differentiate the equation $x^2 + y^2 = 25$, keeping in mind that y is a function of x:

$$x^2 + y^2 = 25 \;\Rightarrow\; 2x + 2y \cdot \frac{dy}{dx} = 0 \;\Rightarrow\; 2y \cdot \frac{dy}{dx} = -2x \;\Rightarrow\; \frac{dy}{dx} = -\frac{x}{y}$$

The result is a differential equation (although we didn't call it that in Section 2.9) and putting $x = 3$ and $y = 4$ into the right side of that ODE yields $\frac{dy}{dx} = -\frac{3}{4}$, which gives us the slope of the line tangent to the circle at the point $(3, 4)$.

What if we start with $\frac{dy}{dx} = -\frac{x}{y}$ and try to solve this ODE? It is not of the form $y' = f(x)$, but we can try to solve it by beginning to reverse the steps in the implicit differentiation process above:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad y \cdot \frac{dy}{dx} = -x$$

Each side of this new ODE is a function of x because y and $\frac{dy}{dx}$ are functions of x, so we can integrate both sides of this equation to get:

Each indefinite integral yields a constant, but these are not necessarily equal.

$$\int y \cdot \frac{dy}{dx}\, dx = \int -x\, dx \;\Rightarrow\; \frac{1}{2}y^2 + C_1 = -\frac{1}{2}x^2 + C_2$$

The integration on the left-hand side of this equation uses the fact that:

$$\frac{d}{dx}\left[\frac{1}{2}y^2\right] = \frac{1}{2} \cdot 2y \cdot \frac{dy}{dx} = y \cdot \frac{dy}{dx}$$

but this is the same result as if we had "cancelled" the dx in the original left-hand integral:

$$\int y \cdot \frac{dy}{dx}\, dx = \int y\, dy = \frac{1}{2}y^2 + C_1$$

so in our first steps above we could have instead rewritten the ODE using differentials:

$$\frac{dy}{dx} = -\frac{x}{y} \;\Rightarrow\; y \cdot \frac{dy}{dx} = -x \;\Rightarrow y\, dy = -x\, dx \;\Rightarrow \int y\, dy = \int -x\, dx$$

and then integrated both sides. (We will follow this procedure as we solve similar ODEs from now on.) We can now move both constants of integration to the same side of the equation:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + [C_2 - C_1] \;\Rightarrow\; \frac{1}{2}y^2 = -\frac{1}{2}x^2 + K$$

where K is another arbitrary constant. Because we can *always* move the left-hand constant to the right side of the equation and combine it with the right-hand constant, in the future we will use a single constant on the right side when integrating both sides of the differential equation.

At this stage we can write:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + K \;\Rightarrow\; y^2 = -x^2 + 2K \;\Rightarrow x^2 + y^2 = C$$

where C is yet another arbitrary constant. We recognize this as an equation for a circle centered at the origin and we call a solution of this type an **implicit solution** to the ODE because we have not explicitly solved for y. If we know our solution curve passes through $(3, 4)$, we can put this information into our solution equation to get:

$$3^2 + 4^2 = C \;\Rightarrow\; C = 25 \;\Rightarrow\; x^2 + y^2 = 25$$

Solving for y, we get $y = \sqrt{25 - x^2}$, an **explicit solution** to the ODE.

We first get $y = \pm\sqrt{25 - x^2}$ but knowing that $y(3) = 4$ tells us to use the $+$ symbol rather than the $-$ symbol.

Separable Equations

In the ODE we solved above, we were able to "separate" the variables so that only y appeared on one side and only x appeared on the other, allowing us to integrate each side separately to solve the ODE.

Definition: A differential equation is called **separable** if we can separate the variables algebraically so that the equation has the form:

$$g(y) \cdot y' = f(x)$$

Example 1. If possible, separate the variables in each ODE by writing the given differential equation in the form $g(y) \cdot y' = f(x)$: (a) $y' = xy$ (b) $xy' = \dfrac{y+1}{x}$ (c) $y' = \dfrac{1 + \sin(x)}{y^2 + y}$ (d) $y' = y$ (e) $y' = x + y$

Solution. (a) Divide each side of $y' = xy$ by y (to do so we must assume $y \neq 0$) to get $\dfrac{1}{y} \cdot y' = x$ so $g(y) = \dfrac{1}{y}$ and $f(x) = x$.

(b) Divide each side by $x(y+1)$ (to do so we must assume $x \neq 0$ and $y \neq -1$) to get $\dfrac{1}{y+1} \cdot y' = \dfrac{1}{x^2}$ so $g(y) = \dfrac{1}{y+1}$ and $f(x) = \dfrac{1}{x^2}$.

(c) Multiply each side by $y^2 + y$ to get $\left(y^2 + y\right) \cdot y' = 1 + \sin(x)$ so $g(y) = y^2 + y$ and $f(x) = 1 + \sin(x)$.

(d) Divide each side by y (to do so we must assume $y \neq 0$) to get $\frac{1}{y} \cdot y' = 1$ so $g(y) = \frac{1}{y}$ and $f(x) = 1$.

(e) We cannot write this ODE in the form $g(y) \cdot y' = f(x)$.
The first four ODEs are separable, but the last one is not. ◀

Practice 1. If possible, separate the variables in each ODE by writing the given differential equation in the form $g(y) \cdot y' = f(x)$:

(a) $y' = x^3(y-5)$ (b) $y' = \dfrac{3}{2x + x \cdot \sin(y+2)}$

Solving Separable ODEs

To solve a separable differential equation, we will follow the steps outlined above when solving $\frac{dy}{dx} = -\frac{x}{y}$:

- Use algebra to separate the variables in the ODE: $g(y) \cdot y' = f(x)$
- Put into an equivalent form using differentials: $g(y)\,dy = f(x)\,dx$
- Integrate each side of the equation: $\int g(y)\,dy = \int f(x)\,dx$
- Find antiderivatives, if possible: $G(y) = F(x) + C$
- If given an initial condition (x_0, y_0), find C: $C = G(y_0) - F(x_0)$
- If possible, solve explicitly for y.

Example 2. Find the general solution of $\frac{1}{x} \cdot y' = \frac{x}{2y}$.

Solution. Multiply each side by $2xy$ to get $2y \cdot y' = x^2$, so the ODE is separable and we can translate it into differential form:

$$2y \cdot \frac{dy}{dx} = x^2 \quad \Rightarrow \quad 2y\,dy = x^2\,dx$$

Integrating each side, we get:

$$\int 2y\,dy = \int x^2\,dx \quad \Rightarrow \quad y^2 = \frac{1}{3}x^3 + C$$

which is an implicit form of the general solution. Solving for y, we get:

$$y = \pm\sqrt{\frac{1}{3}x^3 + C}$$

which is an explicit form of the general solution. ◀

Example 3. Find the solution of $y' = \dfrac{6x+1}{2y}$ that satisfies $y(2) = 3$.

Solution. We can rewrite this ODE as $2y \cdot y' = 6x + 1$, so it is separable. Next rewrite using differentials:

$$2y \cdot \frac{dy}{dx} = 6x + 1 \quad \Rightarrow \quad 2y\,dy = (6x+1)\,dx$$

and then integrate each side:

$$\int 2y\,dy = \int (6x+1)\,dx \quad \Rightarrow \quad y^2 = 3x^2 + x + C$$

Putting $x = 2$ and $y = 3$ into the general solution $y^2 = 3x^2 + x + C$:

$$3^2 = 3 \cdot 2^2 + 2 + C \quad \Rightarrow \quad 9 = 12 + 2 + C \quad \Rightarrow \quad C = -5$$

In an initial value problem, it is often easier to solve for C immediately after finding the antiderivatives.

So $y^2 = 3x^2 + x - 5 \quad \Rightarrow \quad y = \pm\sqrt{3x^2 + x - 5}$. Because $y(2) = 3$, we need the $+$ value of the square root: $y = \sqrt{3x^2 + x - 5}$. ◀

Practice 2. Find the general solution of $y' = \dfrac{1 - \sin(x)}{3y^2}$ and then find the solution that passes through $(0, 2)$.

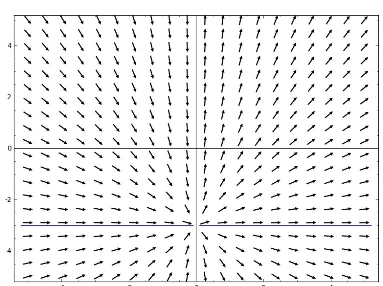
Sometimes algebra is the hardest part of solving an ODE, and often logarithms crop up when solving separable equations.

Example 4. Solve $x \cdot y' = y + 3$.

Solution. To write the ODE in the form $g(y) \cdot y' = f(x)$ we can divide by x and by $y + 3$ (so we must assume that $x \neq 0$ and $y \neq -3$):

$$x \cdot \frac{dy}{dx} = y + 3 \;\Rightarrow\; \frac{1}{y+3}\,\frac{dy}{dx} = \frac{1}{x} \;\Rightarrow\; \frac{1}{y+3}\,dy = \frac{1}{x}\,dx$$

Study the direction field for $x \cdot y' = y + 3$ shown below. What do you notice about its behavior near the lines $x = 0$ and $y = -3$?



Now integrate both sides:

$$\int \frac{1}{y+3}\,dy = \int \frac{1}{x}\,dx \quad \Rightarrow \quad \ln(|y+3|) = \ln(|x|) + C$$

which is an implicit form of the general solution. To explicitly solve for y, recall that $e^{\ln(a)} = a$ so:

$$e^{\ln(|y+3|)} = e^{\ln(|x|)+C} = e^{\ln(|x|)} \cdot e^C \quad \Rightarrow \quad |y+3| = e^C \cdot |x|$$

Removing the absolute value signs we get:

$$y + 3 = \pm e^C \cdot x \quad \Rightarrow \quad y = \pm e^C x - 3 \quad \Rightarrow \quad y = Ax - 3$$

The domain of any particular solution of the form $y = Ax - 3$ is either $(-\infty, 0)$ or $(0, \infty)$, which is related to the assumption that $x \neq 0$ in our solution. Putting $x = 0$ into the original ODE yields $0 = y + 3 \Rightarrow y - 3$, so the only solution for which $x = 0$ is the single point $(0, -3)$ rather than a function defined on an open interval.

where A is any nonzero constant. In the first step above, we had to assume that $y \neq -3$. You can check that the constant function $y = -3$ is also a solution to the ODE: $x \cdot [-3]' = -3 + 3$. So $y = Ax - 3$ is a solution of the ODE even when $A = 0$, and the general solution of the ODE is $y = Ax - 3$ for any value of A. ◀

Existence and Uniqueness

In Section 6.2, we saw that any IVP of the form $y' = f(x)$, $y(x_0) = y_0$ had a unique solution: that is, at least one solution to the IVP exists, and this solution is the only possible solution. Unfortunately, for other types of IVPs things can be more complicated.

Can you see why?

The ODE $y^2 + (y')^2 = -1$ has *no* solution, no matter what initial value condition we might specify. The ODE $y^2 + (y')^2 = 0$ has only one solution (the constant solution $y = 0$) so the IVP $y^2 + (y')^2 = 0$, $y(0) = 0$ has one unique solution, but the IVP $y^2 + (y')^2 = 0$, $y(0) = 5$ has none.

Even separable equations can behave in unexpected ways. Consider the IVP $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$. The ODE is separable (if $y \neq 0$):

$$\frac{dy}{dx} = 3y^{\frac{2}{3}} \quad \Rightarrow \quad \frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{1}{3}y^{-\frac{2}{3}}\,dy = dx$$

Integrating both sides:

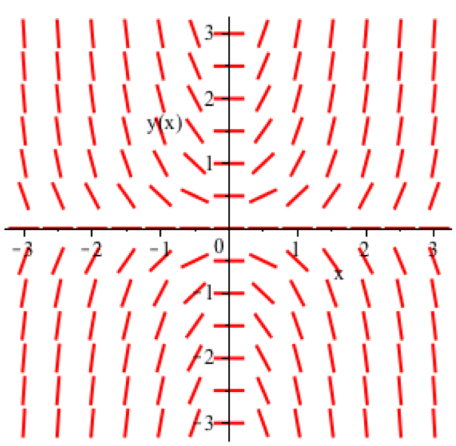
$$\int \frac{1}{3}y^{-\frac{2}{3}}\,dy = \int 1\,dx \quad \Rightarrow \quad y^{\frac{1}{3}} = x + C \quad \Rightarrow \quad y = (x+C)^3$$

In a course on differential equations, you will learn about a theorem that specifies restrictions on a first-order IVP that will guarantee a unique solution.

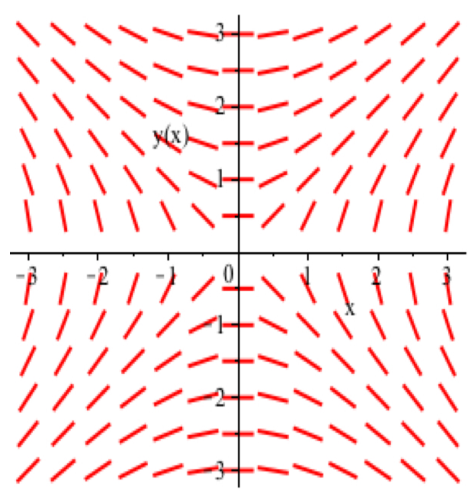
Using the initial condition, $0 = y(0) = (0+C)^3 = C^3 \Rightarrow C = 0$ so $y = x^3$ solves the IVP. But we had to assume that $y \neq 0$; checking the constant function $y = 0$, $[0]' = 3 \cdot 0^{\frac{2}{3}}$ and $y(0) = 0$, so $y(x) = 0$ also satisfies the IVP, giving the IVP *two* solutions: $y(x) = 0$ and $y(x) = x^3$.

6.3 Problems

1. The direction field of the separable ODE $y' = 2xy$ appears below. Sketch solutions of the ODE that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$ and $y(1) = 2$.



2. The direction field of the separable ODE $y' = x/y$ appears below. Sketch solutions of the ODE that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$ and $y(1) = 2$.



In Problems 3–10, solve the separable ODE.

3. $y' = 2xy$
4. $y' = \dfrac{x}{y}$
5. $\left(1+x^2\right) \cdot y' = 3$
6. $xy' = y + 3$
7. $y' \cdot \cos(x) = e^y$
8. $y' = x^2 y + 3y$
9. $y' = 4y$
10. $y' = 5(2-y)$

In Problems 11–18, solve the separable ODEs subject to the given initial conditions.

11. $y' = 2xy$ for $y(0) = 3$, $y(0) = 5$ and $y(1) = 2$.
12. $y' = \dfrac{x}{y}$ for $y(0) = 3$, $y(0) = 5$ and $y(1) = 2$.
13. $y' = 3y$ for $y(0) = 4$, $y(0) = 7$ and $y(1) = 3$.
14. $y' = -2y$ for $y(0) = 4$, $y(0) = 7$ and $y(1) = 3$.
15. $y' = 5(2-y)$ for $y(0) = 5$ and $y(0) = -3$.
16. $y' = 7(1-y)$ for $y(0) = 4$ and $y(0) = -2$.
17. $\left(1+x^2\right) \cdot y' = 3$ for $y(1) = 4$ and $y(0) = 2$.
18. $xy' = y + 3$ for $y(1) = 20$.
19. For $xy' = y+3$, can $y(0) = 2$?
20. Find all solutions to the IVP $y' = \sqrt[3]{y}$, $y(0) = 0$.

6.3 Practice Answers

1. (a) If $y \neq 5$, divide the ODE $y' = x^3(y-5)$ by $y-5$ to get $\dfrac{1}{y-5} \cdot \dfrac{dy}{dx} = x^3$ so $g(y) = \dfrac{1}{y-5}$ and $f(x) = x^3$.

   (b) Factor x out of the denominator on the right-hand side and multiply both sides of the ODE by $2 + \sin(y+2)$ to get:

$$\frac{dy}{dx} = \frac{3}{x\,[2+\sin(y+2)]} \quad \Rightarrow \quad [2+\sin(y+2)] \cdot \frac{dy}{dx} = \frac{3}{x}$$

   so $g(y) = 2 + \sin(y+2)$ and $f(x) = \dfrac{3}{x}$.

2. Multiply both sides of the ODE by $3y^2$ to get:

$$3y^2 \cdot \frac{dy}{dx} = 1 - \sin(x) \quad \Rightarrow \quad 3y^2\,dy = [1-\sin(x)]\,dx$$

   and integrate:

$$\int 3y^2\,dy = \int [1-\sin(x)]\,dx \quad \Rightarrow \quad y^3 = x + \cos(x) + C$$

   This is an implicit form of the general solution; an explicit general solution is $y = \sqrt[3]{x + \cos(x) + C}$.

   Using the initial condition $y(0) = 2$ with the implicit form of the general solution:

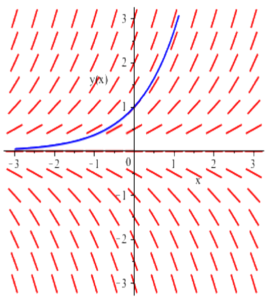
$$2^3 = 0 + \cos(0) + C \;\Rightarrow 8 = 1 + C \quad \Rightarrow \quad C = 7$$

   so the solution to the IVP is $y = \sqrt[3]{x + \cos(x) + 7}$.

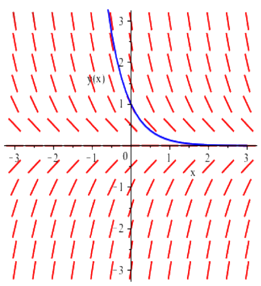
6.4 *Exponential Growth and Decay*

The separable differential equation $y' = ky$ is relatively simple to solve, but it can model a wealth of important situations, including population growth, radioactive decay and drug absorption in the bloodstream. In this section we will solve this ODE and explore some related applications.

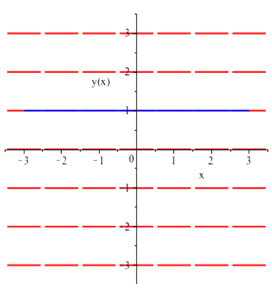
The Differential Equation $y' = ky$



y ' = 1y, y(0)=1



y ' =-2y, y(0)=1



y ' =0y, y(0)=1

Direction fields for y ' = ky

The differential equation $y' = ky$ says that the rate of change of a quantity y is proportional to the value of y. The margin figures show direction fields for $y' = 1y$ (growth) and $y' = -2y$ (decay). The ODE $y' = ky$ can model the behavior of populations (the rate at which babies are born is proportional to the number of people currently in the population), radioactive decay (the rate at which atoms decay is proportional to the number of atoms present), the absorption of some medicines by our bodies, and many other situations. The solutions of $y' = ky$ will help us determine how long it takes a population to double in size, the age of some prehistoric artifacts, and even how often some medicines should be taken in order to maintain a safe and effective concentration of that medicine in a patient's body.

$$\boxed{\text{If } y' = ky \text{ (for } y > 0\text{), then } y(t) = y(0) \cdot e^{kt}.}$$

Proof. The ODE $y' = ky$ is separable, so we can employ the method of Section 6.3 to solve it:

$$\frac{dy}{dt} = k \cdot y \;\Rightarrow\; \frac{1}{y} \cdot \frac{dy}{dt} = k \;\Rightarrow\; \frac{1}{y}\,dy = k\,dt \;\Rightarrow\; \int \frac{1}{y}\,dy = \int k\,dt$$
$$\Rightarrow\; \ln(|y|) = kt + C \;\Rightarrow\; e^{\ln(|y|)} = e^{kt+C}$$
$$\Rightarrow\; |y| = e^{kt} \cdot e^{C} \;\Rightarrow\; y = \pm e^{C} e^{kt}$$

Because we assumed that $y > 0$, we didn't need to worry about dividing by y, and we didn't really need the absolute values (or the $\pm$) in the solution above. But $y = 0$ is also a solution to $y' = ky$, so $y = Ae^{kt}$ solves $y' = ky$ for any value of A.

We've found an infinite family of solutions for $y' = ky$, but how do we know that we've found *all* solutions to that ODE? Let $f(t)$ be any solution to $y' = ky$ so that $f'(t) = k \cdot f(t)$. Then define another function $g(t) = f(t) \cdot e^{-kt}$ so that:

$$g(t) = \frac{f(t)}{e^{kt}} \;\Rightarrow\; g'(t) = \frac{e^{kt} \cdot f'(t) - f(t) \cdot e^{kt}}{\left[e^{kt}\right]^2} = \frac{e^{kt}\left[f'(t) - k \cdot f(t)\right]}{e^{2kt}}$$

The last expression in brackets is 0 (because $f'(t) = k \cdot f(t)$), so:

$$g'(t) = 0 \;\Rightarrow\; g(t) = C \;\Rightarrow\; \frac{f(t)}{e^{kt}} = C \;\Rightarrow\; f(t) = Ce^{kt}$$

We now know that any function of the form $y = Ae^{kt}$ solves $y' = ky$ *and* that any solution of $y' = ky$ must have the form $y = Ae^{kt}$.

Finally, putting $t = 0$ into the general solution:

$$y(0) = Ae^{k \cdot 0} = A \Rightarrow A = y(0) \Rightarrow y(t) = y(0) \cdot e^{kt}$$

which holds for any value of $y(0)$, but in particular for any $y(0) > 0$. $\square$

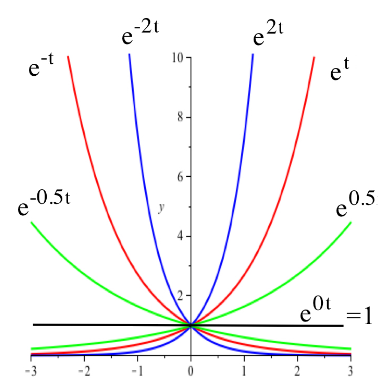
Exponential Growth

A population of people, a chunk of radioactive material and the amount of money in a bank account can all share a common trait. In each situation, the rate at which an amount changes at a particular time is often proportional to the value of that amount at that time. For example:

- the number of births per year is proportional to the number of people in the population
- the number of atoms per hour that release a particle is proportional to the number of atoms present
- the number of dollars of interest per year added to a bank account is proportional to the amount of money in that bank account

These situations can all be modeled with the separable ODE solved above. Our focus in this section will be on using those equations and their solutions to answer questions about applied problems. The applications here all involve the rate of change of some quantity with respect to time, so the input variable will generally be time t (instead of x). We might also write the output quantity as $f(t)$ (instead of y). The ODE $y' = ky$ then becomes $f'(t) = k \cdot f(t)$ and the solution $y = y_0 \cdot e^{kx}$ becomes $f(t) = f(0) \cdot e^{kt}$.

When $k > 0$, $f(t) = f(0) \cdot e^{kt}$ represents **exponential growth** and we call k the **growth constant**. When $k < 0$, $f(t) = f(0) \cdot e^{kt}$ represents **exponential decay** and we call k the **decay constant**. The margin figure shows the graphs of $f(t) = e^{kt}$ for several values of k.



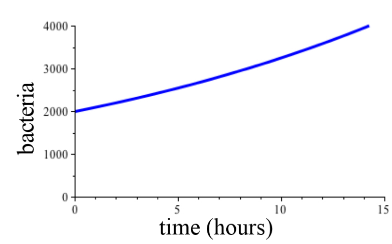
Example 1. The number of bacteria on a Petri plate t hours after an experiment starts is $2000 \cdot e^{0.0488t}$.

(a) How many bacteria are on the plate after one hour? Two hours?

(b) What is the percentage growth of the population from $t = 0$ to $t = 1$? From $t = 1$ to $t = 2$?

(c) How long does it take for the population to reach 3000? To double?

When you know the initial population $f(0)$ and the growth constant k, you can write an equation for $f(t)$, the population at any time t, and use it to answer questions about the population.



Solution. See margin figure for a graph of $f(t) = 2000 \cdot e^{0.0488t}$.

(a) $f(1) = 2000 \cdot e^{0.0488} \approx 2100$; $f(2) = 2000 \cdot e^{0.0976} \approx 2205$

(b) The percentage growth from $t = 0$ to $t = 1$ is:

$$\frac{f(1) - f(0)}{f(0)} = \frac{2100 - 2000}{2000} = \frac{100}{2000} = 0.05 = 5\%$$

The percentage growth from $t = 1$ to $t = 2$ is:

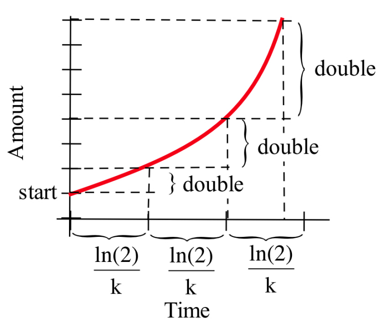
$$\frac{f(2) - f(1)}{f(1)} = \frac{2205 - 2100}{2100} = \frac{105}{2100} = 0.05 = 5\%$$

During the first hour, the population grows by 100 and during the second hour it grows by 105, but the percentage growth during each hour remains constant at 5%.

(c) We need the value of T so that $3000 = f(T) = 2000 \cdot e^{0.0488T}$:

$$1.5 = e^{0.0488T} \Rightarrow \ln(1.5) = \ln\left(e^{0.0488T}\right) = 0.0488T$$
$$\Rightarrow T = \frac{\ln(1.5)}{0.0488} \approx 8.31 \text{ hours}$$

The original population is 2000, so the doubled population is 4000 and the doubling time is $\frac{\ln(2)}{0.0488} \approx 14.2$ hours. ◀



Constant doubling times

When we know the growth constant k, the doubling time is simple to find (as in the preceding Example). If $f(t) = f(0) \cdot e^{kt}$ then the doubling time is the time t_d so that:

$$2f(0) = f(0) \cdot e^{kt_d} \Rightarrow 2 = e^{kt_d} \Rightarrow \ln(2) = kt_d \Rightarrow t_d = \frac{\ln(2)}{k}$$

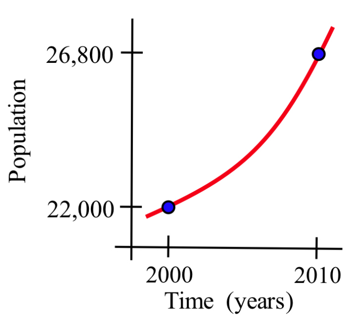
An important aspect of exponential growth is that the doubling time depends only on the growth constant k.

Practice 1. Use the information from the previous Example to:

(a) determine the population at $t = 5$.

(b) find how long it takes for the population to reach 5,000. To triple.

If you do not know the value of the growth constant k, your first step will typically be to use other information to find it.



Example 2. The population of a community was 22,000 in 2000 and 26,800 in 2010. Assuming that the community maintains the same rate of exponential growth (see margin figure):

(a) Find a formula for the population t years after 2000.

(b) Find the annual percentage rate of growth of the community.

Solution. Let t represent the number of years since 2000, so the year 2000 corresponds to $t = 0$ and the year 2010 corresponds to $t = 10$. Then $f(0) = 22000$, $f(10) = 26800$ and $f(t) = 22000 \cdot e^{kt}$.

(a) To find the value for k:

$$26800 = f(10) = 22000 \cdot e^{k(10)} \Rightarrow 1.218 = e^{10k} \Rightarrow \ln(1.218) = 10k$$
$$\Rightarrow k = 0.1 \ln(1.218) \approx 0.0197$$

so $f(t) \approx 22000 \cdot e^{0.0197t}$.

(b) $f(0) = 22000$ and $f(1) \approx 22000 \cdot e^{(0.0197)1} \approx 22438$ so the annual percentage increase was

$$\frac{f(1) - f(0)}{f(0)} = \frac{438}{22000} \approx 0.0199 = 1.99\%$$

during the first year. ◀

Practice 2. A scientist released 12,000 free neutrons into a material. Two seconds later, the material contained 18,000 free neutrons. If the number of free neutrons grows exponentially:

(a) Find a formula for the number of neutrons present t seconds after the beginning of the experiment.

(b) Find the doubling time for the number of free neutrons.

Compound interest provides another example of exponential growth.

Example 3. How long does it take \$1,000 to double when invested in a savings account with interest compounded continuously at an annual rate of 5%? At an effective annual rate of return of 5% (compounded continuously)?

Solution. If the interest is compounded continuously at an annual rate of 5% and $A(t)$ is the amount of money in the account t years after the initial deposit of \$1,000, then:

$$A'(t) = 0.05A(t),\ A(0) = 1000 \quad \Rightarrow \quad A(t) = 1000e^{0.05t}$$

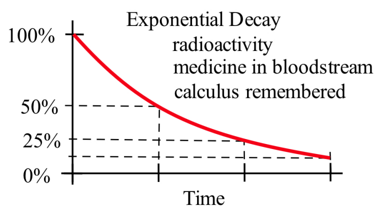
and the doubling time is $t_d = \frac{\ln(2)}{0.05} \approx 13.86$ years.

If the effective annual rate of return is 5%, then $A(1) = 1000 + 0.05(1000) = 1.05(1000) = 1050$ so:

$$1050 = 1000e^{k \cdot 1} \quad \Rightarrow \quad 1.05 = e^k \quad \Rightarrow \quad k = \ln(1.05) \approx 0.0488$$

hence the doubling time is $t_d = \frac{\ln(2)}{\ln(1.05)} \approx 14.21$ years. (An effective annual return of 5% corresponds to a continuously compounded annual interest rate of 4.88%.) ◀

Practice 3. How long does it take an investment to double if the effective annual rate of return is 2%?



Exponential Decay

Exponential decay occurs when the rate of loss of something is proportional to the amount present. One example of exponential decay is radioactive decay: the number of atoms of a radioactive substance that "decay" (split into nonradioactive atoms and release particles) during a short time interval is proportional to the number of radioactive atoms present at that time. Exponential decay (see margin) also models how quickly some medicines are absorbed from the bloodstream — and even how quickly you forget calculus concepts.

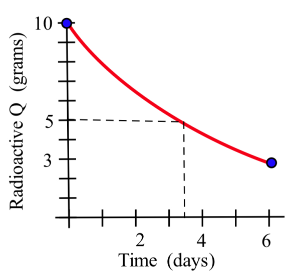
Exponential decay calculations are similar to those for growth, but the value of k is negative and we talk about "**half-life**," the time for half of the material to decay or be absorbed, instead of the doubling time. The margin table shows the half-lives of some isotopes.

element	half-life
iodine-131	8.07 days
strontium-90	29 years
argon-39	265 years
carbon-14	5700 years
plutonium-239	24400 years
uranium-238	4.51×10^5 years
uranium-234	2.47×10^9 years

Example 4. You started an experiment with 10 g of a radioactive substance, but after 6 days of decay only 3 g remained.

(a) Find a formula for the amount of radioactive material present t days after beginning the experiment.

(b) Find the half-life for the radioactive substance.



Solution. Let $f(t)$ represent the amount of the radioactive substance present after t days. Then $f(t) = 10e^{kt}$ (see margin figure).

(a) $3 = f(6) = 10e^{6k} \Rightarrow 0.3 = e^{6k} \Rightarrow \ln(0.3) = 6k$ so $k = \frac{1}{6}\ln(0.3) \approx -0.2007$ and $f(t) = 10e^{-0.2007t}$.

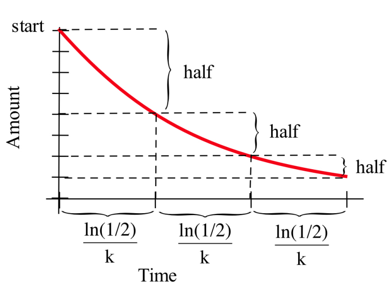
(b) The half-life t_h is the time required for half of the material to decay, so we need to solve $5 = 10e^{-0.2007t_h}$ for t_h:

$$0.5 = e^{-0.2007t_h} \Rightarrow \ln(0.5) = -0.2007t_h \Rightarrow t_h = \frac{\ln(0.5)}{-0.2007} \approx 3.5 \text{ days}$$

Note that $t_h = \dfrac{\ln(0.5)}{k}$, which will hold true generally. ◀

When you know the decay constant k, the half-life t_h is simple to find, as in the preceding Example:

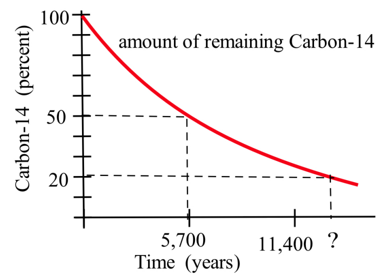
$$t_h = \frac{\ln(0.5)}{k} = \frac{\ln\left(\frac{1}{2}\right)}{k} = \frac{-\ln(2)}{k}$$



The half-life depends only on the decay constant k (see margin figure).

If you know the half-life of a substance and you know how much of the substance is present in a sample now, you can determine how much was present at some past time or determine how long ago the sample contained a particular amount of the substance.

Scientists use radioactive **carbon-14**, with a half-life of about 5,700 years, in this way to estimate how long ago plants and animals lived. A living plant continually exchanges carbon-14 and ordinary carbon with the atmosphere so that the ratio of carbon-14 to non-radioactive carbon remains relatively constant. But once the plant dies, this exchange stops. The ordinary carbon remains in the material, but the carbon-14 decays, so the ratio of carbon-14 to ordinary carbon decreases at a known rate. By measuring the ratio of carbon-14 to ordinary carbon in a sample of plant tissue, scientists can determine how long ago the plant died and obtain an estimate for the age of the sample.



Example 5. The amount of carbon-14 in plant fiber of a woven basket is 20% of the amount present in a living plant (see margin figure). Estimate the age of the basket.

Solution. Let $f(t)$ represent the relative amount of carbon-14 in a sample with age t years. Because the half-life of carbon-14 is 5,700 years:

$$t_h = \frac{-\ln(2)}{k} \Rightarrow 5700 = \frac{-\ln(2)}{k} \Rightarrow k = \frac{-\ln(2)}{5700} \approx -0.0001216$$

so that $f(t) = f(0) \cdot e^{-0.0001216t}$. Because 20% of the carbon-14 remains in our sample, we want the value of T so that:

$$\begin{aligned} 0.20f(0) = f(T) &\Rightarrow 0.20f(0) = f(0) \cdot e^{-0.0001216T} \\ &\Rightarrow 0.20 = e^{-0.0001216T} \Rightarrow \ln(0.20) = -0.0001216T \\ &\Rightarrow T = \frac{\ln(0.2)}{-0.0001216} \approx 13235 \end{aligned}$$

We can conclude that the basket was made from a plant that died about 13,200 years ago. (Does that mean the basket was made then?) ◀

This dating method is very sensitive to small changes in the measured amount of carbon-14.

Practice 4. The half-life of an isotope is 8 days. Write a formula for the amount of the isotope present t days after you begin an experiment with 10 mg of the isotope.

The rate at which many medicines are absorbed from the blood is proportional to the concentration of the medicine in the blood: the higher the concentration in the blood, the faster it is absorbed.

Example 6. Suppose a certain medicine has an absorption (decay) constant of -0.17 (determined experimentally) and that the lowest "effective"concentration of the medicine is 0.3 mg/l (milligrams of medicine per liter of blood). If a patient who has 5 liters of blood is injected with 20 mg of the medicine, how long will the medicine be effective?

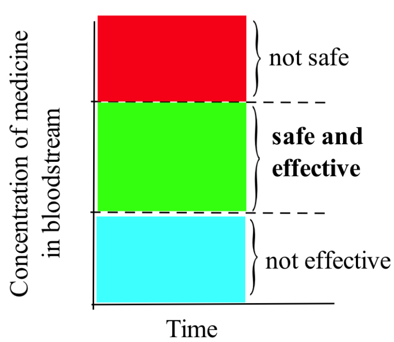
Solution. because the patient is starting with 20 mg of the medicine in 5 liters of blood, the initial concentration is $\frac{20 \text{ mg}}{5 \text{ L}} = 4$ mg/L. The amount of medicine in the blood t hours later is thus $f(t) = 4e^{-0.17t}$ and we want to find T so that $f(T) = 0.3$ mg/L

$$0.3 = 4e^{-0.17T} \Rightarrow \frac{0.3}{4} = e^{-0.17T} \Rightarrow \ln\left(\frac{0.3}{4}\right) = -0.17T \Rightarrow T \approx 15.2$$

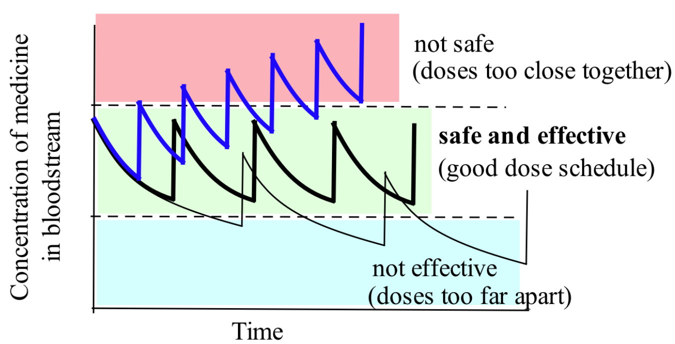
so a patient should receive a new does of the medicine about 15 hours after the first dose. ◀

Practice 5. Should the amount of the second dose in the preceding Example be the same as the initial dose?

Many medicines have a "safe and effective" interval of concentrations (see figure below), so the goal of a schedule for taking the medicine is to keep the concentration near the middle of that range. Taking doses too close together in time can result in an overdose, while taking them too far apart is eventually ineffective.



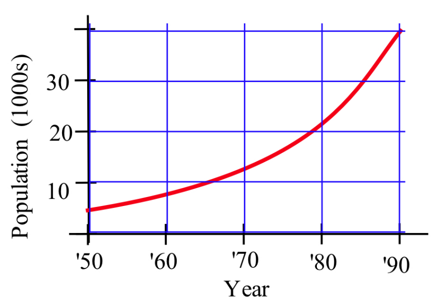
Safe and effective region



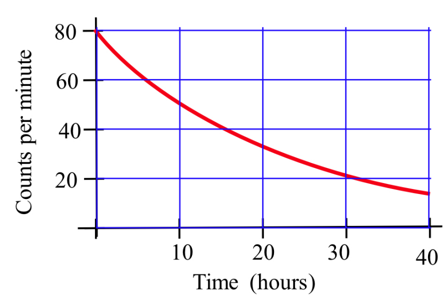
Safe and effective dosing schedule

6.4 Problems

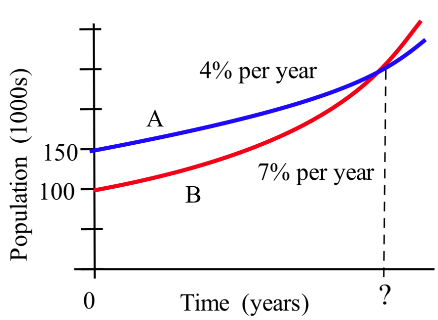
1. How long did it take the population of a city (see below) to double from 10,000 to 20,000? From 15,000 to 30,000? Approximate the doubling time.



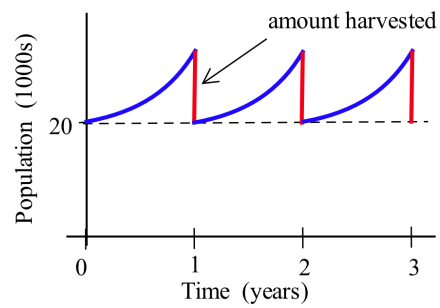
2. How long did it take the counts for a radioactive material to decay from 80 per minute to 40? From 60 to 30? From 40 to 20? What is the half-life?



3. The population of a community in 1990 was 48,000 people and in 2010 it was 64,000 people.
   (a) Write a formula for the population of the community t years after 1990.
   (b) Estimate the population in the year 2020.
   (c) When will the population be 100,000?
   (d) What is the doubling time of the population?
4. Repeat Problem 3 if the population of another community was was 40,000 people in 1990 and 60,000 people in 2010.
5. A terrific investment pays interest at an effective annual rate of 15%. How long will it take for a \$5,000 investment to double? To triple?
6. You have invested \$3,000 for 10 years at an effective annual rate of 7.5% and a friend has invested the same amount invested at an effective annual interest rate of 7.75%. Your friend will get back how much more money than you at the end of 10 years? At the end of 20 years?
7. Find a formula for the population in Problem 1.
8. Each bacterium of a certain species splits into two bacteria at the end of each minute. If we start with a few bacteria in a bowl at 3:00 p.m. and the bowl is full of bacteria at 4:30 p.m., when was the bowl half full? (Calculus is not required.)
9. A newscaster reports that the world's population is now doubling every 50 years. What annual rate of growth results in a 50-year doubling time?
10. Group A has a population of 150,000 and an annual growth rate of 4%; group B has a population of 100,000 and an annual growth rate of 7% (see figure below). After how many years will the two groups be the same size?

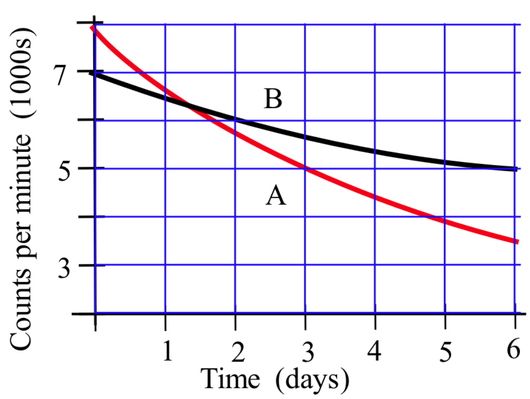


11. Group A has a population of 600,000 and an annual growth rate of 3%; group B has a population of 400,000 and an annual growth rate of 6%. After how many years will the two groups be the same size?
12. The unregulated population of fish in a certain lake grows by 30% per year under optimum conditions. A fish census reveals there are approximately 20,000 fish in the lake. How many fish can be harvested (see figure below) at the end of each year in order to maintain a stable population? (This is an example of calculating the yield for a "renewable resource." In practice, more sophisticated calculations also take into account the distribution of species, ages and genders.)



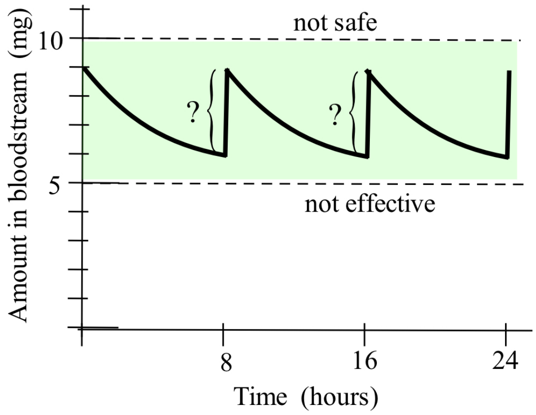
13. The annual exponential growth constant for a population of snails is $k = 0.14$. Currently you have 8,000 snails.
   (a) Determine the size of the population over the next 20 months if you harvest 2,000 snails every 2 months.
   (b) What happens if you harvest 3,000 snails every 2 months?
   (c) How many can we harvest every 2 months in order to maintain a stable population?
14. An exponential function $f(t) = Ae^{kt}$ has constant doubling time, but some non-exponential functions also have constant doubling times.
   (a) Show that the exponential function $f(t) = 2^t = e^{\ln(2)t}$ has a constant doubling time of 1. (Show that $f(t+1) = 2f(t)$.)
   (b) Graph $g(t) = 2^t\,[1 + A\sin(2\pi t)]$ for $A = 0.5$ and $A = 1.5$. Show that g has a constant doubling time 1 for any choice of A.

15. An experiment starts with 10 grams of a radioactive material and 14 days later 2 grams remain.
    (a) Find a formula for the amount of material remaining t days after the experiment starts.
    (b) Find the half-life of the material.
    (c) When will 0.7 grams of the material remain?
16. You start with 8 mg of a radioactive substance and 10 days later determine that 6.3 mg remains.
    (a) Find a formula for the amount left t days later.
    (b) Find the half-life of the substance.
    (c) When will 1 mg of the substance remain?
17. A Geiger counter initially recorded 187 counts per minute from a radioactive material, but 2 days later the count was down to 143 counts per minute. (The count per minute is proportional to the amount of radioactive material present.)
    (a) What is the half-life of the material?
    (b) When will the count be down to 20 counts per minute?
18. The initial Geiger counter measurement from a radioactive substance was 540 counts per minute, and a week later it was 500 counts per minute.
    (a) What is the half-life of the substance?
    (b) When will the count be down to 100 counts per minute?
19. Find a formula for the counts per minute for the radioactive material A in the figure below.



20. Find a formula for the counts per minute for the radioactive material B in the figure above.
21. Your friend plans to purchase a letter reputedly written by Isaac Newton (1642–1727), but an analysis of the paper shows that it contains 97.5% of the proportion of carbon-14 present in new paper of the same type. Can you be certain the letter is a forgery? If the age of the paper is consistent with time frame during which Newton lived, can you be certain the letter is genuine?
22. For several centuries, the Shroud of Turin was widely believed to be the shroud of Jesus. Three independent laboratories in England, Switzerland and the United States used carbon-14 dating on a few square centimeters of the cloth, and in 1988 they reported (*Science*, October 21, 1988) that the Shroud of Turin was probably made during the early 1300s and certainly after 1200 A.D.
    (a) If the Shroud was made in 1300 A.D., what percentage of the original carbon-14 was still present in 1988?
    (b) If the Shroud was made in 30 A.D., what percentage of the original carbon-14 was still present in 1988?
23. Half of a particular medicine is used up by the body every 6 hours, and the medicine is not effective if the concentration in the blood is less than 10 mg/l. If an ill person is given an initial dose of medicine to raise the concentration to 30 mg/l, for how long will the medicine be effective?
24. A particular controlled substance has a half-life of 12 hours, and it can be detected in concentrations as low as 0.002 mg/l in the blood.
    (a) If a person has an initial concentration of the substance of 15 mg/l in the blood, for how long can it be detected?
    (b) If the detection test is improved by a factor of 100, so it can detect a concentration of 0.00002 mg/l, for how long can an initial concentration of 15 mg/l be detected?
25. A doctor gave a patient 9 mg of a medicine that has half-life of 15 hours in the body. How much of the medicine does the patient need to take every 8 hours in order to maintain between 6 and 9 mg

of the medicine in the body all of the time? (See figure below.)



26. Each layer of a dark film transmits 40% of the light that strikes it.
    (a) How many layers are needed for an eye shield to transmit only 10% of the light?
    (b) How many layers are needed to transmit only 2% of the light?

27. A region has been contaminated with radioactive iodine-131 to a level five times the safe level. How long will it be until the area is safe?

28. A region has been contaminated with radioactive strontium-90 to a level five times the safe level. How long will it take until the area is safe?

29. The population of a country is 4 million and is growing at 5% per year. Currently the country has 10 million acres of forests that are being cut down (and not replanted) at a rate of 300,000 acres per year.
    (a) Find a formula for the number of acres of forest per person.
    (b) At what rate is the number of acres of forest per person changing?
    (c) If the population and harvest rates remain constant, in approximately how many years will there be one acre of forest per person?

6.4 Practice Answers

1. (a) $f(5) = 2000e^{0.0488(5)} \approx 2552$
   (b) $f(T) = 5000 \Rightarrow 5000 = 2000e^{0.0488T} \Rightarrow 2.5 = e^{0.0488T} \Rightarrow \ln(2.5) = 0.0488T$
   $\Rightarrow T = \dfrac{\ln(2.5)}{0.0488} \approx 18.78$ hours; tripling time is $\dfrac{\ln(3)}{0.0488} \approx 22.51$ hours.

2. (a) $f(0) = 12000$ so $f(t) = 12000e^{kt}$ and $f(2) = 18000$ so:
   $$18000 = 12000e^{2k} \Rightarrow 1.5 = e^{2k} \Rightarrow \ln(1.5) = 2k \Rightarrow k = \frac{1}{2}\ln(1.5) \approx 0.2027$$
   and thus $f(t) \approx 12000e^{0.2027t}$.
   (b) Doubling time is $t_d = \dfrac{\ln(2)}{k} \approx \dfrac{\ln(2)}{0.2027} \approx 3.42$ seconds.

3. After 1 year, each \$1 invested will become $\$1 + (0.02)(\$1) = \$1.02$ so:
   $$f(1) = 1.02 = 1 \cdot e^{k \cdot 1} \Rightarrow \ln(1.02) = k \Rightarrow k \approx 0.0198$$
   The doubling time is therefore: $t_d = \dfrac{\ln(2)}{0.0198} \approx 35$ years.

4. $8 = t_h = -\dfrac{\ln(2)}{k} \Rightarrow k = -\dfrac{\ln(2)}{8} \approx -0.0866$ so $f(t) = 10e^{-0.0866t}$.

5. No. After 15.2 hours, the patient still has 0.3 mg/l of medicine in his blood, or $5(0.3) = 1.5$ mg. A dose of $20 - 1.5 = 18.5$ mg would return the medicine in his blood to the original level of 20 mg.

6.5 Heating, Cooling and Mixing

The previous section examined applications of exponential growth and decay. This section examines other applications involving separable IVPs that can be solved using the methods of Section 6.3.

Newton's Law of Cooling

Some rates of change depend on how far the value of a variable quantity is from a fixed value. The rate at which a hot cup of soup cools (or a cool cup of water warms up) is proportional to the difference in temperature between the soup (or water) and the surrounding air. This principle, called Newton's Law of Cooling (or Warming), provides an approximate model for temperature change when the object in question is subject to forced convection (a steady breeze, for example).

It also applies to natural convection (no breeze or draft) if the temperature difference between the object and the surrounding air is not too great.

Newton's Law of Cooling/Warming

If $f(t)$ is the temperature at time t of an object in an atmosphere with constant temperature a,

then the rate of change of f is proportional to the difference between f and a: $f'(t) = k\,[f(t) - a]$ and the solution to this ODE is:

$$f(t) = a + [f(0) - a] \cdot e^{kt}$$

The statement that the rate of change is proportional to the difference:

$$f'(t) = k\,[f(t) - a]$$

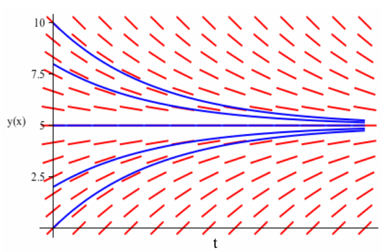
can be observed by experimentation.

This ODE is separable, so we can solve it using the methods of Section 6.3. Or we can let $g(t) = f(t) - a$ so that $g'(t) = f'(t)$ and $g(0) = f(0) - a$, transforming the ODE for $f(t)$ into the IVP:

$$g'(t) = kg(t), \qquad g(0) = f(0) - a$$

From Section 6.4 we know that $g(t) = g(0) \cdot e^{kt}$ so that:

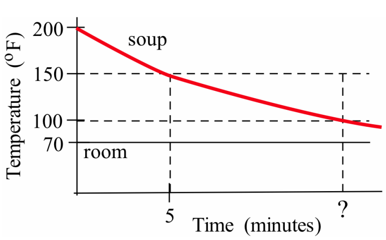
$$f(t) - a = [f(0) - a]\,e^{kt} \quad \Rightarrow \quad f(t) = a + [f(0) - a]\,e^{kt}$$



Solutions of f'(t)= (-1)*(f(t) - 5)

The margin figure shows some functions with different initial values that all satisfy the differential equation $f'(t) = f(t) - 5$.

Example 1. A cup of soup sits on a counter in a room with temperature 70° F. When first poured into the cup, the soup had a temperature of 200° F, and five minutes later its temperature was 150° F (see margin).



(a) Find a formula for the soup's temperature at any time $t > 0$.

(b) How long does it take for the soup to cool to 100° F?

(c) What will the temperature of the soup be after a "long" time?

Solution. (a) If $f(t)$ is the soup's temperature t minutes after being poured and the room temperature is $a = 70°$ F, then we know that $f(0) = 200°$ F so $f'(t) = k\,[f(t) - 70]$ and hence:

$$f(t) = 70 + [200 - 70] \cdot e^{kt} = 70 + 130 \cdot e^{kt}$$

We can now use the information that $f(5) = 150°$ F to find the value of k and an equation for $f(t)$:

$$150 = f(5) = 70 + 130.e^{k \cdot 5} \Rightarrow k = \frac{1}{5} \cdot \ln\left(\frac{80}{130}\right) \approx -0.0971$$

so $f(t) \approx 70 + 130 \cdot e^{-0.0971t}$.

(b) We now want to find the time t so that $f(t) = 100$:

$$100 = f(t) \approx 70 + 130 \cdot e^{-0.0971t} \Rightarrow 30 = 130 \cdot e^{-0.0971t}$$

$$\Rightarrow t = \frac{\ln\left(\frac{30}{130}\right)}{-0.0971} \approx 15.1 \text{ minutes}$$

(c) After a "long time" means for very large values of t:

$$\lim_{t\to\infty} \left[70 + 130 \cdot e^{-0.0971t}\right] = 70$$

so (eventually) the temperature of the soup approaches 70° F, the temperature of the room. ◀

Practice 1. After eating some of the soup in the preceding Example, you refrigerate the leftover soup, cooling it to a temperature of 37° F. At noon the next day, you move the container of soup from the refrigerator back to the 70° F room. One hour later, the temperature of the soup is 48° F. At what time will the temperature of the soup be 60° F?

Mixing

Many important applications involve one substance being mixed into another: chlorine being slowly introduced into a swimming pool, a toxic gas being mixed into the air in a closed room, or (in a classic example) salt (NaCl) being mixed into water (H_2O).

Example 2. A tank contains 350 grams of salt dissolved in 10 liters of water. Pure water begins flowing into the tank at 2 liters per hour, and the well-mixed solution flows out of the tank at the same rate.

(a) Find a formula for the amount of salt in the tank t hours after this process begins.

(b) Find a formula for the concentration salt in the tank.

(c) What will the concentration be after a "long" time?

Solution. (a) Let $A(t)$ be the amount of salt (in grams) t hours after this process begins. We know that $A(0) = 350$. Now consider $A'(t)$, the rate at which amount of salt in the tank is changing with respect to time. No salt is flowing in, but the rate at which salt is leaving the tank is:

$$\frac{A(t)\text{ g}}{10\text{ L}} \cdot 2\ \frac{\text{L}}{\text{hr}} = 0.2A(t)\ \frac{\text{g}}{\text{hr}}$$

which gives us the IVP:

$$\frac{dA}{dt} = -0.2A, \quad A(0) = 350$$

This is a separable IVP with solution $A(t) = 350e^{-0.2t}$.

(b) The concentration is:

$$C(t) = \frac{A(t)\text{ g}}{10\text{ L}} = \frac{350e^{-0.2t}\text{ g}}{10\text{ L}} = 35e^{-0.2t}\ \frac{\text{g}}{\text{L}}$$

(c) After a "long" time:

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} 35e^{-0.2t} = 0$$

This seems reasonable: after a very long time, most of the original salt in the tank will be flushed out, replaced by pure water. ◀

Practice 2. A tank contains 300 grams of salt dissolved in 10 liters of water. Water containing 15 grams of salt per liter begins flowing into the tank at a rate of 4 liters per hour, and the well-mixed solution flows out of the tank at the same rate.

(a) Find a formula for the amount of salt in the tank t hours after this process begins.

(b) Find a formula for the concentration salt in the tank.

(c) What will the concentration be after a "long" time?

If the volume of liquid in the tank does not remain constant, the differential equation can become a bit more complicated.

Example 3. A 20-liter tank contains 250 grams of salt dissolved in 10 liters of water. Pure water begins flowing into the tank at a rate of 5 liters per hour, and the well-mixed solution flows out of the tank at a rate of 3 liters per hour.

(a) Find a formula for the amount of salt in the tank t hours after this process begins.

(b) Find a formula for the concentration salt in the tank.

(c) What will the concentration be at the moment the tank overflows?

Solution. (a) Let $A(t)$ be the amount of salt (in grams) t hours after this process begins. We know that $A(0) = 250$ and that the volume of water in the tank is $V(t) = 10 + 2t$ (because 5 liters of water flow in each hour, but only 3 liters flow out). No salt is flowing into the tank, but the rate at which salt is leaving the tank is:

$$\frac{A(t) \text{ g}}{10 + 2t \text{ L}} \cdot 3 \frac{\text{L}}{\text{hr}} = \frac{3A(t)}{10 + 2t} \frac{\text{g}}{\text{hr}}$$

which gives us the IVP:

$$\frac{dA}{dt} = -\frac{3A}{10 + 2t}, \quad A(0) = 250$$

This is a separable IVP:

$$\frac{1}{A} \cdot \frac{dA}{dt} = -\frac{3}{10 + 2t} \Rightarrow \ln(A) = -\frac{3}{2} \ln(10 + 2t) + C$$

Using the initial condition:

$$\ln(250) = -\frac{3}{2} \ln(10) + C \Rightarrow C = \ln(250) + \frac{3}{2} \ln(10)$$

Solving for A:

$$\begin{aligned} \ln(A) &= \ln(10 + 2t)^{-\frac{3}{2}} + \ln(250) + \ln(10^{\frac{3}{2}}) \\ \Rightarrow \quad A(t) &= (10 + 2t)^{-\frac{3}{2}} \cdot 250 \cdot 10\sqrt{10} = \frac{2500\sqrt{10}}{(10 + 2t)^{\frac{3}{2}}} \end{aligned}$$

(b) The concentration is:

$$C(t) = \frac{A(t) \text{ g}}{10 + 2t \text{ L}} = \frac{2500\sqrt{10}}{(10 + 2t)^{\frac{5}{2}}} \frac{\text{g}}{\text{L}}$$

(c) The tank overflows when $20 = 10 + 2t \Rightarrow t = 5$, at which time the concentration is:

$$C(5) = \frac{2500\sqrt{10}}{(10 + 2 \cdot 5)^{\frac{5}{2}}} = \frac{25}{4\sqrt{2}} \frac{\text{g}}{\text{L}}$$

or about 4.42 grams per liter. ◀

Practice 3. A 20-liter tank contains 250 grams of salt dissolved in 10 liters of water. Water containing 35 grams of salt per liter begins flowing into the tank at a rate of 5 liters per hour, and the well-mixed solution flows out of the tank at a rate of 3 liters per hour. Set up an initial value problem to model the amount of salt in the tank t hours after this process begins. Is the ODE separable?

6.5 Problems

1. You remove a cheesecake from the oven with an ideal internal temperature of 165° F and place it into a 35° F refrigerator. After 10 minutes, the cheesecake has cooled to 150° F. If you must wait until the cheesecake has cooled to 70° before you eat it, how long will you have to wait?

2. When a pot of hot (200° F) water is removed from the stove in a 70° F kitchen, it takes 15 minutes to cool to a temperature of 150° F.
   (a) Find a formula for the temperature of the water t minutes after it is removed from the stove.
   (b) When will the water be 100° F?
   (c) When will the water be 80° F?
   (d) When will the water be 60° F?

3. When you remove the pot of 200° F water from the previous problem from the stove and take it outside on a 40° F day, it only takes 4 minutes to cool to a temperature of 150° F.
   (a) Find a formula for the temperature of the water t minutes after it is removed from the stove.
   (b) When will the water be 100° F?
   (c) When will the water be 80° F?
   (d) When will the water be 60° F?

4. When you remove a pitcher of orange juice from a 40° F refrigerator into a 70° F kitchen, it takes 25 minutes to warm up to a temperature of 60° F.
   (a) Estimate the temperature of the juice t minutes after you removed it from the refrigerator.
   (b) When was the juice 50° F?
   (c) When will the juice be 65° F?

5. A tank initially holds 100 gallons of brine (water mixed with salt) containing 50 pounds of dissolved salt. Pure water flows into the tank at a rate of 3 gallons per minute and flows out at the same rate, with the brine concentration kept roughly uniform by constant stirring. How much salt remains in the tank after one hour?

6. A tank initially holds 1000 liters of brine containing 8 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters per minute and brine (with concentration kept roughly uniform by constant stirring) leaves the tank at the same rate.
   (a) Find the concentration of salt in the tank after half an hour.
   (b) Determine the limiting concentration of salt in the tank after "a very long time."

7. A large tank initially contains 100 gallons of brine (water mixed with salt) containing 50 pounds of dissolved salt. Pure water flows into the tank at a rate of 3 gallons per minute and flows out at 2 gallons per minute, with the brine concentration kept roughly uniform by constant stirring. How much salt remains in the tank after one hour?

8. A tank initially holds 1000 liters of pure water. Brine containing 8 g of dissolved salt per liter enters the tank at a rate of 10 liters per minute and the well-mixed solution leaves the tank at the same rate.
   (a) Find the concentration of salt in the tank after half an hour.
   (b) Determine the limiting concentration of salt in the tank after "a very long time."

6.5 Practice Answers

1. The temperature of the soup t minutes after being removed from the refrigerator is:

$$f(t) = 70 + [37 - 70] \cdot e^{kt} = 70 - 33e^{kt}$$

Using the fact that $f(60) = 48$:

$$48 = 70 - 33e^{kt} \;\Rightarrow\; \frac{22}{33} = e^{60k} \;\Rightarrow\; k = \frac{1}{60} \cdot \ln\left(\frac{22}{33}\right) \approx -0.006758$$

so that $f(t) \approx 70 - 33e^{-0.006758t}$. If $60 = 70 - 33e^{-0.006758T}$ then:

$$\frac{10}{33} = e^{-0.006758T} \;\Rightarrow\; T = -\frac{1}{0.006758} \cdot \ln\left(\frac{10}{33}\right) \approx 176.7$$

so the soup will warm up to 60° F after roughly three hours.

2. If $A(t)$ is the amount of salt (in grams) t hours after this process begins, then:

$$\frac{dA}{dt} = 15\,\frac{\text{g}}{\text{L}} \cdot 4\,\frac{\text{L}}{\text{hr}} - \frac{A\ \text{g}}{10\ \text{L}} \cdot 4\,\frac{\text{L}}{\text{hr}}$$

so $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = 60 - \frac{2}{5}A, \quad A(0) = 300$$

(a) This IVP is separable, so we can solve it:

$$\int \frac{1}{60 - \frac{2}{5}A}\,dA = \int 1\,dt \;\Rightarrow\; -\frac{5}{2}\ln\left(\left|60 - \frac{2}{5}A\right|\right) = t + C$$

$$\Rightarrow\; \ln\left(\left|60 - \frac{2}{5}A\right|\right) = -\frac{2}{5}t + K$$

Using the initial condition:

$$\ln\left(\left|60 - \frac{2}{5} \cdot 300\right|\right) = -\frac{2}{5} \cdot 0 + K \;\Rightarrow\; K = \ln(60)$$

so that:

$$\ln\left(\left|60 - \frac{2}{5}A\right|\right) = -\frac{2}{5}t + \ln(60) \;\Rightarrow\; 60 - \frac{2}{5}A = -60e^{-\frac{2}{5}t}$$

where we need to use a $-$ sign when "undoing" the absolute value to ensure that the initial condition $A(0) = 300$ will still hold. Solving for A yields $A(t) = 150\left[1 + e^{-\frac{2}{5}t}\right]$.

(b) Divide by the volume to get $C(t) = 15\left[1 + e^{-\frac{2}{5}t}\right]$.

(c) $\lim\limits_{t\to\infty} 15\left[1 + e^{-\frac{2}{5}t}\right] = 15$, so the limiting concentration is 15 g/L (the concentration of solution flowing into the tank, as expected).

3. If $A(t)$ is the amount of salt (in grams) t hours after this process begins, then:

$$\frac{dA}{dt} = 35\,\frac{\text{g}}{\text{L}} \cdot 5\,\frac{\text{L}}{\text{hr}} - \frac{A\text{ g}}{(10+2t)\text{ L}} \cdot 3\,\frac{\text{L}}{\text{hr}}$$

so $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = 175 - \frac{3A}{10+2t}, \quad A(0) = 250$$

This is not a separable equation. (You will learn how to solve equations of this type in a course about differential equations. For now, you could construct a direction field to analyze the behavior of solutions to the ODE.)

7
Transcendental Functions

An **algebraic number** is any number that can be a solution to a polynomial equation with integer coefficients. Any rational number is algebraic: for example, $\frac{5}{7}$ is a solution of $7x - 5 = 0$. Any square root or cube root is an algebraic number: $\sqrt{3}$ is a solution of $x^2 - 3 = 0$ and $-\sqrt[3]{5}$ is a solution of $x^3 + 5 = 0$. In fact, any number that can be expressed as the sum, difference, product, quotient or rational power of integers is an algebraic number, including something as horrible-looking as:

$$\sqrt[5]{\frac{\sqrt[4]{93} - \sqrt[3]{436 + \sqrt{2}}}{17 + \sqrt{37}}}$$

Any real number that is *not* an algebraic number is called a **transcendental number**. We have already met (and used extensively) two very important transcendental numbers: π and e.

It turns out that *proving* that these numbers are transcendental is rather difficult, although by the end of Chapter 8 we will have most of the tools necessary to prove that e is transcendental.

Functions defined using sums, differences, products, quotients or rational powers of rational coefficients and a real-valued variable x are called **algebraic functions**. Any non-algebraic functions of a real variable are called **transcendental functions**. Examples of transcendental functions with which you should be very familiar are $\sin(x)$, $\cos(x)$, $\tan(x)$ and the other trigonometric functions, as well as e^x.

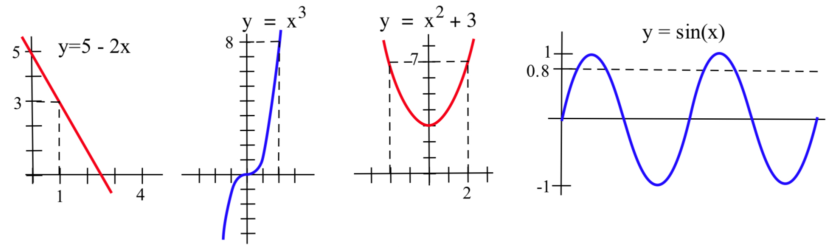
The geometric definitions of the trigonometric functions involve π, while the exponential function obviously involves e.

The inverse functions of these transcendental functions are also transcendental functions: $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$ and the other inverse trigonometric functions, as well as $\ln(x)$.

Because many important transcendental functions are defined as inverse functions, this chapter begins with a review of inverse functions and a discussion of finding derivatives of inverse functions. It continues with a review of inverse trigonometric functions and a discussion of their derivatives (and the usefulness of these derivative patterns in finding certain antiderivatives), then ties up some (very important) loose ends related to the definitions and properties of e^x and $\ln(x)$. The chapter concludes by introducing some new transcendental functions: the hyperbolic functions and their inverses, which play important roles in calculus and in applications.

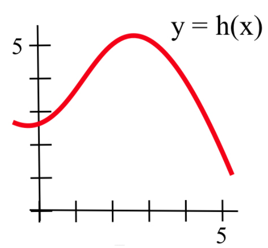
7.1 One-to-One Functions

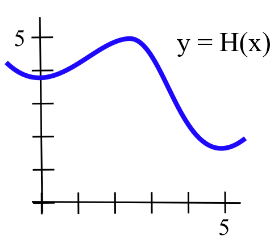
You've seen that some equations have only one solution (for example, $5 - 2x = 3$ and $x^3 = 8$), while some have two solutions ($x^2 + 3 = 7$) and some even have an infinite number of solutions ($\sin(x) = 0.8$). The graphs of $y = 5 - 2x$, $y = x^3$, $y = x^2 + 3$ and $y = \sin(x)$ and the solutions of the equations mentioned above appear below:



Functions f for which equations of the form $f(x) = k$ have at most one solution for each value of k (that is, each outcome k comes from only one input x) arise often in applications and possess a number of useful mathematical properties. This brief section focuses on those functions and examines some of their properties.

x	0	1	2	3	4	5
$g(x)$	5	7	3	5	0	7





Example 1. How many solutions does each equation have?

(a) $f(x) = 0$ for $f(x) = x(x-4)$

(b) $g(x) = 3$ for g given in the margin table

(c) $h(x) = 4$ for h given by the graph in the margin

(d) $f(x) = k$ for $f(x) = e^x$.

Solution. (a) Two: $x(4-x) = 0 \Rightarrow x = 0$ or $x = 4$. (b) One: $g(x) = 3$ only if $x = 2$. (c) Two: $h(x) = 4$ if $x \approx 1.2$ or if $x \approx 4$. (d) If $k > 0$, it has one solution: $x = \ln(k)$. If $k \leq 0$, it has no solutions. ◀

Practice 1. How many solutions does each equation have?

(a) $f(x) = 4$ for $f(x) = x(4-x)$

(b) $g(x) = 7$ for g given by the margin table

(c) $H(x) = 3$ for H given by the graph in margin

(d) $f(x) = 5$ for $f(x) = \ln(x)$

Horizontal Line Test

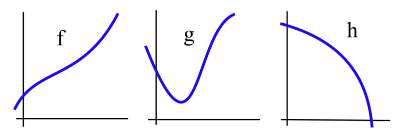
If not, review Section 0.3.

You should be familiar with the Vertical Line Test, a graphical tool you can use to help determine whether or not a curve in the xy-plane is the graph of a function. A similar geometrical test leads to the definition of a **one-to-one function** and provides a tool for helping to determine when a function is one-to-one.

> **Horizontal Line Test (Definition of One-to-One)**:
> A function is **one-to-one** if each horizontal line intersects the graph of the function at most once.

Equivalently, a function $y = f(x)$ is one-to-one if two *distinct* x-values always produce two *distinct* y-values: that is, $a \neq b \Rightarrow f(a) \neq f(b)$. This immediately tells us that every strictly increasing function is one-to-one, and that every strictly decreasing function is one-to-one. (Why?)

For any function, if we know an input value we can calculate the output, but an output may arise from any of several different inputs. With a one-to-one function, each output comes from only one input.

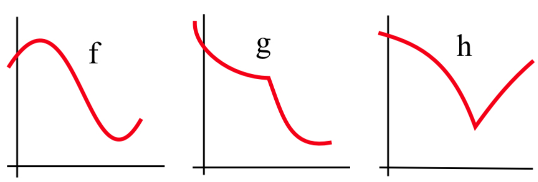


x	$f(x)$	$g(x)$	$h(x)$
0	5	7	2
1	2	3	-1
2	3	0	5
3	5	1	4
4	0	6	3
5	1	3	0

Example 2. (a) Which functions in the first margin figure are one–to-one? (b) Which functions in the first margin table are one-to-one?

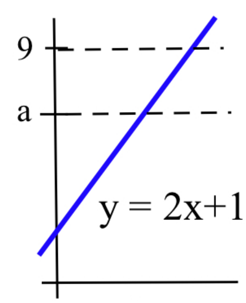
Solution. (a) In the figure, f and h are one-to-one; g fails the Horizontal Line Test, so g is not one-to-one. (b) In the table, h is one-to-one, while f and g are not one-to-one because $f(0) = f(3)$ and $g(1) = g(5)$. ◀

Practice 2. (a) Which functions graphed below are one-to-one? (b) Which functions in the second margin table are one-to-one?



x	$f(x)$	$g(x)$	$h(x)$
0	4	2	-2
1	2	3	5
2	-2	0	1
3	5	4	14
4	3	6	3
5	1	7	1

Example 3. Let $f(x) = 2x + 1$ (see margin). Find the values of x so that (a) $f(x) = 9$ and (b) $f(x) = a$ and then (c) solve $f(y) = x$ for y.



Solution. (a) $9 = f(x) = 2x + 1 \Rightarrow 8 = 2x \Rightarrow x = \frac{8}{2} = 4$ (b) $a = 2x + 1 \Rightarrow 2x = a - 1 \Rightarrow x = \frac{a-1}{2}$ (c) $x = f(y) = 2y + 1 \Rightarrow 2y = x - 1$ so $y = \dfrac{x-1}{2}$. Notice that this new function reverses the operations of $f(x)$, applied in reverse order: $f(x)$ multiplies x by 2, then adds 1; the new function subtracts 1, then divides by 2. ◀

Practice 3. Let $g(x) = 3x - 5$. Find the values of x so that (a) $g(x) = 7$ and (b) $g(x) = b$ and then (c) solve $g(y) = x$ for y.

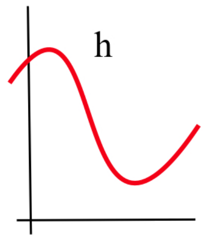
Practice 4. Show that exponential growth, for example $f(x) = e^{3x}$, and exponential decay, for example $g(x) = e^{-2x}$, are both one-to-one.

7.1 Problems

In Problems 1–4, explain why each given function is (or is not) one-to-one.

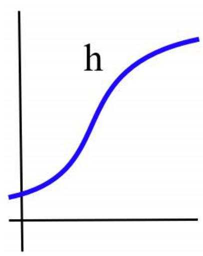
1. $f(x) = 3x - 5$, $y = 3 - x$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	3
1	4
2	5
3	2
4	4



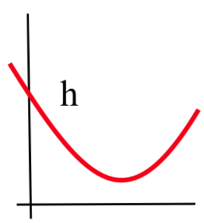
2. $f(x) = \frac{x}{4}$, $y = x^2 + 3$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	3
1	2
2	0
3	-2
4	1



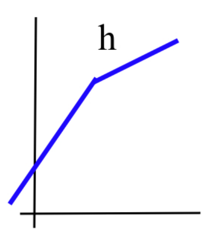
3. $f(x) = \sin(x)$, $y = e^x - 2$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	-1
1	5
2	3
3	1
4	0



4. $f(x) = 17$, $y = x^3 - 1$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	2
1	5
2	4
3	1
4	2



5. Is the relation between people and Social Security numbers a function? A one-to-one function?

6. Is the relation between people and phone numbers a function? If so, is it one-to-one?

7. What would it mean if the scores on a calculus test were one-to-one?

8. The relation given below represents "y is married to x." (a) Is this relation a function? (b) Is it one-to-one? (c) Is P breaking the law? (d) Is A breaking the law?

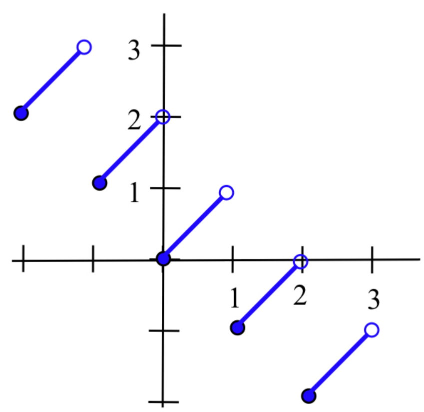
x	A	B	C	D
y	P	Q	P	R

9. In how many places can a one-to-one function touch the x-axis?

10. Can a continuous one-to-one function have the values given below? Explain.

x	1	3	5
$f(x)$	2	7	3

11. The graph of $f(x) = x - 2 \cdot \lfloor x \rfloor$ for $-2 \leq x \leq 3$ appears below.



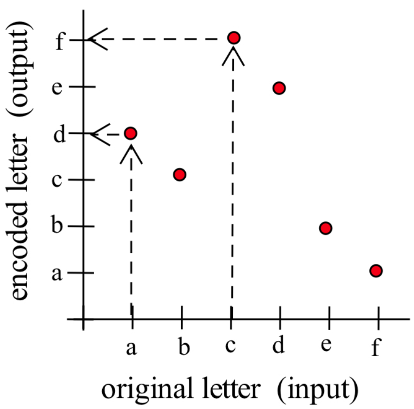
(a) Is f a one-to-one function?
(b) Is f an increasing function?
(c) Is f a decreasing function?

12. Is every linear function $L(x) = ax + b$ one-to-one? If not, which linear functions *are* one-to-one?

13. Show that $f(x) = \ln(x)$ is one-to-one for $x > 0$.
14. Show that $g(x) = e^x$ is one-to-one.
15. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	c	f	e	b	a

    (a) Is the encoding rule a function?
    (b) Is the encoding rule one-to-one?
    (c) Encode the word "bad."
    (d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
    (e) A graph of the encoding rule appears below. Create a graph of the decoding rule.



    (f) Compare the encoding and decoding graphs.
16. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
b	d	b	b	a	c

    (a) Is the encoding rule a function?
    (b) Is the encoding rule one-to-one?
    (c) Encode the word "bad."
    (d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
    (e) Create a graph of the encoding rule.
    (f) Create a graph of the decoding rule.
    (g) Compare the encoding and decoding graphs.
17. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	f	e	a	c	b

    (a) Is the encoding rule a function?
    (b) Is the encoding rule one-to-one?
    (c) Encode the word "bad."
    (d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
    (e) Create a graph of the encoding rule.
    (f) Create a graph of the decoding rule.
    (g) Compare the encoding and decoding graphs.
    (h) What happens if you encode a word, then encode the encoded word? For example, encode (encode ("bad")) = ?
18. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
e	a	f	c	b	d

    (a) Is the encoding rule a function?
    (b) Is the encoding rule one-to-one?
    (c) Encode the word "bad."
    (d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
    (e) Create a graph of the encoding rule.
    (f) Create a graph of the decoding rule.
    (g) Compare the encoding and decoding graphs.
    (h) What happens if you apply this encoding rule three times in succession? For example, encode (encode (encode ("bad"))) = ?

7.1 Practice Answers

1. (a) One: solve $x(4-x)=4$ to get $x=2$.
   (b) Two: $x=1$ and $x=5$.
   (c) One: $x \approx 3.5$.
   (d) One: solve $5=\ln(x)$ to get $x=e^5 \approx 148.4$.

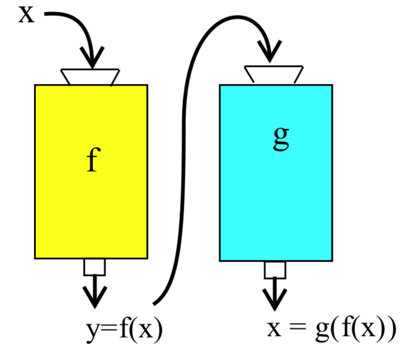
2. (a) Only g is one-to-one; f and h fail the Horizontal Line Test.
   (b) Both f and g are one-to-one; h is not, because $h(2)=h(5)$.

3. (a) $3x-5=7 \Rightarrow 3x=12 \Rightarrow x=4$
   (b) $3x-5=a \Rightarrow 3x=a+5 \Rightarrow x=\dfrac{a+5}{3}$
   (c) $f(x)=3x-5 \Rightarrow f(y)=3y-5$ so $f(y)=x \Rightarrow 3y-5=x \Rightarrow 3y=x+5 \Rightarrow y=\dfrac{x+5}{3}$

4. If $f(x)=e^{kx}$ where $k>0$ then $f'(x)=k\cdot e^{kx}>0$ so $f(x)$ is strictly increasing, hence one-to-one. If $g(x)=e^{rx}$ where $r<0$ then $g'(x)=r\cdot e^{rx}<0$ so $g(x)$ is strictly decreasing, hence one-to-one.

7.2 *Inverse Functions*

If f is any one-to-one function then equations of the form $f(x) = k$ have (at most) a single solution. Such functions can be uniquely "undone": if f is a one-to-one function, then there is another function g that "undoes" the effect of f, so that $g(f(x)) = x$. When g and f are composed in this manner, g retrieves the original input of f (see margin). We call this function g that "undoes" the effect of f the **inverse function** of f or simply **the inverse** of f.



If f is a function that encodes a message, then the inverse of f is the function that decodes an encoded message to retrieve the original message. The functions e^x and $\ln(x)$ "undo" the effects of each other:

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x \text{ (for } x > 0)$$

so the functions e^x and $\ln(x)$ are inverses of each other.

This section will examine some of the properties of inverse functions and explain how to find the inverse of a function given by a table of data, a graph or a formula.

> **Definition**: If f and g are functions that satisfy both $g(f(x)) = x$ and $f(g(y)) = y$ for all x in the domain of f and all y in the domain of g, then g is the **inverse** of f, f is the inverse of g, and f and g are a **pair of inverse functions**.

We often write the inverse function of f as f^{-1} (pronounced "eff inverse") but *you must be very careful*: f^{-1} does **not** mean $\dfrac{1}{f}$.

Example 1. The values of a function f appear in the margin table. Evaluate $f^{-1}(0)$ and $f^{-1}(1)$.

x	$f(x)$
0	3
1	2
2	0
3	1

Solution. For every x, $f^{-1}(f(x)) = x$ so the value of $f^{-1}(0)$ is the solution x of the equation $f(x) = 0$. The value we want is $x = 2$, and we can check that $f^{-1}(0) = f^{-1}(f(2)) = 2$.

The value of $f^{-1}(1)$ is the solution x of the equation $f(x) = 1$, which is $x = 3$, and we can check that $f^{-1}(1) = f^{-1}(f(3)) = 3$.

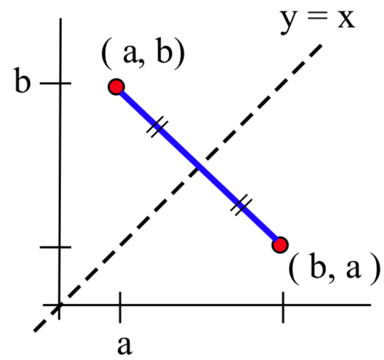
Similarly, $f^{-1}(2) = 1$ and $f^{-1}(3) = 0$. These results appear in the second margin table. You should notice that if the ordered pair (a, b) is in the table for f, then the reversed pair (b, a) is in the table for f^{-1}. ◀

x	$f^{-1}(x)$
0	2
1	3
2	1
3	0

Practice 1. The values of the function g appear in the margin table. Create a table of values for g^{-1}.

x	$g(x)$
0	2
1	1
2	3
3	4
4	0

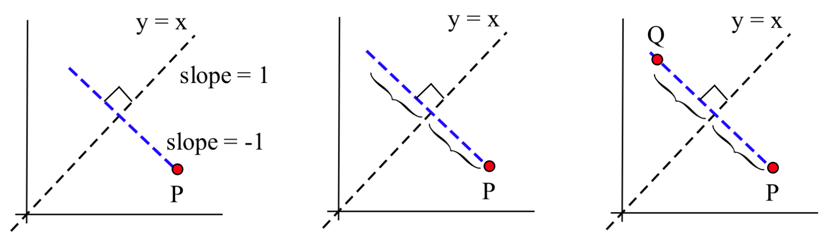
The method of interchanging the coordinates of a point on the graph (or in a table of values) of f to get a point on the graph (or in a table of values) of f^{-1} provides an efficient way to graph f^{-1}.



> **Theorem**: If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

Proof. If (a, b) is on the graph of f, then $b = f(a)$, so $f^{-1}(b) = f^{-1}(f(a))$. By definition, $f^{-1}(f(a)) = a$, so $f^{-1}(b) = a$, which tells us that (b, a) is on the graph of f^{-1}. □

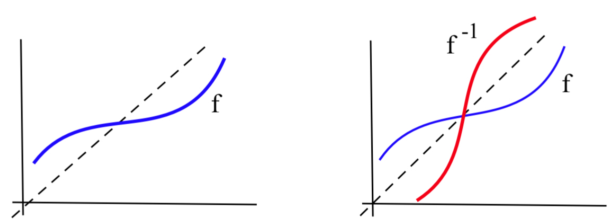
Graphically, when you interchange the coordinates of a point (a, b) to get a new point (b, a), the old point and the new point are symmetric about the line $y = x$. If you put a spot of wet ink at the point (a, b) on a piece of graph paper (see margin) and fold the paper along the line $y = x$, a new spot of ink will appear at the point (b, a). The figures below demonstrate another graphical method for finding the location of the point (b, a):



Draw the line $y = x$ along with a line of slope -1 that passes through the given point P (above left), then measure the distance from P to the line $y = x$ and move that same distance on the other side of $y = x$ (above center), and plot the new point Q at that location.

> **Corollary**: The graphs of f and f^{-1} are symmetric about the line $y = x$.

Example 2. A graph of f appears below left; sketch a graph of f^{-1}.



Solution. Imagine the graph of f is drawn with wet ink and fold the xy-plane along the line $y = x$. When you unfold the plane, the new graph is f^{-1} (see figure above right). ◀

You could also proceed point by point: Pick several points (a, b) on the graph of f and plot the symmetric points (b, a), then use the new (b, a) points as a guide for sketching the graph of f^{-1}.

Practice 2. A graph of g appears in the margin. Sketch a graph of g^{-1}. What happens to points on the graph of g that lie on the line $y = x$?

Finding a Formula for $f^{-1}(x)$

When you have a table of values for f, you can create a table of values for f^{-1} by interchanging the values of x and y in the table for f (as in Example 1). When you have a graph of f, you can sketch a graph of f^{-1} by reflecting the graph of f about the line $y = x$ (as in Example 2). When you have a formula for f, you can try to find a formula for f^{-1}.

Example 3. The steps for wrapping a gift are (1) put the gift in a box, (2) cover the box with paper and (3) attach a ribbon. What are the steps for opening the gift — the inverse of the wrapping operation?

Solution. (i) Remove the ribbon (undo step 3), (ii) remove the paper (undo step 2) and (iii) remove the gift from the box (undo step 1). ◀

Then show happiness and gratitude.

The reason for the preceding trivial example is to point out that the first unwrapping step undoes the last wrapping step, the second unwrapping step undoes the second-to-last wrapping step ... and the last unwrapping step undoes the first wrapping step. This pattern holds for functions and their inverses too.

Example 4. The steps to evaluate $f(x) = 9x + 6$ are (1) multiply the input by 9 and (2) add 6 to the result. Write the steps, in words, for the inverse of this function, and then translate the verbal steps for the inverse into a formula for the inverse function.

Solution. (i) Subtract 6, undoing (2), and (ii) divide by 9, undoing (1):

$$x \quad \longrightarrow \quad x - 6 \quad \longrightarrow \quad \frac{x-6}{9}$$

so $f^{-1}(x) = \dfrac{x-6}{9}$ provides a formula for the inverse function. ◀

If (x, y) is a point on the graph of f, we know that (y, x) is on the graph of f^{-1}, so interchanging the roles of x and y in the formula for f should lead us to a formula for f^{-1}. Applying this idea to the function $f(x) = 9x + 6$ from the previous Example, we swap x and y in the formula $y = 9x + 6$ to get $x = 9y + 6 \Rightarrow x - 6 = 9y \Rightarrow y = \dfrac{x-6}{9}$, yielding the formula $f^{-1}(x) = \dfrac{x-6}{9}$ as in Example 4.

The following algorithm provides a general "recipe" for finding a formula for an inverse function:

- Start with a formula for f: $y = f(x)$.
- Interchange the roles of x and y: $x = f(y)$.
- Solve $x = f(y)$ for y.
- The resulting formula for y is the inverse of f: $y = f^{-1}(x)$.

The "interchange" and "solve" steps in the algorithm effectively undo the original operations in reverse order.

Example 5. Find formulas for the inverses $f^{-1}(x)$ and $g^{-1}(x)$ of $f(x) = \dfrac{7x-5}{4}$ and $g(x) = 2e^{5x}$.

Solution. Starting with $y = f(x) = \dfrac{7x-5}{4}$ and interchanging the roles of x and y yields:

$$x = \frac{7y-5}{4} \Rightarrow 4x = 7y - 5 \Rightarrow 4x + 5 = 7y \Rightarrow y = \frac{4x+5}{7}$$

so $f^{-1}(x) = \dfrac{4x+5}{7}$. Starting with $y = g(x) = 2e^{5x}$ and interchanging the roles of x and y yields:

$$x = 2e^{5y} \Rightarrow \frac{x}{2} = e^{5y} \Rightarrow \ln\left(\frac{x}{2}\right) = 5y \Rightarrow y = \frac{1}{5}\ln\left(\frac{x}{2}\right)$$

so $g^{-1}(x) = \dfrac{1}{5}\ln\left(\dfrac{x}{2}\right)$. ◀

Practice 3. Find formulas for the inverses $f^{-1}(x)$, $g^{-1}(x)$ and $h^{-1}(x)$ of $f(x) = 2x - 5$, $g(x) = \dfrac{2x-1}{x+7}$ and $h(x) = 2 + \ln(3x)$.

Sometimes it is easy to "solve $x = f(y)$ for y," but often it is not. When we try to find a formula for the inverse of $y = f(x) = x + e^x$, the first step is easy: interchanging the roles of x and y yields $x = y + e^y$. At this point, unfortunately, there is no way to algebraically solve the equation $x = y + e^y$ explicitly for y. The function $y = x + e^x$ has an inverse function, but we cannot find an explicit formula for that inverse.

Which Functions Have Inverse Functions?

We have seen how to find the inverse function for some functions given by tables of values, by graphs and by formulas, but there are functions that do not have inverse functions. The only way a graph and its reflection about the line $y = x$ can *both* be function graphs (so that f and f^{-1} are both functions) is if the graph of f passes both the Vertical Line Test (so that f is a function) and the Horizontal Line Test (so that the graph of f^{-1} passes the Vertical Line Test, verifying that f^{-1} is a function). This idea leads to the next theorem (stated without proof).

> **Theorem**: The function f has an inverse function if and only if f is one-to-one.

Two useful corollaries follow from this theorem.

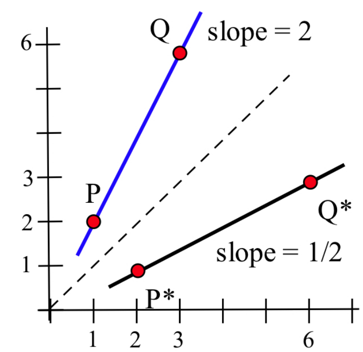
> **Corollary 1**: If f is strictly increasing or strictly decreasing, then f has an inverse function.

Corollary 2: If $f'(x) > 0$ for all x or $f'(x) < 0$ for all x, then f has an inverse function.

Applying this last Corollary to $f(x) = x + e^x$, we know that $f'(x) = 1 + e^x > 0$ for all x, so f must have an inverse (even though we can't find an explicit formula for f^{-1}).

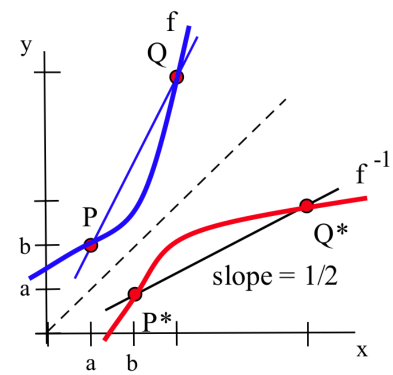
Slopes of Inverse Functions

When a function f has an inverse, the symmetry of the graphs of f and f^{-1} also provides us with information about slopes and derivatives.



Example 6. Suppose the points $P = (1, 2)$ and $Q = (3, 6)$ are on the graph of a function f. (a) Sketch the line passing through P and Q. (b) Compute the slope of that line. (c) Graph the reflected points P^* and Q^* on the graph of f^{-1}. (d) Sketch the line passing through P^* and Q^*. Find the slope of the line through P^* and Q^*.

Solution. (a) See margin figure. (b) The slope through P and Q is $m = \dfrac{6-2}{3-1} = \dfrac{4}{2} = 2$. (c) The reflected points, obtained by interchanging the first and second coordinates of each point on the graph of f, are $P^* = (2, 1)$ and $Q^* = (6, 3)$. (d) See margin figure. (e) The slope of the line though P^* and Q^* is $\dfrac{3-1}{6-2} = \dfrac{1}{2}$ ◀



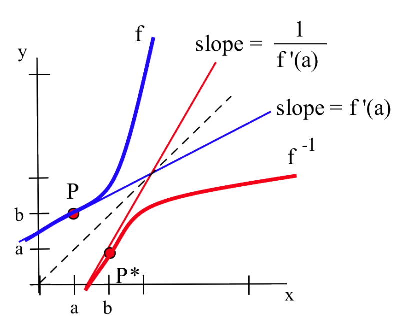
In the preceding Example, you may have noticed that:

$$\text{slope of line through } P^* \text{ and } Q^* = \frac{1}{\text{slope of line through } P \text{ and } Q}$$

This was not a coincidence. In general, if $P = (a, b)$ and $Q = (x, y)$ are points on the graph of f (see margin), then the reflected points $P^* = (b, a)$ and $Q^* = (y, x)$ are on the graph of f^{-1} and:

$$\text{slope of segment } P^*Q^* = \frac{1}{\text{slope of segment } PQ}$$

Because the slope of a tangent line is the limit of slopes of secant lines, a similar relationship holds between the slope of the tangent line to f at the point (a, b) and slope of the tangent line to f^{-1} at the point (b, a). If we let the point Q^* approach the point P^* along the graph of f^{-1} (as in the margin figure) then:



$$\left(f^{-1}\right)'(b) = \lim_{Q^* \to P^*} [\text{slope of segment } P^*Q^*]$$
$$= \lim_{Q \to P} \frac{1}{\text{slope of segment } PQ} = \frac{1}{f'(a)}$$

This geometric idea leads to the following result:

Derivative of an Inverse Function

If $b = f(a)$, f is differentiable at the point (a, b) and $f'(a) \neq 0$

then $a = f^{-1}(b)$, f^{-1} is differentiable at the point (b, a) and:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Example 7. The point $\left(e^2, 2\right)$ is on the graph of $f(x) = \ln(x)$ and $f'(x) = \frac{1}{x} \Rightarrow f'\left(e^2\right) = \frac{1}{e^2}$. Let g be the inverse function of f. Give one point on the graph of g and evaluate g' at that point.

Solution. The point $(2, e^2)$ is on the graph of g and:

$$g'(2) = \frac{1}{f'(e^2)} = \frac{1}{\frac{1}{e^2}} = e^2$$

In fact, the inverse of $f(x) = \ln(x)$ is the exponential function $g(x) = e^x$ and we can check that $g'(x) = e^x \Rightarrow g'(2) = e^2$. ◀

Example 8. In the table below, fill in the values of $f^{-1}(x)$ and $\left(f^{-1}\right)'(x)$ for $x = 0$ and $x = 1$.

Solution. $f(3) = 0$, so $f^{-1}(0) = 3$, while $\left(f^{-1}\right)'(0) = \frac{1}{f'(3)} = \frac{1}{2}$; $f(2) = 1$, so $f^{-1}(1) = 2$ while $\left(f^{-1}\right)'(1) = \frac{1}{f'(2)} = \frac{1}{-1} = -1$. ◀

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
0	2	3		
1	3	-2		
2	1	-1		
3	0	2		

Practice 4. Fill in the missing values for $x = 2$ and $x = 3$.

7.2 Problems

1. Given the values of f and f' in the table below, compute the specified values of f^{-1} and $\left(f^{-1}\right)'$.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
1	3	-3		
2	1	2		
3	2	3		

2. Given the values of g and g' in the table below, compute the specified values of g^{-1} and $\left(g^{-1}\right)'$.

x	$g(x)$	$g'(x)$	$g^{-1}(x)$	$\left(g^{-1}\right)'(x)$
1	2	-2		
2	1	4		
3	3	2		

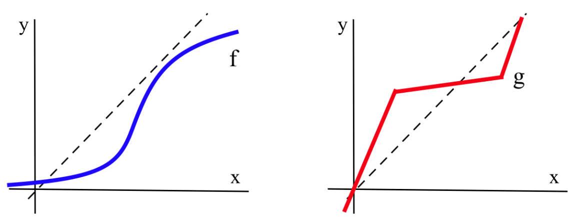
3. Given the values of h and h' in the table below, compute the specified values of h^{-1} and $\left(h^{-1}\right)'$.

x	$h(x)$	$h'(x)$	$h^{-1}(x)$	$\left(h^{-1}\right)'(x)$
1	2	2		
2	3	-2		
3	1	0		

4. Given the values of w and w' in the table below, compute the specified values of w^{-1} and $\left(w^{-1}\right)'$.

x	$w(x)$	$w'(x)$	$w^{-1}(x)$	$\left(w^{-1}\right)'(x)$
1	1	2		
2	3	0		
3	2	5		

5. The figure below left shows a graph of f. Sketch a graph of f^{-1}.



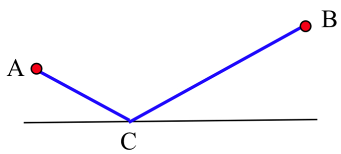
6. The figure above right shows a graph of g. Sketch a graph of g^{-1}.
7. If the graphs of f and f^{-1} intersect at the point (a,b), how are a and b related?
8. If the graph of f intersects the line $y = x$ at $x = a$, does the graph of f^{-1} intersect the line $y = x$? If so, where?
9. The steps to evaluate the function $f(x) = \dfrac{7x-5}{4}$ are (1) multiply by 7, (2) subtract 5 and (3) divide by 4. Write the steps, in words, for the inverse of this function, and then translate these verbal steps into a formula for the inverse function.
10. Find a formula for the inverse function of $f(x) = 3x - 2$. Verify that $f^{-1}(f(5)) = 5$ and $f\left(f^{-1}(2)\right) = 2$.
11. Find a formula for the inverse function of $g(x) = 2x + 1$. Verify that $g^{-1}(g(1)) = 1$ and $g\left(g^{-1}(7)\right) = 7$.
12. Find a formula for the inverse of $h(x) = 2e^{3x}$. Verify that $h^{-1}(h(0)) = 0$.
13. Find a formula for the inverse of $w(x) = 5 + \ln(x)$. Verify that $w^{-1}(w(1)) = 1$.
14. If the graph of f goes through the point $(2,5)$ and $f'(2) = 3$, then the graph of f^{-1} goes through the point (____, ____) and $\left(f^{-1}\right)'(5) =$ ____.
15. If the graph of f goes through the point $(1,3)$ and $f'(1) > 0$, then the graph of f^{-1} goes through the point (____, ____). What can you say about the value of $\left(f^{-1}\right)'(3)$?
16. If $f(6) = 2$ and $f'(6) < 0$, then the graph of f^{-1} goes through the point (____, ____). What can you say about $\left(f^{-1}\right)'(2)$?
17. If $f'(x) > 0$ for all values of x, what can you say about $\left(f^{-1}\right)'(x)$? What does this tell you about the graphs of f and f^{-1}?
18. If $f'(x) < 0$ for all values of x, what can you say about $\left(f^{-1}\right)'(x)$? What does this tell you about the graphs of f and f^{-1}?
19. Find a linear function $f(x) = ax + b$ so the graphs of f and f^{-1} are parallel and do not intersect.
20. Does $f(x) = 3 + \sin(x)$ have an inverse function? Justify your answer.
21. Does $f(x) = 3x + \sin(x)$ have an inverse function? Justify your answer.
22. For which positive integers n is $f(x) = x^n$ one-to-one? Justify your answer.
23. Some functions are their own inverses. For which four of these functions does $f^{-1}(x) = f(x)$?

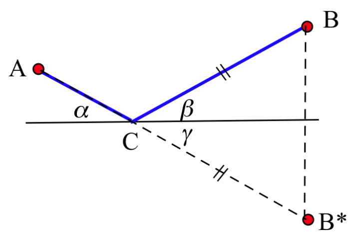
(a) $f(x) = \dfrac{1}{x}$ (b) $f(x) = \dfrac{x+1}{x-1}$

(c) $f(x) = \dfrac{3x-5}{7x-3}$ (d) $f(x) = \dfrac{ax+b}{cx-a}$

(e) $f(x) = x + a \ (a \neq 0)$

Reflections on Folding





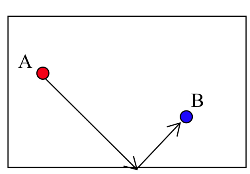
The symmetry of the graphs of a function and its inverse about the line $y = x$ make sketching a graph of the inverse function relatively easy if you have a graph of f: fold your graph paper along the line $y = x$ so the graph of f^{-1} is the "folded" image of f. This simple idea of obtaining a new image of something by folding along a line can enable us to quickly "see" solutions to some otherwise difficult problems.

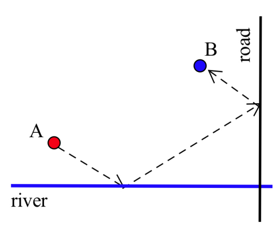
The minimization problem of finding the shortest path from town A to town B with an intermediate stop at a (straight) river, as depicted in the margin figure, is moderately difficult to solve using derivatives (see Problem 11 in Section 3.5). Geometry allows us to solve the problem much more easily (see second margin figure):

- obtain the point B^* by folding the image of B across the river line
- connect A and B^* with a line segment (the shortest path connecting A and B^*)
- fold the C-to-B^* segment back across the river to obtain the A-to-C-to-B solution.

As an almost-free bonus, we see that—for the minimum path—the angle of incidence at the river equals the angle of reflection.

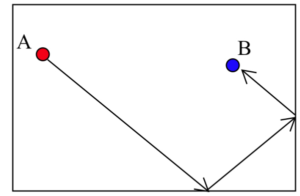
24. Devise an algorithm using "folding" to locate the point at the bottom edge of the billiards table you should aim ball A in the figure below left so that ball A will hit ball B after bouncing off the bottom edge of the table. (Assume that the angle of incidence equals the angle of reflection.)





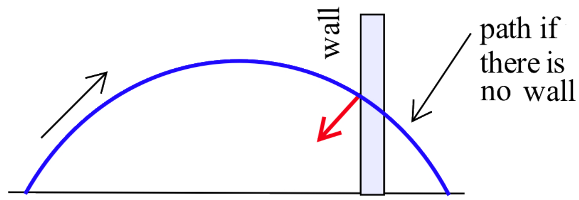
25. Devise an algorithm using "folding" to sketch the shortest path in the figure above right from town A to town B that includes a stop at the river *and* at the road. (One fold may not be enough.)

26. Devise an algorithm using "folding" to find where you should aim ball A at the bottom edge of the billiards table in the figure below so that ball A will hit ball B after bouncing off the bottom edge *and* the right edge of the table. (Assume that the angle of incidence equals the angle of reflection. Unfortunately, in a real-life game of billiards, the ball picks up a spin, called "English," when it bounces off the first bank, so at the second bounce the angle of incidence does not equal the angle of reflection.)



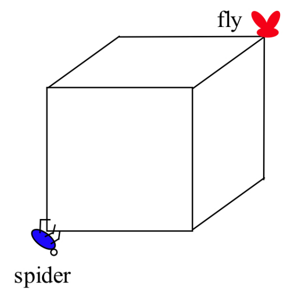
The "folding" idea can even be useful if the path is not a straight line.

27. The first figure below shows the parabolic path of a thrown ball. If the ball bounces off the tall vertical wall in the second margin figure, where will it hit the ground? (Assume that the angle of incidence equals the angle of reflection and that the ball does not lose energy during the bounce.)



Sometimes "unfolding" a problem is useful too.

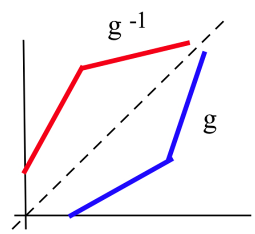
28. A spider and a fly are located at opposite corners of the cube as shown below. Sketch the shortest path the spider can travel along the surface of the cube to reach the fly.



7.2 Practice Answers

1. See table below left.

x	$g^{-1}(x)$
0	4
1	1
2	0
3	2
4	3



2. See figure above right. If a point lies on the line $y = x$, it must have the form (a, a) so that $g(a) = a \Rightarrow a = g^{-1}(a)$: the point (a, a) is also on the graph of g^{-1}.

3. (a) Swapping x and y in $y = 2x - 5$ yields $x = 2y - 5 \Rightarrow x + 5 = 2y \Rightarrow y = \dfrac{x+5}{2}$ so $f^{-1}(x) = \dfrac{x+5}{2}$.

(b) Swapping x and y in $y = \dfrac{2x-1}{x+7}$: $x = \dfrac{2y-1}{y+7} \Rightarrow xy + 7x = 2y - 1 \Rightarrow 7x + 1 = 2y - xy \Rightarrow y = \dfrac{7x+1}{2-x}$ so $g^{-1}(x) = \dfrac{7x+1}{2-x}$.

(c) Interchanging x and y in $y = 2 + \ln(3x)$ yields $x = 2 + \ln(3y) \Rightarrow x - 2 = \ln(3y) \Rightarrow e^{x-2} = 3y \Rightarrow y = \frac{1}{3}e^{x-2}$ so $h^{-1}(x) = \frac{1}{3}e^{x-2}$.

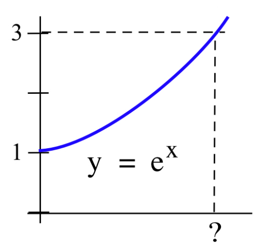
4. $f^{-1}(2) = 0$ and $f^{-1}(3) = 1$ while $\left(f^{-1}\right)'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$ and $\left(f^{-1}\right)'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{-2}$.

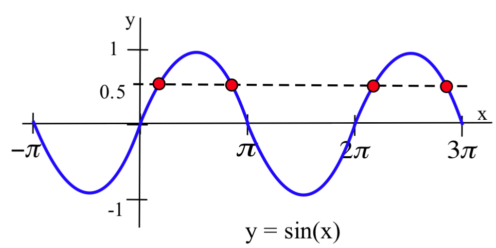
7.3 *Inverse Trigonometric Functions*

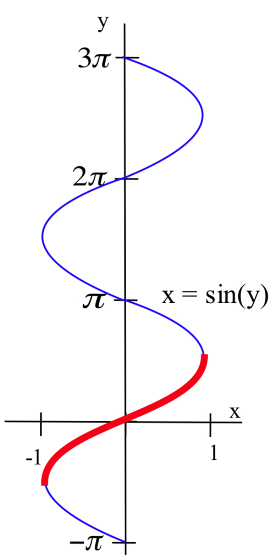
We now turn our attention to the inverse trigonometric functions, their properties and their graphs, focusing on properties and techniques needed to investigate derivatives and integrals of these functions. We will concentrate on the inverse sine and inverse tangent functions, the two inverse trigonometric functions that arise most often in calculus.

Inverse Sine: Solving $k = \sin(x)$ for x

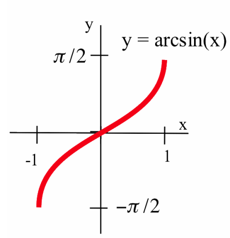
It is straightforward to solve the equation $3 = e^x$ (see margin figure below left): simply apply the natural logarithm function, the inverse of the exponential function e^x, to each side of the equation to get $\ln(3) = \ln(e^x) = x$. Because the function $f(x) = e^x$ is one-to-one, the equation $3 = e^x$ has only the one solution $x = \ln(3) \approx 1.1$.

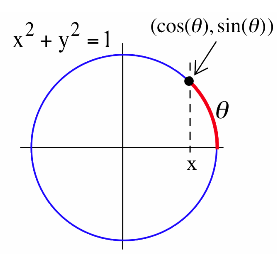


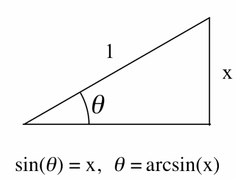




The solution of the equation $0.5 = \sin(x)$ presents more difficulties. As the figure above right illustrates, the function $f(x) = \sin(x)$ is *not* one-to-one: its graph reflected about the line $y = x$ (see margin figure) is *not* the graph of a function. Sometimes, however, it is necessary to "undo" the sine function, and we can do so by restricting its domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the function $f(x) = \sin(x)$ is one-to-one and has an inverse function—and the graph of the inverse function (see below left) is the reflection about the line $y = x$ of the (restricted) graph of $y = \sin(x)$.







We call this inverse of the (restricted) sine function the **arcsine** and denote it $\arcsin(x)$. The name "arcsine" comes from the unit-circle definition of the sine function. On the unit circle (above center), if θ is the length of the arc whose sine is x, then $\sin(\theta) = x$ and $\theta = \arcsin(x)$. Using the right-triangle definition of sine (above right), θ represents an angle whose sine is x.

Many textbooks—and most calculators—use the notation $\sin^{-1}(x)$ for $\arcsin(x)$. You must be very careful to **never** interpret $\sin^{-1}(x)$ to mean:

$$(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x)$$

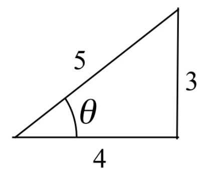
We avoid the $\sin^{-1}(x)$ notation for this reason and suggest that you do as well.

> **Definition of Inverse Sine**
> For $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$:
> $$y = \arcsin(x) \iff x = \sin(y)$$
> The domain of $\arcsin(x)$ is $[-1, 1]$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The (restricted) sine function and the arcsine are inverses of each other:

$$\begin{aligned} -1 \le x \le 1 &\Rightarrow \sin(\arcsin(x)) = x \\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} &\Rightarrow \arcsin(\sin(y)) = y \end{aligned}$$

Right Triangles and Arcsine



$\sin(\theta) = \dfrac{3}{5}$

$\theta = \arcsin\left(\dfrac{3}{5}\right)$

$\theta \approx 0.6435$

For the right triangle shown in the margin, $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$ so $\theta = \arcsin\left(\frac{3}{5}\right)$. It is possible to evaluate other trigonometric functions (such as cosine and tangent) of an angle expressed as an arcsine without explicitly solving for the value of the angle. For example:

$$\begin{aligned} \cos\left(\arcsin\left(\frac{3}{5}\right)\right) &= \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \\ \tan\left(\arcsin\left(\frac{3}{5}\right)\right) &= \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} \end{aligned}$$

Once you know the sides of the right triangle, you can compute the values of the other trigonometric functions using their standard right-triangle definitions:

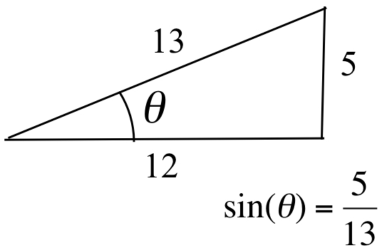
$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$$

If you are given an angle θ as the arcsine of a number, but not given the sides of a right triangle, you can construct your own triangle with the given angle: select values for the opposite side and hypotenuse so the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$ is the value whose arcsine we want: $\arcsin\left(\dfrac{\text{opposite}}{\text{hypotenuse}}\right)$. You can calculate the length of the third ("adjacent") side using the Pythagorean Theorem.

Example 1. Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin\left(\frac{5}{13}\right)$. Use the triangle to determine the values of $\tan\left(\arcsin\left(\frac{5}{13}\right)\right)$ and $\csc\left(\arcsin\left(\frac{5}{13}\right)\right)$.

Solution. We want the sine of θ, the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$, to be $\frac{5}{13}$ so we can choose the opposite side to be 5 and the hypotenuse to be 13 (see



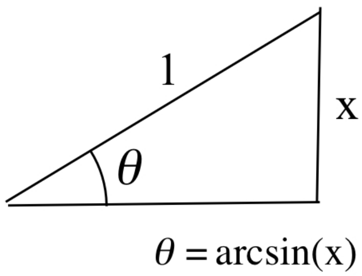
margin figure). Then $\sin(\theta) = \frac{5}{13}$, as desired. Using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{13^2 - 5^2} = 12$. So:

$$\tan(\theta) = \tan\left(\arcsin\left(\frac{5}{13}\right)\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$

$$\csc(\theta) = \csc\left(\arcsin\left(\frac{5}{13}\right)\right) = \frac{1}{\sin\left(\arcsin\left(\frac{5}{13}\right)\right)} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$$

Any choice of values for the opposite side and the hypotenuse will work (for example opposite = 500 and hypotenuse = 1300), as long as the ratio of the opposite side to the hypotenuse is $\frac{5}{13}$. ◀

Practice 1. Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin\left(\frac{6}{11}\right)$. Use the triangle to determine the values of $\tan\left(\arcsin\left(\frac{6}{11}\right)\right)$, $\csc\left(\arcsin\left(\frac{6}{11}\right)\right)$ and $\cos\left(\arcsin\left(\frac{6}{11}\right)\right)$.



Example 2. Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(x)$. Use the triangle to determine the values of $\tan(\arcsin(x))$ and $\cos(\arcsin(x))$.

Solution. We want the sine of θ, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, to be x so we can choose the opposite side to be x and the hypotenuse to be 1 (see margin figure). Then $\sin(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{1-x^2}$ so that:

$$\tan(\arcsin(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$$

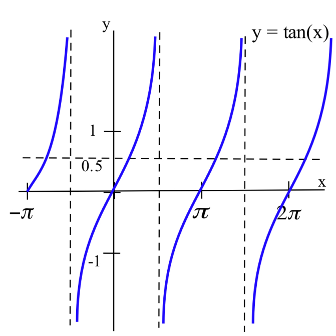
$$\cos(\arcsin(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

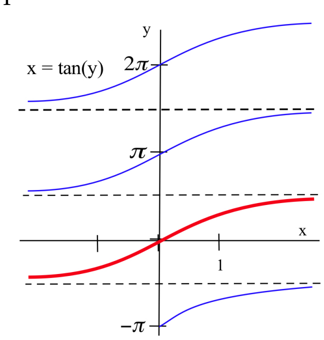
Other choices for the lengths of the opposite side and hypotenuse, such as $3x$ and 3, will work, but x and 1 are the simplest choices. ◀

Practice 2. Evaluate $\sec(\arcsin(x))$ and $\csc(\arcsin(x))$.

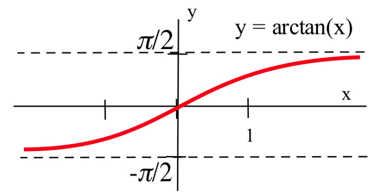
Inverse Tangent: Solving $k = \tan(x)$ for x

The equation $0.5 = \tan(x)$ (see below left) has many solutions: the function $f(x) = \tan(x)$ is not one-to-one, and its graph reflected across the line $y = x$ (below right) is not the graph of a function.





If, however, we restrict the domain of the tangent function to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the restricted $f(x) = \tan(x)$ *is* one-to-one and has an inverse function. The graph of this inverse tangent function (see margin figure) is the reflection about the line $y = x$ of the (restricted) graph of $y = \tan(x)$. We call this inverse of the (restricted) tangent function the **arctangent** and denote it $\arctan(x)$. For $x > 0$, the number $\arctan(x)$ is the length of the arc on the unit circle whose tangent is x, and $\arctan(x)$ is the angle whose tangent is x: $\tan\left(\arctan(x)\right) = x$.



Many textbooks — and most calculators — use the notation $\tan^{-1}(x)$ for $\arctan(x)$. You must be very careful to **never** interpret $\tan^{-1}(x)$ to mean:

$$(\tan(x))^{-1} = \frac{1}{\tan(x)} = \cot(x)$$

We avoid the $\tan^{-1}(x)$ notation for this reason and suggest that you do as well.

Definition of Inverse Tangent
For all x and $-\frac{\pi}{2} < y < \frac{\pi}{2}$:

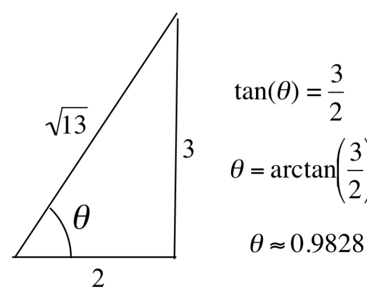
$$y = \arctan(x) \iff x = \tan(y)$$

The domain of $\arctan(x)$ is $(-\infty, \infty)$ and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The (restricted) tangent function and the arctangent are inverses:

$$-\infty < x < \infty \Rightarrow \tan\left(\arctan(x)\right) = x$$
$$-\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow \arctan\left(\tan(y)\right) = y$$

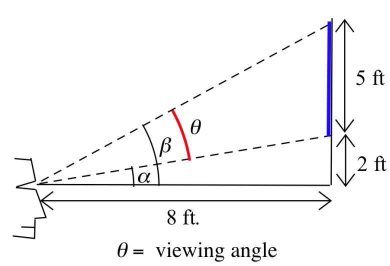
Right Triangles and Arctangent



For the right triangle in the margin, $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$ so that $\theta = \arctan\left(\frac{3}{2}\right)$, hence:

$$\sin\left(\arctan\left(\frac{3}{2}\right)\right) = \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{\sqrt{13}} \approx 0.832$$
$$\cot\left(\arctan\left(\frac{3}{2}\right)\right) = \frac{1}{\tan\left(\arctan\left(\frac{3}{2}\right)\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3} \approx 0.667$$

Practice 3. Determine the lengths of the sides of a right triangle so that one angle is $\theta = \arctan\left(\frac{3}{4}\right)$, then use the triangle to determine the values of $\sin\left(\arctan\left(\frac{3}{4}\right)\right)$, $\cot\left(\arctan\left(\frac{3}{4}\right)\right)$ and $\cos\left(\arctan\left(\frac{3}{4}\right)\right)$.



Example 3. On a wall 8 feet in front of you, the lower edge of a 5-foot-tall painting rests 2 feet above your eye level (see margin). Represent your viewing angle θ using arctangents.

Solution. The viewing angle α to the bottom of the painting satisfies:

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{8} \Rightarrow \alpha = \arctan\left(\frac{1}{4}\right)$$

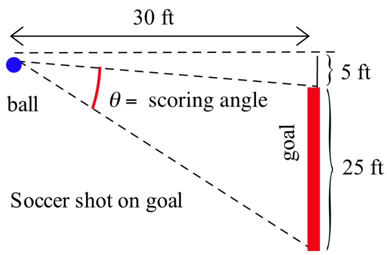
Similarly, the angle β to the top of the painting satisfies:

$$\tan(\beta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{8} \Rightarrow \alpha = \arctan\left(\frac{7}{8}\right)$$

The viewing angle θ for the painting is therefore:

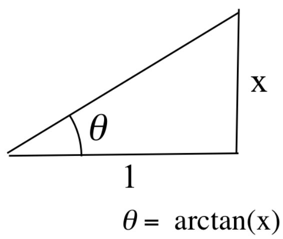
$$\theta = \beta - \alpha = \arctan\left(\frac{7}{8}\right) - \arctan\left(\frac{1}{4}\right) \approx 0.719 - 0.245 = 0.474$$

or about 27°. ◀



Practice 4. Determine the scoring angle for the soccer player in the margin figure.

Example 4. Determine the lengths of the sides of a right triangle so that one angle is $\theta = \arctan(x)$, then use the triangle to determine the values of $\sin(\arctan(x))$ and $\cos(\arctan(x))$.



Solution. We want the tangent of θ (the ratio of opposite to adjacent) to be x, so we can choose the opposite side to be x and the adjacent side to be 1 (see margin). Then $\tan(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1+x^2}$ so that:

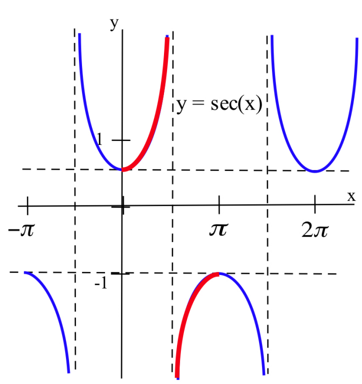
$$\sin(\arctan(x)) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}}$$
$$\cos(\arctan(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}$$

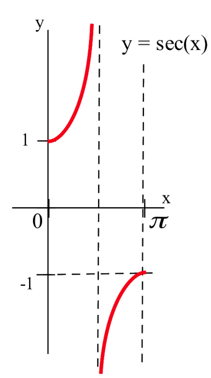
We could have chosen other values for the opposite and adjacent sides (such as x^2 and x), but x and 1 provide the simplest option. ◀

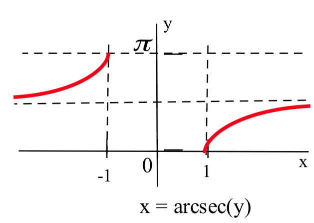
Practice 5. Evaluate $\sec(\arctan(x))$ and $\cot(\arctan(x))$.

Inverse Secant: Solving $k = \sec(x)$ for x

The equation $2 = \sec(x)$ (see figure below left) has many solutions, but we can create an inverse function for secant—much the same way we did for sine and tangent—by suitably restricting the domain of the secant function so that it becomes a one-to-one function:







The figure above center shows the restriction $0 \le x \le \pi$ ($x \ne \frac{\pi}{2}$), which results in a one-to-one function that has an inverse. The graph of the inverse function (above right) is the reflection about the line $y = x$ of the (restricted) graph of $y = \sec(x)$. We call this inverse of the (restricted) secant function the **arcsecant** and denote it $\text{arcsec}(x)$.

There are alternate ways to restrict the secant function to get a one-to-one function, and they lead to slightly different definitions of the inverse secant. We chose to use this restriction because it seems more "natural" than the alternatives, it is easier to evaluate on a calculator, and it is the most commonly used.

Definition of Inverse Secant
If $|x| \geq 1$ and $0 \leq y \leq \pi$ with $y \neq \frac{\pi}{2}$:

$$y = \text{arcsec}(x) \iff x = \sec(y)$$

The domain of $\text{arcsec}(x)$ is $(-\infty, -1] \cup [1, \infty)$ and the range of $\text{arcsec}(x)$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Many textbooks use the notation $\sec^{-1}(x)$ for $\text{arcsec}(x)$. You must be very careful to **never** interpret $\sec^{-1}(x)$ to mean:

$$(\sec(x))^{-1} = \frac{1}{\sec(x)} = \cos(x)$$

We avoid the $\sec^{-1}(x)$ notation for this reason and suggest that you do as well.

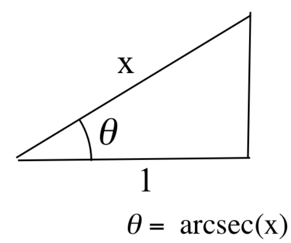
The (restricted) secant function and the arcsecant are inverses:

$$|x| \geq 1 \Rightarrow \sec(\text{arcsec}(x)) = x$$
$$0 \leq y \leq \pi \left(y \neq \frac{\pi}{2}\right) \Rightarrow \text{arcsec}(\sec(y)) = y$$

Example 5. Evaluate $\tan(\text{arcsec}(x))$.

Solution. We want the secant of θ (the ratio of hypotenuse to adjacent) to be x, so we can choose the hypotenuse to be x and the adjacent side to be 1 (see margin). Then $\sec(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the opposite side is $\sqrt{x^2 - 1}$, so:

$$\tan(\text{arcsec}(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$



As usual, x and 1 are the simplest—but not the only—choices. ◀

Practice 6. Evaluate $\sin(\text{arcsec}(x))$ and $\cot(\text{arcsec}(x))$.

The Other Inverse Trigonometric Functions

The inverse tangent and inverse sine functions are by far the most commonly used of the six inverse trigonometric functions in calculus. The inverse secant function turns up less often. The other three inverse trigonometric functions ($\text{arccos}(x)$, $\text{arccot}(x)$ and $\text{arccsc}(x)$) can be defined as the inverses of restricted versions of $\cos(x)$, $\cot(x)$ and $\csc(x)$, respectively, but these functions are almost dispensable in calculus.

The reasons for this will become apparent in the next section.

Calculators and Inverse Trigonometric Functions

Most calculators only have keys for $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$, but the following identities allow you to compute values of the other inverse trigonometric functions.

If $x \neq 0$ and x is in the appropriate domain
then $\text{arccot}(x) = \arctan\left(\frac{1}{x}\right)$, $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Proof. If $x \neq 0$, then:

$$\tan(\operatorname{arccot}(x)) = \frac{1}{\cot(\operatorname{arccot}(x))} = \frac{1}{x}$$

Applying the arctangent function to each side of this equation:

$$\arctan(\tan(\operatorname{arccot}(x))) = \arctan\left(\frac{1}{x}\right) \Rightarrow \operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Proofs of the other two identities are left to you. □

If	$x \neq 0$ and x is in the appropriate domain
then	$\arcsin(x) + \arccos(x) = \frac{\pi}{2}$, $\arctan(x) + \operatorname{arccot}(x) = \frac{\pi}{2}$ and $\operatorname{arcsec}(x) + \operatorname{arccsc}(x) = \frac{\pi}{2}$.

Proof. If α and β are complementary angles in a right triangle, so that $\alpha + \beta = \frac{\pi}{2}$, then $\sin(\alpha) = \cos(\beta)$. Let $x = \sin(\alpha) = \cos(\beta)$ so that $\alpha = \arcsin(x)$ and $\beta = \arccos(x)$, hence:

$$\alpha + \beta = \arcsin(x) + \arccos(x) = \frac{\pi}{2}$$

This proves the first result for $0 < x < 1$. You can easily check that the result also holds for $x = 0$, $x = 1$ and $x = -1$. To check that it holds for $-1 < x < 0$, we need the the next set of identities listed below. Proofs of the other two identities above are left to you. □

If	x is in the appropriate domain
then	$\arcsin(-x) = -\arcsin(x)$, $\arccos(-x) = \pi - \arccos(x)$, $\arctan(x) = -\arctan(x)$, $\operatorname{arcsec}(x) = \pi - \operatorname{arcsec}(x)$, $\operatorname{arccsc}(-x) = -\operatorname{arccsc}(x)$ and $\operatorname{arccot}(-x) = -\operatorname{arccot}(x)$.

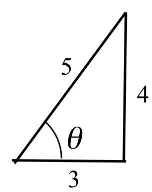
Proof. If $-1 \leq x \leq 1$, let $\theta = \arcsin(-x)$ so that $\sin(\theta) = -x \Rightarrow x = -\sin(\theta) = \sin(-\theta) \Rightarrow \arcsin(x) = -\theta \Rightarrow -\arcsin(x) = \theta = \arcsin(-x)$. This proves the first identity; the others are left to you. □

Some programming languages only include a *single* inverse trigonometric function, $\arctan(x)$, but it suffices to enable you to evaluate the other five inverse trigonometric functions:

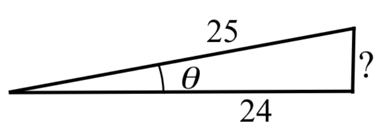
- $\arcsin(x) = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$
- $\arccos(x) = \dfrac{\pi}{2} - \arcsin(x) = \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$
- $\operatorname{arccot}(x) = \arctan\left(\dfrac{1}{x}\right)$
- $\operatorname{arcsec}(x) = \arctan\left(\sqrt{x^2-1}\right)$
- $\operatorname{arccsc}(x) = \dfrac{\pi}{2} - \operatorname{arcsec}(x) = \dfrac{\pi}{2} - \arctan\left(\sqrt{x^2-1}\right)$

7.3 Problems

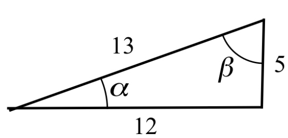
1. (a) List the three smallest positive angles θ that are solutions of the equation $\sin(\theta) = 1$.
   (b) Evaluate $\arcsin(1)$ and $\text{arccsc}(1)$.
2. (a) List the three smallest positive angles θ that are solutions of the equation $\tan(\theta) = 1$.
   (b) Evaluate $\arctan(1)$ and $\text{arccot}(1)$.
3. Find all x between 1 and 7 so that: (a) $\sin(x) = 0.3$ (b) $\sin(x) = -0.4$ (c) $\sin(x) = 0.5$
4. Find all values of x between 1 and 7 so that: (a) $\sin(x) = 0.3$ (b) $\sin(x) = -0.4$
5. Find all values of x between 2 and 7 so that: (a) $\tan(x) = 3.2$ (b) $\tan(x) = -0.2$
6. Find all values of x between 1 and 5 so that: (a) $\tan(x) = 8$ (b) $\tan(x) = -3$
7. In the figure below, angle θ is (a) the arcsine of what number? (b) the arctangent of what number? (c) the arcsecant of what number? (d) the arccosine of what number?



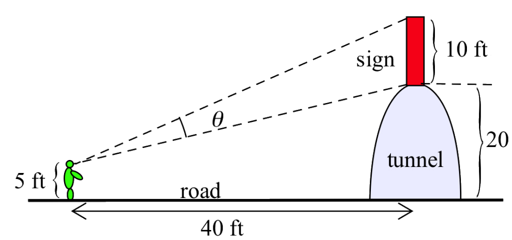
8. In the figure below, angle θ is (a) the arcsine of what number? (b) the arctangent of what number? (c) the arcsecant of what number? (d) the arccosine of what number?



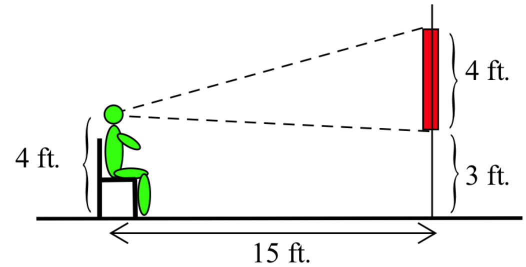
9. For the angle α in the triangle below, evaluate: (a) $\sin(\alpha)$ (b) $\tan(\alpha)$ (c) $\sec(\alpha)$ (d) $\cos(\alpha)$



10. For the angle β in the triangle above, evaluate: (a) $\sin(\beta)$ (b) $\tan(\beta)$ (c) $\sec(\beta)$ (d) $\cos(\beta)$
11. For $\theta = \arcsin\left(\frac{2}{7}\right)$, find the exact values of: (a) $\tan(\theta)$ (b) $\cos(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
12. For $\theta = \arctan\left(\frac{9}{2}\right)$, find the exact values of: (a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
13. For $\theta = \arccos\left(\frac{1}{5}\right)$, find the exact values of: (a) $\tan(\theta)$ (b) $\sin(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
14. For $\theta = \arcsin\left(\frac{a}{b}\right)$ with $0 < a < b$, find the exact values of: (a) $\tan(\theta)$ (b) $\cos(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
15. For $\theta = \arctan\left(\frac{a}{b}\right)$ with $0 < a < b$, find the exact values of: (a) $\tan(\theta)$ (b) $\sin(\theta)$ (c) $\cos(\theta)$ (d) $\cot(\theta)$
16. For $\theta = \arctan(x)$, find the exact values of: (a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\sec(\theta)$ (d) $\cot(\theta)$
17. Find the exact values of (a) $\sin(\arccos(x))$ (b) $\cos(\arcsin(x))$ (c) $\sec(\arccos(x))$
18. Find the exact values of (a) $\tan(\arccos(x))$ (b) $\cos(\arctan(x))$ (c) $\sec(\arcsin(x))$
19. (a) Does $\arcsin(1) + \arcsin(1) = \arcsin(2)$?
    (b) Does $\arccos(1) + \arccos(1) = \arccos(2)$?
20. (a) What is the viewing angle for the tunnel sign in the figure below?
    (b) Use arctangents to describe the viewing angle when the observer is x feet from the entrance of the tunnel.



21. (a) What is the viewing angle for the whiteboard in the figure below?
    (b) Use arctangents to describe the viewing angle when the student is x feet from the wall.



22. Graph $y = \arcsin(2x)$ and $y = \arctan(2x)$.

23. Graph $y = \arcsin\left(\frac{x}{2}\right)$ and $y = \arctan\left(\frac{x}{2}\right)$.

24. Which curve is longer, $y = \sin(x)$ from $x = 0$ to $x = \pi$, or $y = \arcsin(x)$ from $x = -1$ to $x = 1$?

For Problems 25–28, $\left.\frac{d\theta}{dt}\right|_{\theta=1.3} = 12$, and θ and h are related by the given formula. Find $\left.\frac{dh}{dt}\right|_{\theta=1.3}$.

25. $\sin(\theta) = \frac{h}{20}$

26. $\tan(\theta) = \frac{h}{50}$

27. $\cos(\theta) = 3h + 20$

28. $3 + \tan(\theta) = 7h$

For Problems 29–32, $\left.\frac{dh}{dt}\right|_{\theta=1.3} = 4$, and θ and h are related by the given formula. Find $\left.\frac{d\theta}{dt}\right|_{\theta=1.3}$.

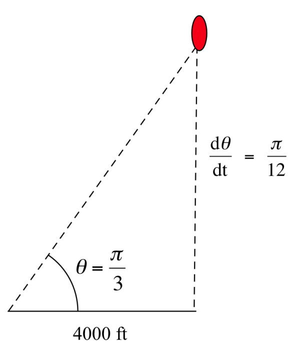
29. $\sin(\theta) = \frac{h}{38}$

30. $\tan(\theta) = \frac{h}{40}$

31. $\cos(\theta) = 7h - 23$

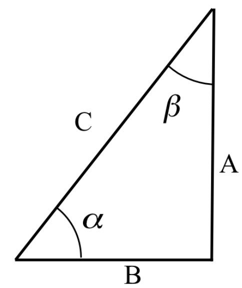
32. $\tan(\theta) = h^2$

33. You are observing a rocket launch from a position located 4000 feet from the launch pad (see below). When your observation angle of the rocket is $\frac{\pi}{3}$, the angle is increasing at $\frac{\pi}{12}$ feet per second. How fast is the rocket traveling?



34. You are observing a rocket launch from a position 3000 feet from the launch pad. NASA's Twitter feed reports that when the rocket is 5000 feet high, its velocity is 100 feet per second.
    (a) What is the angle of elevation of the rocket when it is 5000 feet above the launch pad?
    (b) How fast is the angle of elevation increasing when the rocket is 5000 feet high?

35. Refer to the right triangle shown below.
    (a) Angle α is arcsine of what number?
    (b) Angle β is arccosine of what number?
    (c) For positive numbers A and C, evaluate $\arcsin\left(\frac{A}{C}\right) + \arccos\left(\frac{A}{C}\right)$.

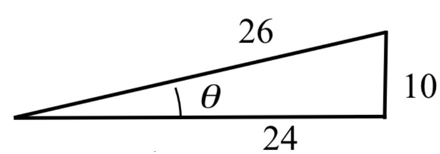


36. Refer to the right triangle shown above.
    (a) Angle α is arctangent of what number?
    (b) Angle β is arccotangent of what number?
    (c) For positive numbers A and B, evaluate $\arctan\left(\frac{A}{B}\right) + \text{arccot}\left(\frac{A}{B}\right)$.

37. Refer to the triangle from Problems 35–36.
    (a) Angle α is arcsecant of what number?
    (b) Angle β is arccosecant of what number?
    (c) For positive numbers B and C, evaluate $\text{arcsec}\left(\frac{C}{B}\right) + \text{arccsc}\left(\frac{C}{B}\right)$.

38. Describe the pattern apparent in your results from the previous three problems.

39. Refer to the right triangle shown below.
    (a) Angle θ is arctangent of what number?
    (b) Angle θ is arccotangent of what number?



40. Refer to the right triangle shown above.
    (a) Angle θ is arcsine of what number?
    (b) Angle θ is arccosecant of what number?

41. Refer to the triangle from Problems 39–40.
    (a) Angle θ is arccosine of what number?
    (b) Angle θ is arcsecant of what number?

42. Describe the pattern apparent in your results from the previous three problems.

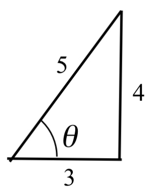
In 43–51, use a calculator (as necessary) and appropriate identities to compute the given values.

43. $\text{arcsec}(3)$ 44. $\text{arcsec}(-2)$ 45. $\text{arcsec}(-1)$

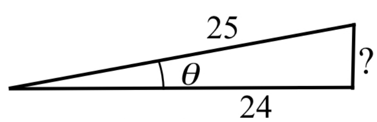
46. $\arccos(0.5)$ 47. $\arccos(-0.5)$ 48. $\arccos(1)$

49. $\text{arccot}(1)$ 50. $\text{arccot}(0.5)$ 51. $\text{arccot}(-3)$

52. For the triangle shown below:
   (a) $\theta = \arctan\left(\underline{\quad}\right)$
   (b) $\theta = \text{arccot}\left(\underline{\quad}\right)$
   (c) $\text{arccot}\left(\underline{\quad}\right) = \arctan\left(\underline{\quad}\right)$



53. For the triangle shown below:
   (a) $\theta = \arcsin\left(\underline{\quad}\right)$
   (b) $\theta = \arccos\left(\underline{\quad}\right)$
   (c) $\arccos\left(\underline{\quad}\right) = \arcsin\left(\underline{\quad}\right)$



54. Prove that $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$.

55. Prove that $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Using a right triangle you can show that:

$$\tan\left(\arcsin(x)\right) = \frac{x}{\sqrt{1-x^2}}$$
$$\Rightarrow \arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Imitate this reasoning in Problems 56–57.

56. Evaluate $\tan\left(\text{arccot}(x)\right)$ and use the result to find a formula for $\text{arccot}(x)$ in terms of arctangent.

57. Evaluate $\tan\left(\text{arcsec}(x)\right)$ and use the result to find a formula for $\text{arcsec}(x)$ in terms of arctangent.

58. Let $a = \arctan(x)$ and $b = \arctan(y)$. Use the identity:

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\cdot\tan(b)}$$

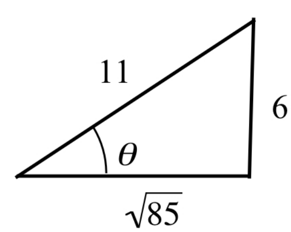
to show that:

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

7.3 Practice Answers

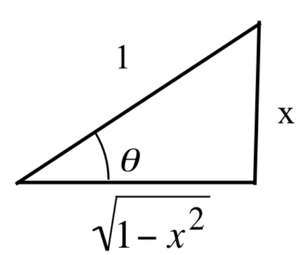
1. See the margin figure; with hypotenuse 11 and opposite side 6, the adjacent side must have length $\sqrt{11^2 - 6^2} = \sqrt{85}$, so:

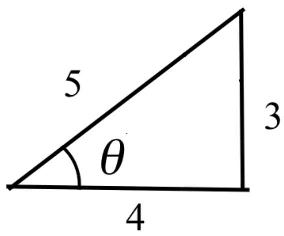
$$\tan\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{6}{\sqrt{85}}$$
$$\csc\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{11}{6}$$
$$\cos\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{\sqrt{85}}{11}$$



2. See the margin figure; with hypotenuse 1 and opposite side x, the adjacent side must have length $\sqrt{1^2 - x^2} = \sqrt{1-x^2}$, so:

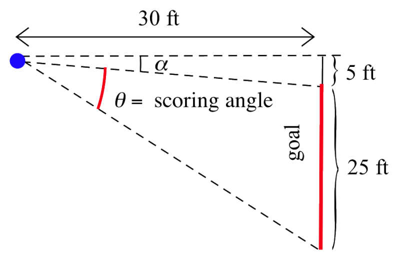
$$\sec\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\csc\left(\arcsin(x)\right) = \frac{1}{x}$$





3. See the margin figure; with opposite side 3 and adjacent side 4, the hypotenuse must have length $\sqrt{3^2+4^2} = \sqrt{25} = 5$, so:

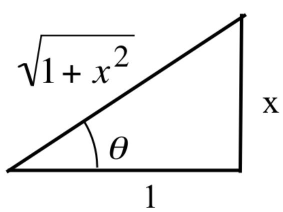
$$\sin\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{3}{5}$$
$$\cot\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{4}{3}$$
$$\cos\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{4}{5}$$



4. See margin figure; $\tan(\alpha) = \frac{5}{30} = \frac{1}{6}$ so $\alpha = \arctan\left(\frac{1}{6}\right) \approx 0.165$ (or about 9.46°). Likewise, $\tan(\alpha+\theta) = \frac{30}{30} = 1$ so $\alpha+\theta = \arctan(1) = \frac{\pi}{4} \approx 0.785$ (or 45°). Finally:

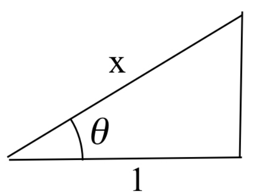
$$\theta = (\alpha+\theta) - \alpha \approx 0.785 - 0.165 = 0.62$$

or about 35.54°.



5. See the margin figure; with opposite side x and adjacent side 1, the hypotenuse must have length $\sqrt{1^2+x^2} = \sqrt{1+x^2}$, so:

$$\sec(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$
$$\cot(\arctan(x)) = \frac{1}{x}$$



θ = arcsec(x)

6. See the margin figure; with hypotenuse x and adjacent side 1, the opposite side must have length $\sqrt{x^2-1^2} = \sqrt{x^2-1}$, so:

$$\sin(\operatorname{arcsec}(x)) = \frac{\sqrt{x^2-1}}{x}$$
$$\cot(\operatorname{arcsec}(x)) = \frac{1}{\sqrt{x^2-1}}$$

7.4 *Derivatives of Inverse Trigonometric Functions*

The three previous sections introduced the ideas of one-to-one functions and inverse functions, then used those concepts to define arcsine, arctangent and the other inverse trigonometric functions. In this section, we obtain derivative formulas for the inverse trigonometric functions and consider an important application and some of its variations.

Derivative Formulas for Inverse Trigonometric Functions

$$\mathbf{D}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1) \qquad \mathbf{D}\left(\arccos(x)\right) = \frac{-1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1)$$

$$\mathbf{D}\left(\arctan(x)\right) = \frac{1}{1+x^2} \quad (\text{for all } x) \qquad \mathbf{D}\left(\operatorname{arccot}(x)\right) = \frac{-1}{1+x^2} \quad (\text{for all } x)$$

$$\mathbf{D}\left(\operatorname{arcsec}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x| > 1) \qquad \mathbf{D}\left(\operatorname{arccsc}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x| > 1)$$

Proof. If $y = \arctan(x)$ then $\tan(y) = x$, so differentiating with respect to x and applying the Chain Rule yields:

$$\sec^2(y) \cdot \frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

Recalling the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, we can write:

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

Notice that the derivative of $\arctan(x)$ is defined for all x, which corresponds to the domain of $\arctan(x)$.

proving the second formula listed above. To prove the first formula, put $y = \arcsin(x)$ so that $\sin(y) = x$. Implicitly differentiating with respect to x yields:

$$\cos(y) \cdot \frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Solving the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\cos(\theta)$:

$$\cos^2(\theta) = 1 - \sin^2(\theta) \;\Rightarrow\; \cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}$$

The range of $y = \arcsin(x)$ is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, so $\cos(y) \ge 0$ and:

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

Notice that the derivative of $\arcsin(x)$ is defined only when $1 - x^2 > 0$ or, equivalently, if $|x| < 1$, corresponding to the domain of $\arcsin(x)$ (omitting endpoints).

For the third formula, if $y = \operatorname{arcsec}(x)$ then $\sec(y) = x$, so differentiation yields:

$$\sec(y) \cdot \tan(y) \cdot \frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\sec(y) \cdot \tan(y)}$$

Solving the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ for $\tan(\theta)$:

$$\tan^2(\theta) = \sec^2(\theta) - 1 \;\Rightarrow\; \tan(\theta) = \pm\sqrt{\sec^2(\theta) - 1}$$

If $x \geq 1$ then $0 \leq y < \dfrac{\pi}{2}$ and on this interval $\tan(y) \geq 0$, so:

$$\frac{dy}{dx} = \frac{1}{\sec(y) \cdot \tan(y)} = \frac{1}{\sec(y) \cdot \sqrt{\sec^2(y) - 1}} = \frac{1}{x \cdot \sqrt{x^2 - 1}}$$

If $x \leq -1$ then $\dfrac{\pi}{2} < y \leq \pi$ and on this interval $\tan(y) \leq 0$ so:

$$\frac{dy}{dx} = \frac{1}{\sec(y) \cdot \tan(y)} = \frac{1}{\sec(y) \cdot \left[-\sqrt{\sec^2(y) - 1}\right]} = \frac{1}{-x \cdot \sqrt{x^2 - 1}}$$

Recognizing that $x = |x|$ when $x \geq 1$ and that $-x = |x|$ when $x \leq -1$, we can combine these two cases into a single compact form:

Notice that the derivative of $\text{arcsec}(x)$ is defined only when $x^2 - 1 > 0$ or, equivalently, if $|x| > 1$, corresponding to the domain of $\text{arcsec}(x)$ (omitting endpoints).

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Proofs of the the remaining three formulas are nearly identical to those given above and are left for you as Problems 24–26. □

Example 1. Compute $\mathbf{D}\left(\arcsin\left(e^x\right)\right)$, $\mathbf{D}\left(\arctan\left(x-3\right)\right)$, $\mathbf{D}\left(\arctan^3\left(5x\right)\right)$ and $\mathbf{D}\left(\ln\left(\arcsin(x)\right)\right)$.

Solution. Each of these functions involves a composition of an inverse trigonometric function with other functions, so we need to use the Chain Rule:

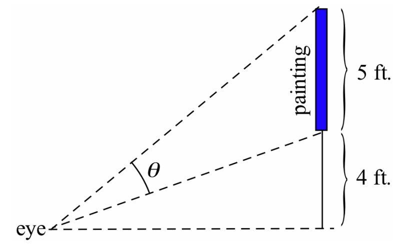
$$\begin{aligned}
\mathbf{D}\left(\arcsin\left(e^x\right)\right) &= \frac{1}{\sqrt{1 - \left(e^x\right)^2}} \cdot \mathbf{D}\left(e^x\right) = \frac{e^x}{\sqrt{1 - e^{2x}}} \\
\mathbf{D}\left(\arctan\left(x-3\right)\right) &= \frac{1}{1 + (x-3)^2} \cdot \mathbf{D}\left(x - 3\right) \\
&= \frac{1}{1 + (x-3)^2} = \frac{1}{x^2 - 6x + 10} \\
\mathbf{D}\left(\arctan^3\left(5x\right)\right) &= 3\arctan^2\left(5x\right) \cdot \mathbf{D}\left(\arctan(5x)\right) \\
&= 3\arctan^2\left(5x\right) \cdot \frac{1}{1 + (5x)^2} \cdot 5 = \frac{15\arctan^2\left(5x\right)}{1 + 25x^2} \\
\mathbf{D}\left(\ln\left(\arcsin(x)\right)\right) &= \frac{1}{\arcsin(x)} \cdot \mathbf{D}\left(\arcsin(x)\right) = \frac{1}{\arcsin(x)} \cdot \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

You have already practiced some of these differentiation patterns in Chapter 2 (see Problems 72–83 in Section 2.4). ◀

Practice 1. Compute $\mathbf{D}\left(\arcsin\left(5x\right)\right)$, $\mathbf{D}\left(\arctan\left(x+2\right)\right)$, $\mathbf{D}\left(\text{arcsec}\left(7x\right)\right)$ and $\mathbf{D}\left(e^{\arctan(7x)}\right)$.

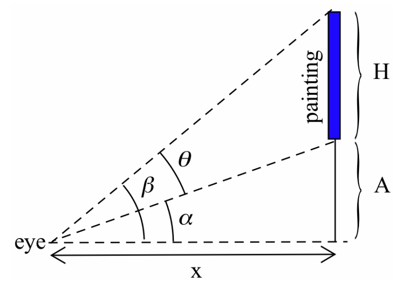
A Classic Application

Mathematics is the study of patterns, and one of the pleasures of mathematics is that patterns can crop up in some unexpected places. The mathematician Johannes Müller first posed the version of the classical Museum Problem below in 1471; it is one of the oldest known maximization problems.



Museum Problem: The lower edge of a 5-foot-tall painting is 4 feet above your eye level (see margin). At what distance should you stand from the wall so that your viewing angle of the painting is maximum?

On a typical sunny weekend, many people might rather be watching or playing a football or soccer game than visiting a museum or solving a calculus problem about a painting. But the pattern of the Museum Problem even appears in football and soccer. Because we also want to examine the Museum Problem in these other contexts, let's solve the general version.



Example 2. The lower edge of an H-foot-tall painting is A feet above your eye level (see margin). At what distance x should you stand from the painting so that your viewing angle is maximum?

Solution. Let $B = A + H$. Then $\tan(\alpha) = \dfrac{A}{x}$ and $\tan(\beta) = \dfrac{B}{x}$ so $\alpha = \arctan\left(\dfrac{A}{x}\right)$ and $\beta = \arctan\left(\dfrac{B}{x}\right)$. The viewing angle is thus:

$$\theta = \beta - \alpha = \arctan\left(\frac{B}{x}\right) - \arctan\left(\frac{A}{x}\right)$$

We can maximize θ by calculating the derivative $\dfrac{d\theta}{dx}$ and finding where that derivative is 0. Differentiation yields:

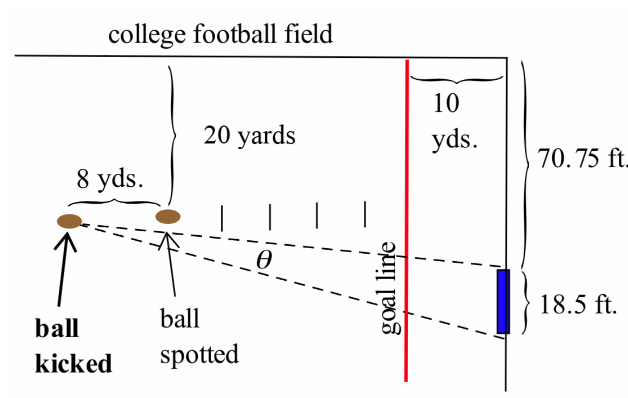
$$\begin{aligned}
\frac{d\theta}{dx} &= \mathbf{D}\left[\arctan\left(\frac{B}{x}\right)\right] - \mathbf{D}\left[\arctan\left(\frac{A}{x}\right)\right] \\
&= \frac{1}{1+\left(\frac{B}{x}\right)^2} \cdot \left(\frac{-B}{x^2}\right) - \frac{1}{1+\left(\frac{A}{x}\right)^2} \cdot \left(\frac{-A}{x^2}\right) \\
&= \frac{-B}{x^2+B^2} + \frac{A}{x^2+A^2}
\end{aligned}$$

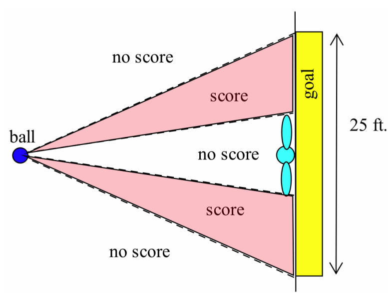
Setting $\dfrac{d\theta}{dx} = 0$ and solving for x yields $x = \sqrt{AB} = \sqrt{A(A+H)}$. ◀

We can disregard the endpoints, as you clearly do not have a maximum viewing angle with your nose pressed against the wall, or from an infinite distance far away from the wall.

Now the original Museum Problem above becomes straightforward (as do the football and soccer variations below). With $A = 4$ and $H = 5$, the maximum viewing angle occurs when $x = \sqrt{4(4+5)} = 6$ feet. The maximum angle is therefore $\theta = \arctan\left(\dfrac{9}{6}\right) - \arctan\left(\dfrac{4}{6}\right) \approx 0.983 - 0.588 \approx 0.395$, or about $22.6°$.

Practice 2. In a football game, a kicker is attempting a field goal by kicking the football between the goal posts (see figure below). At what distance from the goal line should the ball be "spotted" so the kicker has the largest angle for making the field goal? The goal posts are 18.5 feet wide and located 10 yards beyond the goal line. (Assume that the ball is placed on a "hash mark" that is 20 yards from the edge of the field and is actually kicked from a point about 8 yards farther from the goal line than where the ball is spotted.)





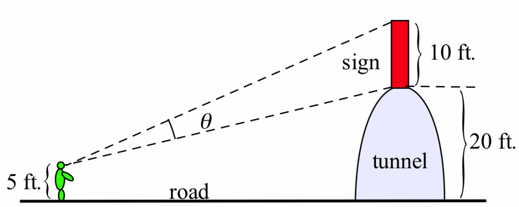
Practice 3. Kelcey is bringing a ball down the middle of a soccer field toward the 25-foot wide goal, which is defended by a goalie (see margin figure) positioned in the center of the goal. The goalie can stop a shot that is within four feet of the center of the goal. At what distance from the goal should Kelcey shoot so the scoring angle is maximum?

7.4 Problems

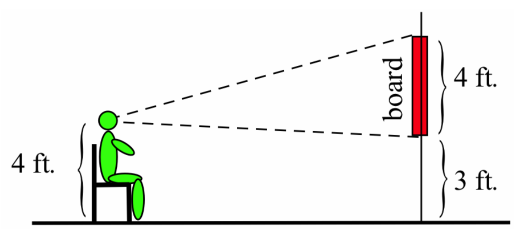
In Problems 1–21, compute the derivative.

1. $\mathbf{D}(\arcsin(3x))$
2. $\mathbf{D}(\arctan(7x))$
3. $\mathbf{D}(\arctan(x+5))$
4. $\mathbf{D}\left(\arcsin\left(\frac{x}{2}\right)\right)$
5. $\mathbf{D}\left(\arctan\left(\sqrt{x}\right)\right)$
6. $\mathbf{D}\left(\operatorname{arcsec}\left(x^2\right)\right)$
7. $\mathbf{D}(\ln(\arctan(x)))$
8. $\mathbf{D}(x \cdot \arctan(x))$
9. $\mathbf{D}\left((\operatorname{arcsec}(x))^3\right)$
10. $\mathbf{D}\left(\arctan\left(\frac{5}{x}\right)\right)$
11. $\mathbf{D}(\arctan(\ln(x)))$
12. $\mathbf{D}(\arcsin(x+2))$
13. $\mathbf{D}(e^x \cdot \arctan(2x))$
14. $\mathbf{D}(\tan(x) \cdot \arctan(x))$
15. $\mathbf{D}(\arcsin(x) + \arccos(x))$
16. $\mathbf{D}\left(\frac{1}{\arcsin(x)}\right)$
17. $\mathbf{D}\left(\sqrt{\arcsin(x)}\right)$
18. $\mathbf{D}\left((1+\operatorname{arcsec}(x))^3\right)$
19. $\mathbf{D}(\sin(3+\arctan(x)))$
20. $\mathbf{D}\left(\frac{\arcsin(x)}{\arccos(x)}\right)$
21. $\mathbf{D}\left(x \cdot \arctan\left(\frac{1}{x}\right)\right)$

22. (a) Use arctangents to describe the viewing angle for the sign in the figure below when the observer is x feet from the tunnel entrance.
    (b) At what distance x is the angle maximized?



23. (a) Use arctangents to describe the viewing angle for the whiteboard in figure below when the student is x feet from the front wall.
    (b) At what distance x is the angle maximized?



24. Mimic the first part of the proof at the beginning of this section to show that (for all x):

$$\mathbf{D}\left(\operatorname{arccot}(x)\right) = \frac{-1}{1+x^2}$$

25. Mimic the second part of the proof at the beginning of this section to show that (for $|x| < 1$):

$$\mathbf{D}\left(\arccos(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

26. Mimic the third part of the proof at the beginning of this section to show that (for $|x| > 1$):

$$\mathbf{D}\left(\operatorname{arccsc}(x)\right) = \frac{-1}{|x| \cdot \sqrt{x^2-1}}$$

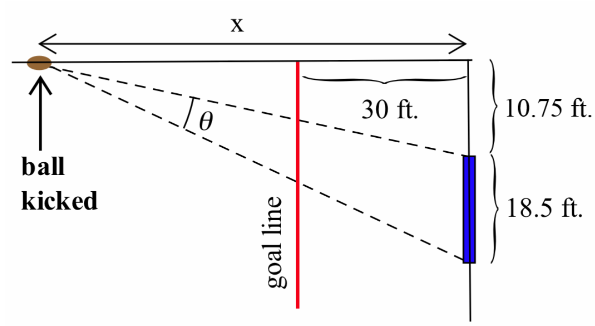
27. Revisit Problem 15 and draw a triangle to arrive at your answer without using any of the derivative patterns for inverse trignometric functions developed in this section.

7.4 Practice Answers

1. Applying the Chain Rule:

$$\begin{aligned}
\mathbf{D}\left(\arcsin(5x)\right) &= \frac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25x^2}} \\
\mathbf{D}\left(\arctan(x+2)\right) &= \frac{1}{1+(x+2)^2} = \frac{1}{x^2+4x+5} \\
\mathbf{D}\left(\operatorname{arcsec}(7x)\right) &= \frac{1}{|7x| \cdot \sqrt{(7x)^2-1}} \cdot 7 = \frac{1}{|x| \cdot \sqrt{49x^2-1}} \\
\mathbf{D}\left(e^{\arctan(7x)}\right) &= e^{\arctan(7x)} \cdot \mathbf{D}\left(\arctan(7x)\right) \\
&= e^{\arctan(7x)} \cdot \frac{1}{1+(7x)^2} \cdot 7 = \frac{7e^{\arctan(7x)}}{1+49x^2}
\end{aligned}$$

2. $A = 10.75$ ft. and $H = 18.5$ ft. (see below) so $x = \sqrt{A(A+H)} = \sqrt{314.4375} \approx 17.73$ feet from the back edge of the endzone. Unfortunately, that is still more than 12 feet into the endzone. Our mathematical analysis shows that to maximize the angle for kicking a field goal, the ball should be kicked from a position 12 feet into the endzone, a touchdown!



If the ball is placed on a hash mark just outside the endzone, then the kicking distance is 54 feet (10 yards for the depth of the endzone plus 8 yards that the ball is hiked) and the scoring angle is:

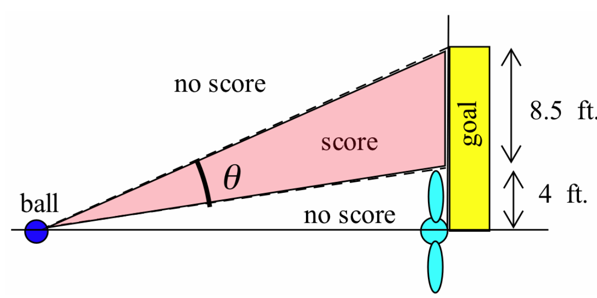
$$\theta = \arctan\left(\frac{29.25}{54}\right) - \arctan\left(\frac{10.75}{54}\right) \approx 17.2^\circ$$

It is somewhat interesting to see investigate how the scoring angle:

$$\theta = \arctan\left(\frac{29.25}{x}\right) - \arctan\left(\frac{10.75}{x}\right)$$

changes with the distance x (in feet), and also to compare the scoring angle for balls placed on the hash mark with those placed in the center of the field.

3. $A = 4$ ft. and $H = 8.5$ ft. so $x = \sqrt{A(A+H)} = \sqrt{50} \approx 7.1$ ft.:



From 7.1 feet, the scoring angle on one side of the goalie is:

$$\theta = \arctan\left(\frac{12.5}{7.1}\right) - \arctan\left(\frac{4}{7.1}\right) \approx 31^\circ$$

For comparison, the margin table gives values of the general scoring angle $\theta = \arctan\left(\frac{12.5}{x}\right) - \arctan\left(\frac{4}{x}\right)$ for some other distances x.

x	θ
5	29.5°
7	31.0°
10	29.5°
15	24.9°
20	20.7°
30	15.0°
40	11.6°

7.5 *Integrals Involving Inverse Trig Functions*

Aside from the Museum Problem and its sporting variations introduced in the previous section, the primary use of the inverse trigonometric functions in calculus involves their role as antiderivatives of rational and algebraic functions. Each of the six differentiation patterns from the previous section provides us with an integral formula, but they give rise to only three essentially different patterns:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

Valid for: $-1 < x < 1$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

Valid for: $-\infty < x < \infty$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C$$

Valid for: $|x| > 1$

Most of the related antiderivative patterns you will need in practice arise from variations of these basic ones. Typically, you need to transform an integrand so that it exactly matches one of the basic patterns.

Why are the derivative patterns for arccos, arccot and arccsc of little use to us when finding antiderivatives of algebraic functions?

Example 1. Evaluate $\displaystyle\int \frac{1}{16+x^2}\,dx$.

Solution. We can transform this integrand into the arctangent pattern by factoring 16 from the denominator

$$\int \frac{1}{16+x^2}\,dx = \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,dx = \frac{1}{16}\int \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

and then using the substitution $u = \dfrac{x}{4} \Rightarrow du = \dfrac{1}{4}\,dx \Rightarrow 4\,du = dx$:

$$\frac{1}{16}\int \frac{1}{1+u^2}\cdot 4\,du = \frac{4}{16}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan(u) + C$$

Replacing u with $\dfrac{x}{4}$ we get $\dfrac{1}{4}\arctan\left(\dfrac{x}{4}\right) + C$ as our final answer. ◀

Practice 1. Evaluate $\displaystyle\int \frac{1}{1+9x^2}\,dx$ and $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$.

The integrands that arise most often contain patterns with the forms a^2-x^2, a^2+x^2 and x^2-a^2, where a is some positive constant, so it is worthwhile to develop general integral patterns for these forms, list them in Appendix I, and refer to them when necessary:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

Valid for: $-a < x < a$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Valid for: $-\infty < x < \infty$

$$\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

Valid for: $|x| > a$

You can arrive at each of these general formulas by factoring the a^2 out of the denominator and making a suitable change of variable (as in Example 1, with a in place of 4). You can then check the result by differentiating. The arctan pattern is, by far, the most common. The arcsin pattern appears occasionally, and the arcsec pattern only rarely.

Example 2. Develop the general formula for $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$ from the known formula for $\int \frac{1}{\sqrt{1-x^2}}\, dx$. (Assume that $a > 0$.)

Solution. Using Example 1 as a guide, factor a^2 out of the denominator:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}}\, dx = \frac{1}{a} \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\, dx$$

Now substitute $u = \frac{x}{a} \Rightarrow du = \frac{1}{a}\, dx \Rightarrow a\, du = dx$ to get:

$$\frac{1}{a} \int \frac{1}{\sqrt{1-u^2}} \cdot a\, du = \frac{a}{a} \int \frac{1}{\sqrt{1-u^2}}\, du = \arcsin(u) + C$$

and replace u with $\frac{x}{a}$ to get $\arcsin\left(\frac{x}{a}\right) + C$, the desired result. ◀

Practice 2. Verify that the derivative of $\frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right)$ is $\frac{1}{a^2 + x^2}$.

Example 3. Evaluate $\int \frac{1}{\sqrt{5 - x^2}}\, dx$ and $\int_1^3 \frac{1}{5 + x^2}\, dx$.

Solution. The constant a needn't be an integer, so take $a^2 = 5 \Rightarrow a = \sqrt{5}$:

$$\int \frac{1}{\sqrt{5 - x^2}}\, dx = \arcsin\left(\frac{x}{\sqrt{5}}\right) + C$$

using the pattern from Example 2, while the general arctan pattern yields:

$$\begin{aligned} \int_1^3 \frac{1}{5 + x^2}\, dx &= \left[\frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right)\right]_1^3 \\ &= \frac{1}{\sqrt{5}} \left[\arctan\left(\frac{3}{\sqrt{5}}\right) - \arctan\left(\frac{1}{\sqrt{5}}\right)\right] \end{aligned}$$

or about 0.228. ◀

The easiest way to integrate certain rational functions is to split the original integrand into two pieces.

Example 4. Evaluate $\int \frac{6x + 7}{25 + x^2}\, dx$.

Solution. The integrand splits nicely into the sum of two other functions that you can integrate more easily:

$$\int \frac{6x+7}{25+x^2}\,dx = \int \frac{6x}{25+x^2}\,dx + \int \frac{7}{25+x^2}\,dx$$

In the first integral, use the subsitution $u = 25 + x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$:

$$\int \frac{6x}{25+x^2}\,dx = \frac{6}{2}\int \frac{1}{u}\,du = 3\ln(|u|) + C_1 = 3\ln\left(25+x^2\right) + C_1$$

Why can we remove the absolute value signs in the last step?

Meanwhile, the second integral matches the general arctangent pattern with $a = 5$:

$$\int \frac{7}{25+x^2}\,dx = 7\int \frac{1}{5^2+x^2}\,dx = 7\cdot\frac{1}{5}\arctan\left(\frac{x}{5}\right) + C_2$$

Combining these two results yields:

$$\int \frac{6x+7}{25+x^2}\,dx = \ln\left(25+x^2\right)^3 + \frac{7}{5}\arctan\left(\frac{x}{5}\right) + C$$

The two constants C_1 and C_2 add up to another arbitrary constant, which we can simply call C.

as our final answer. ◀

The antiderivative of a linear function divided by an irreducible quadratic polynomial will typically result in the sum of a logarithm and an arctangent.

Practice 3. Evaluate $\displaystyle\int \frac{4x+3}{x^2+7}\,dx$.

7.5 Problems

In Problems 1–24, evaluate the integral.

1. $\displaystyle\int \frac{7}{\sqrt{9-x^2}}\,dx$
2. $\displaystyle\int \frac{9}{\sqrt{7-y^2}}\,dy$
3. $\displaystyle\int_0^1 \frac{3}{x^2+25}\,dx$
4. $\displaystyle\int_5^7 \frac{5}{x\sqrt{x^2-16}}\,dx$
5. $\displaystyle\int \frac{9}{\sqrt{49-x^2}}\,dx$
6. $\displaystyle\int_1^4 \frac{2}{7+x^2}\,dx$
7. $\displaystyle\int_6^{10} \frac{3}{x\sqrt{x^2-25}}\,dx$
8. $\displaystyle\int \frac{7}{(x-5)^2+9}\,dx$
9. $\displaystyle\int \frac{1}{(x-1)^2+1}\,dx$
10. $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$
11. $\displaystyle\int_{-1}^1 \frac{e^x}{1+e^{2x}}\,dx$
12. $\displaystyle\int_1^e \frac{1}{x}\cdot\frac{3}{1+[\ln(x)]^2}\,dx$
13. $\displaystyle\int \frac{\cos(\theta)}{\sqrt{9-\sin^2(\theta)}}\,d\theta$
14. $\displaystyle\int \frac{8x}{16+x^2}\,dx$
15. $\displaystyle\int \frac{3x}{\sqrt{9+x^2}}\,dx$
16. $\displaystyle\int \frac{3x}{\sqrt{9-x^2}}\,dx$
17. $\displaystyle\int \frac{6x}{9+x^4}\,dx$
18. $\displaystyle\int \frac{6x}{\sqrt{9-x^4}}\,dx$
19. $\displaystyle\int \frac{1}{1+4x^2}\,dx$
20. $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx$
21. $\displaystyle\int_0^\infty \frac{1}{3+x^2}\,dx$
22. $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$
23. $\displaystyle\int_0^{\sqrt{7}} \frac{1}{\sqrt{7-x^2}}\,dx$
24. $\displaystyle\int_0^\infty \frac{x}{1+x^4}\,dx$

In Problems 25–28, solve the initial value problem.

25. $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{1-x^2}},\ y(0) = e$

26. $\dfrac{dy}{dx} = \dfrac{1}{y(1+x^2)},\ y(0) = 4$

27. $\dfrac{dy}{dx} = \dfrac{y^2}{9+x^2},\ y(1) = 2$

28. $\dfrac{dy}{dx} \cdot \sqrt{16-x^2} = y,\ y(4) = 1$

In Problems 29–32, evaluate the integral by splitting the integrand into two simpler functions.

29. $\displaystyle\int \frac{8x+5}{x^2+9}\,dx$

30. $\displaystyle\int \frac{1-4x}{x^2+1}\,dx$

31. $\displaystyle\int \frac{7x+3}{x^2+10}\,dx$

32. $\displaystyle\int \frac{x+5}{x^2+16}\,dx$

Problems 33–40 illustrate how we can sometimes decompose a difficult integral into simpler ones. (Hints: For 33, complete the square in the denominator; for 34, let u = denominator; for 35, write $4x+20 = (4x+12)+8$.)

33. $\displaystyle\int \frac{8}{x^2+6x+10}\,dx$

34. $\displaystyle\int \frac{4x+12}{x^2+6x+10}\,dx$

35. $\displaystyle\int \frac{4x+20}{x^2+6x+10}\,dx$

36. $\displaystyle\int \frac{7}{x^2+4x+5}\,dx$

37. $\displaystyle\int \frac{12x+24}{x^2+4x+5}\,dx$

38. $\displaystyle\int \frac{12x+31}{x^2+4x+5}\,dx$

39. $\displaystyle\int \frac{6x+15}{x^2+4x+20}\,dx$

40. $\displaystyle\int \frac{2x+5}{x^2-4x+13}\,dx$

7.5 Practice Answers

1. For the first integral, write:

$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$$

and substitute $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ to get:

$$\int \frac{1}{1+u^2}\cdot\frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan(3x) + C$$

For the second integral, factor out 25 to get:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5\sqrt{1-\frac{x^2}{25}}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\,dx$$

and then substitute $u = \dfrac{x}{5} \Rightarrow du = \dfrac{1}{5}\,dx \Rightarrow 5\,du = dx$:

$$\frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\cdot 5\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + K$$

Replacing u with $\dfrac{x}{5}$ yields:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right) + K$$

2. Using the Chain Rule:

$$\mathbf{D}\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]} = \frac{1}{a^2+x^2}$$

3. Split the integrand into two pieces:

$$\int \frac{4x+3}{x^2+7}\,dx = \int \frac{4x}{x^2+7}\,dx + \int \frac{3}{x^2+7}\,dx$$

For the first integral, let $u = x^2 + 7 \;\Rightarrow\; du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$:

$$\int \frac{4x}{x^2+7}\,dx = \int \frac{2}{u}\,du = 2\ln(|u|) + C_1 = 2\ln\left(x^2+7\right) + C_1$$

The second integral matches the arctangent pattern with $a^2 = 7 \Rightarrow a = \sqrt{7}$:

$$\int \frac{3}{x^2+7}\,dx = 3\int \frac{1}{\left(\sqrt{7}\right)^2 + x^2}\,dx = 3 \cdot \frac{1}{\sqrt{7}} \arctan\left(\frac{x}{\sqrt{7}}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x+3}{x^2+7}\,dx = \ln\left(\left[x^2+7\right]^2\right) + \frac{3}{\sqrt{7}} \arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

7.6 Calculus Done Right

This section, currently in preparation, will be included in the official first printing of this edition. It will begin with the function $L(x) = \int_1^x \frac{1}{t}\,dt$ and show that this function possesses all of the properties of the natural logarithm. It will then show $L(x)$ has an inverse function $E(x)$, argue that there is a number e so that $L(e) = 1$ and E(1)= e, show that $E(r) = e^r$ for any rational number r, and define $e^x = E(x)$ for all other values of x.

See `http://contemporarycalculus.com` for a digital version of this section as soon as it becomes available.

7.7 *Hyperbolic Functions*

This (optional) section, currently in preparation, will be included in the official first printing of this edition. It will present the hyperbolic functions, compute their derivatives and compute antiderivatives of related functions.

See `http://contemporarycalculus.com` for a digital version of this section as soon as it becomes available.

7.8 Inverse Hyperbolic Functions

This (optional) section, currently in preparation, will be included in the official first printing of this edition. It will present the inverse hyperbolic functions, compute their derivatives and compute antiderivatives of related functions.

See `http://contemporarycalculus.com` for a digital version of this section as soon as it becomes available.

8
Integration Techniques

In our journey through integral calculus, we have: developed the concept of a Riemann sum that converges to a definite integral; learned how to use the Fundamental Theorem of Calculus to evaluate a definite integral — as long as we can find an antiderivative for the integrand; examined numerical methods to approximate values of definite integrals; applied the concept of a Riemann sum to a variety of geometric and physical situations to compute lengths, areas, volumes, work and more; and employed integration to solve differential equations.

Finding an approximate value for a definite integral is often "good enough," but exact values are sometimes necessary — and this requires us to find antiderivatives of integrand functions. We have already learned how to find antiderivatives of many basic functions, and repeatedly employed substitution to turn complicated integrands into ones that are easier to integrate. This chapter begins with a review of these integration techniques you already know, then develops several new techniques that will allow you to integrate even more functions. It concludes by presenting a way to find "approximate antiderivatives" that will allow you to compute approximate numerical values for certain definite integrals much more efficiently than the techniques introduced in Section 4.9.

8.1 Finding Antiderivatives: A Review

Success at integration is primarily a matter of recognizing standard patterns and being able to manipulate functions into a form that corresponds to one of these patterns. Integral tables — such as the brief one on the next page and the longer one in Appendix I — list antiderivatives for many basic patterns of functions. Often a change of variable (employing the u-substitution method introduced in Section 4.6) will allow you to see a pattern more easily. For most people, developing the skill of recognizing these patterns comes with practice, and this section provides a variety of problems to review and hone your skills.

This is an extended version of the table listed in Section 4.6.

Constant Functions: $\int k\,du = ku + C$

Powers of u: $\int u^p\,du = \dfrac{u^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \dfrac{1}{u}\,du = \ln|u| + C$

Exponential Functions: $\int e^u\,du = e^u + C$, $\int a^u\,du = \dfrac{a^u}{\ln(a)} + C$

Trig Functions: $\int \cos(u)\,du = \sin(u) + C$, $\int \sin(u)\,du = -\cos(u) + C$

$\int \tan(u)\,du = \ln(|\sec(u)|) + C$, $\int \cot(u)\,du = \ln(|\sin(u)|) + C$

These two patterns may be new to you. See the discussion below.

$\int \sec(u)\,du = \ln(|\sec(u) + \tan(u)|) + C$

$\int \csc(u)\,du = -\ln(|\csc(u) + \cot(u)|) + C$

$\int \sec^2(u)\,du = \tan(u) + C$, $\int \csc^2(u)\,du = -\cot(u) + C$

$\int \sec(u)\cdot\tan(u)\,du = \sec(u) + C$, $\int \csc(u)\cdot\cot(u)\,du = -\csc(u) + C$

Inverse-Trig–Related Functions: $\int \dfrac{1}{1+u^2}\,du = \arctan(u) + C$

$\int \dfrac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + C$, $\int \dfrac{1}{|u|\cdot\sqrt{u^2-1}}\,du = \text{arcsec}(u) + C$

The most generally useful and powerful integration technique remains Changing the Variable. The first Problems in this section provide additional practice changing variables to calculate integrals. As we develop more complicated and more specialized techniques for finding antiderivatives, your first thought should still be whether the integral can be simplified by changing the variable. Sometimes the appropriate change of variable is not obvious, and we may need to manipulate the integrand using algebra, trigonometric identities or some clever "tricks" before employing a u-substitution.

Antiderivatives of $\sec(\theta)$ *and* $\csc(\theta)$

In the list of basic antiderivatives at the top of this page, you may have noticed two unfamiliar patterns: those for $\int \sec(\theta)\,d\theta$ and $\int \csc(\theta)\,d\theta$. Antiderivatves for $\cos(\theta)$ and $\sin(\theta)$ essentially came "free" from the derivative patterns we discovered in Chapter 2. Antiderivatives for $\tan(\theta)$ and $\cot(\theta)$ were among the first applications of u-substitution: for example, we can write $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ and put $u = \cos(\theta)$ so that $du = -\sin(\theta)\,d\theta$ and the integral in question becomes:

$$\int \tan(\theta)\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta = \int \frac{-1}{u}\,du = -\ln(|u|) + C$$
$$= -\ln(|\cos(\theta)|) + C = \ln(|\sec(\theta)|) + C$$

Finding an antiderivative of $\sec(\theta)$, however, requires a special "trick."

Before attempting a substitution, write:

$$\sec(\theta) = \frac{\sec(\theta)}{1} \cdot \frac{(\sec(\theta) + \tan(\theta))}{(\sec(\theta) + \tan(\theta))} = \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}$$

This is an example of "multiplying by 1," a tactic often employed in mathematics where we multiply the top and bottom of a fraction by the same expression.

Why would we want to take the nice, simple function $\sec(\theta)$ and rewrite it as this monstrosity? Look at the denominator and notice that the derivative of $\sec(\theta)$ is $\sec(\theta) \cdot \tan(\theta)$, while the derivative of $\tan(\theta)$ is $\sec^2(\theta)$. Both of these derivatives appear in the numerator. So if we use the substitution:

$$u = \sec(\theta) + \tan(\theta) \;\Rightarrow\; du = \left[\sec(\theta)\tan(\theta) + \sec^2(\theta)\right] d\theta$$

the integral of $\sec(\theta)$ becomes:

$$\begin{aligned}\int \sec(\theta)\, d\theta &= \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\, d\theta \\ &= \int \frac{1}{u}\, du = \ln(|u|) + C = \ln\left(|\sec(\theta) + \tan(\theta)|\right) + C\end{aligned}$$

proving the result listed on the previous page.

Practice 1. Prove that $\int \csc(\theta)\, d\theta = -\ln\left(|\csc(\theta) + \cot(\theta)|\right) + C$.

The trick used to integrate $\sec(\theta)$ and $\csc(\theta)$ only applies in these special situations, so rather than remembering the trick, you might want to simply memorize the result if you find yourself needing to integrate $\sec(\theta)$ on a regular basis.

See Problems 41–42 in Section 8.3 (and the related discussion on page 575) for a more intuitive—albeit more tedious—method to obtain antiderivatives for $\sec(\theta)$ and $\csc(\theta)$.

An Irreducible Quadratic Denominator

The following examples review and extend techniques (introduced in Section 7.5) involving variations on the arctangent derivative pattern.

Example 1. Evaluate $\int \frac{18}{1 + (x-3)^2}\, dx$ and $\int \frac{18}{x^2 - 6x + 10}\, dx$.

Solution. The form of the first integrand reminds us of the derivative of the arctangent function:

$$\mathbf{D}\left(\arctan(u)\right) = \frac{1}{1+u^2}$$

If we make the substitution $u = x - 3 \Rightarrow du = dx$ the integral becomes:

$$\int \frac{18}{1 + (x-3)^2}\, dx = 18 \int \frac{1}{1+u^2}\, du = 18\arctan(u) + C$$

Replacing u with $x - 3$, we get $18\arctan(x-3) + C$. The second integrand appears much more complicated, until you notice that:

$$1 + (x-3)^2 = 1 + x^2 - 6x + 9 = x^2 - 6x + 10$$

These integrands are in fact equal, so the second integral also equals $18\arctan(x-3) + C$. ◀

In the preceding Example, we showed that the two integrands were equal by expanding the denominator of the first integrand to get the denominator of the second integrand. If we had started with the second integral in Example 1, we could have rewritten the second denominator employing the method of **completing the square**:

Recall from algebra that to complete the square with a polynomial, you need to take half of the x coefficient:

$$\frac{-6}{2} = -3 \quad \text{or} \quad \frac{1}{2} \cdot \frac{b}{a}$$

and then square it:

$$(-3)^2 = 9 \quad \text{or} \quad \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

to get the number that must be added and subtracted to create a perfect square (plus a leftover constant term). A quadratic polynomial is **irreducible** if it can't be factored into two linear terms using real coefficients; for $ax^2 + bx + c$ this occurs when $b^2 - 4ac < 0$.

$$x^2 - 6x + 10 = \left(x^2 - 6x + 9\right) + (10 - 9) = (x-3)^2 + 1$$

so that we would make the substitution $u = x - 3$. For any irreducible quadratic polynomial of the form $ax^2 + bx + c$, we can write:

$$\begin{aligned} ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}\right] + c = a\left[x^2 + \frac{b}{a} + \frac{b^2}{4a^2}\right] + c - a \cdot \frac{b^2}{4a^2} \\ &= a\left(x - \frac{b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right] \end{aligned}$$

Rather than memorizing this formula, you should remember the *process* of completing the square.

Practice 2. Evaluate $\displaystyle\int \frac{5}{x^2 + 8x + 25}\,dx$ by completing the square and making a substitution.

Example 2. Evaluate $\displaystyle\int \frac{2x}{x^2 - 6x + 10}\,dx$.

Solution. This would be an easy problem if the numerator were $2x - 6$: the numerator would then be the derivative of the denominator and the pattern of the integral would be $\int \frac{1}{u}\,du$ with $u = x^2 - 6x + 10$. Using a bit of cleverness, we can rewrite the numerator as $2x - 6 + 6$. Then the integral becomes:

This is an example of "adding 0," another tactic often employed in mathematics in which we add and subtract the same quantity from an expression.

$$\int \frac{2x - 6 + 6}{x^2 - 6x + 10}\,dx = \int \frac{2x - 6}{x^2 - 6x + 10}\,dx + \int \frac{6}{x^2 - 6x + 10}\,dx$$

For the first integral, substitute $u = x^2 - 6x + 10 \Rightarrow du = (2x - 6)\,dx$:

Why can we omit the absolute values in the last step?

$$\int \frac{2x - 6}{x^2 - 6x + 10}\,dx = \int \frac{1}{u}\,du = \ln(|u|) + C_1 = \ln\left(x^2 - 6x + 10\right) + C_1$$

For the second integral, complete the square in the denominator:

$$x^2 - 6x + 10 = x^2 - 6x + 9 + 1 = (x-3)^2 + 1$$

and use the substitution $w = x - 3 \Rightarrow dw = dx$ to get:

$$\begin{aligned} \int \frac{6}{x^2 - 6x + 10}\,dx &= \int \frac{6}{(x-3)^2 + 1}\,dx = 6\int \frac{1}{w^2 + 1}\,dw \\ &= 6\arctan(w) + C_2 = 6\arctan(x-3) + C_2 \end{aligned}$$

so that the final answer is $\ln\left(x^2 - 6x + 10\right) + 6\arctan(x-3) + C$. ◀

This "logarithm plus an arctangent" pattern that arose in the previous Example turns up quite often with integrals of linear functions divided by irreducible quadratic polynomials. If the quadratic denominator can be factored into a product of two linear factors, we will instead use a technique discussed in Section 8.3 (Partial Fraction Decomposition).

Practice 3. Evaluate $\int \frac{4x+21}{x^2+8x+25}\,dx$.

8.1 Problems

In Problems 1–54, evaluate the integral. A well-chosen substitution will often turn a complicated-looking integral into a much simpler one.

1. $\int 6x\left(x^2+7\right)^2\,dx$
2. $\int 6x\left(x^2-1\right)^3\,dx$
3. $\int_2^4 \frac{6t}{\sqrt{t^2-3}}\,dt$
4. $\int_0^{\pi} 12\cos(\theta)\,[2+\sin(\theta)]^2\,d\theta$
5. $\int \frac{12x}{x^2+3}\,dx$
6. $\int \frac{\cos(\varphi)}{2+\sin(\varphi)}\,d\varphi$
7. $\int \sin(3y+2)\,dy$
8. $\int \cos\left(\frac{x}{5}\right)\,dx$
9. $\int_{-1}^{0} e^x\cdot\sec^2\left(e^x+3\right)\,dx$
10. $\int_0^{\frac{\pi}{2}} \cos(\theta)\sqrt{1+\sin(\theta)}\,d\theta$
11. $\int \frac{\ln(x)}{x}\,dx$
12. $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$
13. $\int \cos(\theta)\cdot e^{\sin(\theta)}\,d\theta$
14. $\int e^z\sin(e^z)\,dz$
15. $\int_1^3 \frac{5}{1+9x^2}\,dx$
16. $\int_0^1 \frac{7}{1+(x+5)^2}\,dx$
17. $\int_1^2 \frac{1}{x^2}\cdot\cos\left(\frac{1}{x}\right)\,dx$
18. $\int_1^e \frac{\sec\left(2+\ln(x)\right)}{x}\,dx$
19. $\int \frac{6\sin(\theta)\cos(\theta)}{5+\sin^2(\theta)}\,d\theta$
20. $\int \frac{6\cos(\alpha)}{5+\sin^2(\alpha)}\,d\alpha$
21. $\int \frac{10}{2x+5}\,dx$
22. $\int \frac{3}{8y+1}\,dy$
23. $\int_1^3 \frac{20x}{5x^2+3}\,dx$
24. $\int_1^5 \frac{4x}{x^2+9}\,dx$
25. $\int_0^1 \frac{7}{(x+3)^2+4}\,dx$
26. $\int_{-2.3}^{-2} \frac{1}{\sqrt{1-(x+2)^2}}\,dx$
27. $\int \frac{e^t}{1+e^{2t}}\,dt$
28. $\int \frac{4x+10}{x^2+5x+9}\,dx$
29. $\int_1^3 \frac{3}{x\,[1+\ln(x)]}\,dx$
30. $\int_0^1 \frac{e^t}{1+e^t}\,dt$
31. $\int_0^1 2x\sqrt{1-x^2}\,dx$
32. $\int_0^3 \frac{2x}{\sqrt{5+x^2}}\,dx$
33. $\int \cos(\theta)\,[1+\sin(\theta)]^3\,d\theta$

34. $\int \cos(\varphi)\sin^4(\varphi)\,d\varphi$

35. $\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx$

36. $\int_1^2 e^x\sqrt{2+e^x}\,dx$

37. $\int \frac{\sec^2(\theta)}{5+\tan(\theta)}\,d\theta$

38. $\int \frac{6x}{(x^2-1)^3}\,dx$

39. $\int \tan(y-5)\,dy$

40. $\int \left(x^3+3\right)^2\,dx$

41. $\int_0^1 e^{5u}\,du$

42. $\int \sec(2+3t)\,dt$

43. $\int t\cdot\sec(2+3t^2)\,dt$

44. $\int_1^1 \arctan\left(\sqrt{5-x^3}\right)\,dx$

45. $\int_e^e \ln\left(\sqrt{5+x^3}\right)\,dx$

46. $\int_1^\infty \frac{1}{1+9x^2}\,dx$

47. $\int_1^\infty \frac{x}{1+9x^4}\,dx$

48. $\int_1^\infty \frac{e^{-x}}{1+e^{-2x}}\,dx$

In 49–54, complete the square in the denominator, make an appropriate substitution, then integrate.

49. $\int \frac{7}{x^2+4x+5}\,dx$

50. $\int \frac{3}{x^2+4x+29}\,dx$

51. $\int \frac{2}{x^2-6x+58}\,dx$

52. $\int \frac{11}{x^2-2x+10}\,dx$

53. $\int \frac{3}{x^2+10x+29}\,dx$

54. $\int \frac{5}{x^2+2x+5}\,dx$

In 55–60, first split the integral into two integrals. (Hint: In Problem 55, $2x+11=(2x+4)+7$.)

55. $\int \frac{2x+11}{x^2+4x+5}\,dx$

56. $\int \frac{4x+11}{x^2+4x+5}\,dx$

57. $\int \frac{4x+7}{x^2-6x+10}\,dx$

58. $\int \frac{6x+28}{x^2+10x+34}\,dx$

59. $\int \frac{6x+5}{x^2-4x+13}\,dx$

60. $\int \frac{4x+9}{x^2+6x+13}\,dx$

In 61–66, remember that completing the square only helps with irreducible quadratic denominators.

61. $\int \frac{1}{x^2+4x+4}\,dx$

62. $\int \frac{x+2}{x^2+4x+4}\,dx$

63. $\int \frac{x+3}{x^2-6x+9}\,dx$

64. $\int_4^\infty \frac{1}{x^2-6x+9}\,dx$

65. $\int_3^\infty \frac{1}{x^2-6x+9}\,dx$

66. $\int_4^\infty \frac{8x-24}{x^2-6x+9}\,dx$

8.1 Practice Answers

1. First use the "multiply by 1" trick to write:

$$\csc(\theta) = \frac{\csc(\theta)}{1} \cdot \frac{\csc(\theta) + \cot(\theta)}{\csc(\theta) + \cot(\theta)} = \frac{\csc^2(\theta) + \csc(\theta)\cot(\theta)}{\csc(\theta) + \cot(\theta)}$$

and then make the substitution $u = \csc(\theta) + \cot(\theta)$ so that $du = \left(-\csc(\theta)\cot(\theta) - \csc^2(\theta)\right) d\theta$ and:

$$\int \csc(\theta)\, d\theta = \int \frac{\csc^2(\theta) + \csc(\theta)\cot(\theta)}{\csc(\theta) + \cot(\theta)}\, d\theta = \int \frac{-1}{u}\, du$$
$$= -\ln(|u|) + C = -\ln(|\csc(\theta) + \cot(\theta)|) + C$$

2. First complete the square in the denominator:

$$x^2 + 8x + 25 = \left(x^2 + 8x + 16\right) + (25 - 16) = (x+4)^2 + 9$$

so the integral becomes:

$$\int \frac{5}{x^2 + 8x + 25}\, dx = \int \frac{5}{(x+4)^2 + 9}\, dx$$

Now make the substitution $u = x + 4 \Rightarrow du = dx$ to get:

$$5 \int \frac{1}{u^2 + 3^2}\, du = 5 \cdot \frac{1}{3} \arctan\left(\frac{u}{3}\right) + C = \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C$$

3. If we substitute $u = x^2 + 8x + 25$ then $du = (2x + 8)\, dx$. We would be in good shape if the numerator of the integrand were $4x + 16 = 2(2x + 8)$, so split the numerator into $4x + 21 = (4x + 16) + 5$ to get:

$$\int \frac{4x + 21}{x^2 + 8x + 25}\, dx = \int \frac{(4x+16) + 5}{x^2 + 8x + 25}\, dx = \int \frac{4x + 16}{x^2 + 8x + 25}\, dx + \int \frac{5}{x^2 + 8x + 25}\, dx$$

The first integral (with $u = x^2 + 8x + 25$) now becomes:

$$\int \frac{4x + 16}{x^2 + 8x + 25}\, dx = \int \frac{2}{u}\, du = 2\ln(|u|) + C_1 = 2\ln\left(x^2 + 8x + 25\right) + C_1$$

The second integral is just the integral from Practice 2:

$$\int \frac{5}{x^2 + 8x + 25}\, dx = \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x + 21}{x^2 + 8x + 25}\, dx = \ln\left(x^2 + 8x + 25\right)^2 + \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C$$

8.2 Integration by Parts

Integration by parts is an integration method that enables us to find antiderivatives of certain functions for which our previous antidifferentiation methods fail, such as $\ln(x)$ and $\arctan(x)$, as well as antiderivatives of certain products of functions, such as $x^2 \ln(x)$ and $e^x \sin(x)$. It leads to many of the general integral formulas in Appendix I and (next to u-substitution) it is one most powerful and frequently used of antidifferentiation techniques.

The Integration by Parts formula for integrals arises from the Product Rule for derivatives. Recall that for functions $u = u(x)$ and $v = v(x)$, the Product Rule says:

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

If we integrate both sides of this equation with respect to x we get:

$$u \cdot v = \int u \cdot \frac{dv}{dx}\, dx + \int v \cdot \frac{du}{dx}\, dx$$

Solving for the first of the two integrals on the right side, yields:

$$\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx$$

At first, this formula may not appear very promising, as it merely exchanges one integration problem for another. But in certain situations, if we choose $u(x)$ and $v(x)$ carefully, this formula exchanges a difficult integral for an easier one. We can restate the formula in slightly more compact form using differentials.

Integration by Parts Formula

If u, v, u' and v' are continuous functions,

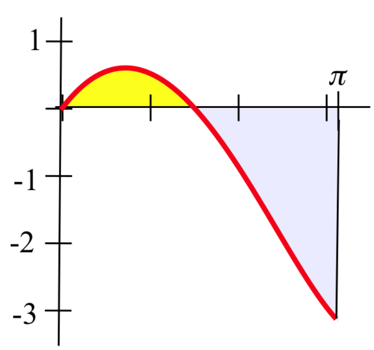
then $\displaystyle\int u\, dv = u \cdot v - \int v\, du$

For definite integrals, the Integration By Parts Formula says:

Integration by Parts Formula (Definite Integrals)

If u, v, u' and v' are continuous functions,

then $\displaystyle\int_a^b u\, dv = \Big[u \cdot v\Big]_a^b - \int_a^b v\, du$



Example 1. Use integration by parts to evaluate $\int x\cos(x)\, dx$ and $\int_0^{\pi} x\cos(x)\, dx$ (see margin for a graphical interpretation).

Solution. Our first step is to write this integral in the form required by the Integration by Parts formula, $\int u\,dv$. If we let $u = x$, then we must have $dv = \cos(x)\,dx$ so that $u\,dv$ completely represents the integrand $x\cos(x)$. We also need to calculate du and v:

$$u = x \ \Rightarrow\ du = dx \qquad \text{and} \qquad dv = \cos(x)\,dx \ \Rightarrow\ v = \sin(x)$$

Putting these pieces into the Integration by Parts formula, we have:

$$\int x\cdot\cos(x)\,dx \;=\; x\cdot\sin(x) - \int \sin(x)\,dx \;=\; x\cdot\sin(x) + \cos(x) + C$$

To check this result, differentiate the answer to verify that:

$$[x\sin(x) + \cos(x)]' = x\cos(x)$$

Now use this result to evaluate the definite integral:

$$\begin{aligned}\int_0^{\pi} x\cdot\cos(x)\,dx &= \Big[x\cdot\sin(x) + \cos(x)\Big]_0^{\pi} \\ &= [\pi\sin(\pi) + \cos(\pi)] - [0\cdot\sin(0) + \cos(0)] = -1 - 1\end{aligned}$$

or -2, which appears reasonable based on the area interpretation of this integral in the margin graph on the previous page. ◀

Integration by parts allowed us to exchange the problem of evaluating $\displaystyle\int x\cos(x)\,dx$ for the much easier problem of evaluating $\displaystyle\int \sin(x)\,dx$.

Practice 1. Use the Integration by Parts formula on $\displaystyle\int x\cos(x)\,dx$ with $u = \cos(x)$ and $dv = x\,dx$. Why does this lead to a poor exchange?

Example 2. Evaluate $\displaystyle\int xe^{3x}\,dx$ and $\displaystyle\int_0^1 xe^{3x}\,dx$.

Solution. The integrand is a product of two functions, so it is reasonable to use integration by parts to search for an antiderivative. If $u = x \ \Rightarrow du = dx$, then $dv = e^{3x}\,dx \ \Rightarrow\ v = \frac{1}{3}e^{3x}$. Inserting these expressions into the Integration by Parts formula, we get:

$$\begin{aligned}&\int xe^{3x}\,dx = x\cdot\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C \\ \Rightarrow &\int_0^1 xe^{3x}\,dx = \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_0^1 = \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}\right] = \frac{2}{9}e^3 + \frac{1}{9}\end{aligned}$$

or about 4.57. ◀

In practice, when you need to use integration by parts to evaluate a definite integral, it is often safest to first evaluate the corresponding indefinite integral and then use that antiderivative pattern to evaluate the definite integral, as we have done here.

In the previous Example another valid choice would have been $u = e^{3x}$ and $dv = x\,dx$, but that choice results in an integral that is more difficult than the original one: $du = 3e^{3x}\,dx$ and $v = \frac{1}{2}x^2$, so the Integration by Parts formula yields:

$$\int xe^{3x}\,dx = e^{3x}\cdot\frac{1}{2}x^2 - \int \frac{1}{2}x^2\cdot 3e^{3x}\,dx$$

which exchanges $\int xe^{3x}\,dx$ for the more difficult integral $\int \frac{3}{2}x^2e^{3x}\,dx$.

Practice 2. Evaluate $\int x\sin(x)\,dx$ and $\int xe^{5x}\,dx$.

Once you have chosen u and dv to represent the integrand as $u\,dv$, you need to calculate du and v. The du calculation is usually easy, but finding v from dv can be difficult for some choices of dv. In practice, you need to select u so that the remaining dv is simple enough that you can find v, the antiderivative of dv.

Example 3. Evaluate $\int 2x\ln(x)\,dx$.

Solution. The choice $u = 2x$ seems fine until we get a little further into the process. If $u = 2x$, then $dv = \ln(x)\,dx$. We now need to find du and v. Computing $du = 2\,dx$ is simple, but then we face the difficult problem of finding an antiderivative v for our choice $dv = \ln(x)\,dx$.

If you cannot find a v for your original choice of dv, try a different u and dv.

The choice $u = \ln(x)$ results in easier calculations. Let $u = \ln(x)$. Then $dv = 2x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = x^2$. Then the Integration by Parts formula gives:

$$\int 2x\ln(x)\,dx = \ln(x)\cdot x^2 - \int x^2\cdot\frac{1}{x}\,dx = x^2\ln(x) - \int x\,dx$$

so the final result is $x^2\ln(x) - \dfrac{1}{2}x^2 + C$. ◀

Antiderivatives of Inverse Functions

So far we have applied the Integration by Parts method to products of simple functions, but it also enables us—perhaps surprisingly—to find antiderivatives of the inverse trigonometric functions and of the logarithm (the inverse exponential function).

Example 4. Evaluate $\int \arctan(x)\,dx$.

Solution. Let $u = \arctan(x)$. Then $dv = dx$, so $du = \dfrac{1}{1+x^2}\,dx$ and $v = x$. Putting these expressions into the Integration by Parts formula:

$$\int \arctan(x)\,dx = x\arctan(x) - \int x\cdot\frac{1}{1+x^2}\,dx$$

We can evaluate the new integral using the substitution $w = 1 + x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}\,dw = x\,dx$:

Why are absolute value signs not needed in the last term?

$$\int x\cdot\frac{1}{1+x^2}\,dx = \int \frac{1}{2}\cdot\frac{1}{w}\,dw = \frac{1}{2}\ln(|w|) + K \;=\; \frac{1}{2}\ln\left(1+x^2\right) + K$$

Combining these results:

We can now include the antiderivative of $\arctan(x)$ in our list of antiderivatives in Appendix I.

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln\left(1+x^2\right) + C$$

or $x\arctan(x) - \ln\left(\sqrt{1+x^2}\right) + C$. ◀

Practice 3. Evaluate $\int \ln(x)\,dx$ and $\int_1^e \ln(x)\,dx$.

Note the following about the Integration by Parts formula:

- Once you choose u, then dv is completely determined.

dv = rest of the integrand

- Because you need to find an antiderivative of dv to get v, pick u and dv with this in mind.
- Integration by parts allows you to trade one integral for another. If the new integral is more difficult than the original integral, then you have made a poor choice of u and dv. Try a different choice for u and dv (or try a different technique).
- To evaluate the new integral $\int v\,du$ you may need to use substitution, integration by parts again, or some other technique (such as the ones discussed later in this chapter).

General Patterns

Sometimes a single application of integration by parts yields a formula that allows us to integrate an entire family of functions.

Example 5. Evaluate $\displaystyle\int x^p \ln(x)\,dx$ for any number $p \neq -1$.

Solution. Set $u = \ln(x)$ so $dv = x^p\,dx$, $du = \frac{1}{x}\,dx$ and $v = \frac{1}{p+1}x^{p+1}$. Putting all of this into the Integration by Parts formula:

$$\int x^p \ln(x)\,dx = \frac{1}{p+1}x^{p+1} \cdot \ln(x) - \int \frac{1}{p+1}x^{p+1} \cdot \frac{1}{x}\,dx$$

This new integral becomes:

$$\frac{1}{p+1}\int x^p\,dx = \frac{1}{p+1} \cdot \frac{1}{p+1}x^{p+1} + K = \frac{x^{p+1}}{(p+1)^2} + K$$

Combining these results yields:

$$\frac{x^{p+1}}{p+1}\ln(x) - \frac{x^{p+1}}{(p+1)^2} + C = \frac{x^{p+1}}{p+1}\left[\ln(x) - \frac{1}{p+1}\right] + C$$

for any number p as long as $p \neq -1$. ◀

What happens when $p = -1$? Can you evaluate that integral?

Practice 4. Use Example 5 to evaluate $\displaystyle\int x^2 \ln(x)\,dx$ and $\displaystyle\int \ln(x)\,dx$.

Reduction Formulas

Sometimes the result of an integration-by-parts procedure still contains an integral, but a simpler one with a smaller exponent. In these situations we can reuse the resulting **reduction formula** until the remaining integral is simple enough to integrate completely.

Example 6. Evaluate $\int x^n e^x \, dx$ and use the result to evaluate $\int x^2 e^x \, dx$.

Solution. Set $u = x^n$ so $dv = e^x \, dx$, $du = nx^{n-1} \, dx$ and $v = e^x$. The Integration by Parts formula gives

$$\int x^n e^x \, dx \;=\; x^n e^x - n \int x^{n-1} e^x \, dx$$

which is a reduction formula because we have reduced the power of x by 1, trading $\int x^n e^x \, dx$ for the "reduced" integral $\int x^{n-1} e^x \, dx$.

Because $\int x^2 e^x \, dx$ matches the general pattern of $\int x^n e^x \, dx$ with $n = 2$, we know that:

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x^1 e^x \, dx$$

The new integral also matches the pattern in the reduction formula (with $n = 1$ this time) so we know that:

$$\int x^1 e^x \, dx = x^1 e^x - 1 \cdot \int x^0 e^x \, dx$$

This last integral is just $\int e^x \, dx = e^x + K$, so combining our results:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x^1 e^x \, dx = x^2 e^x - 2 \left[x e^x - \int e^x \, dx \right] \\ &= x^2 e^x - 2x e^x + 2e^x + C \end{aligned}$$

We can also write the answer as $e^x \left[x^2 - 2x + 2 \right] + C$. ◀

Practice 5. Develop the reduction formula:

$$\int x^n \sin(x) \, dx \;=\; -x^n \cos(x) + n \int x^{n-1} \cos(x) \, dx$$

using integration by parts.

The Reappearing Integral

Sometimes the integral we are trying to evaluate shows up on both sides of the equation during our calculations in such a way that we can solve for the desired integral algebraically.

Example 7. Evaluate $\int e^x \cos(x) \, dx$.

Solution. Let $u = e^x$, so $dv = \cos(x) \, dx$, $du = e^x \, dx$ and $v = \sin(x)$:

$$\int e^x \cos(x) \, dx \;=\; e^x \sin(x) - \int e^x \sin(x) \, dx$$

The new integral does not look any easier than the original one, but it doesn't look any worse, so let's try to evaluate the new integral using integration by parts again. To evaluate $\int e^x \sin(x)\,dx$, let $u = e^x$ and $dv = \sin(x)\,dx$ so that $du = e^x\,dx$ and $v = -\cos(x)$, giving us:

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

Putting this result back into the original problem, we get:

$$\begin{aligned}\int e^x \cos(x)\,dx &= e^x \sin(x) - \int e^x \sin(x)\,dx \\ &= e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x)\,dx\right] \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx\end{aligned}$$

Note that $\int e^x \cos(x)\,dx$ appears on both sides of this last equation, so we can solve for that expression algebraically:

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + K$$

We need an arbitrary constant on the right side of this equation because the left side is an indefinite integral.

and, finally, divide both sides by 2 to get:

$$\int e^x \cos(x)\,dx = \frac{1}{2}\left[e^x \sin(x) + e^x \cos(x)\right] + C$$

$C = \frac{K}{2}$

which we can also write as $\frac{1}{2}e^x\left[\sin(x) + \cos(x)\right] + C$. ◀

Practice 6. Evaluate $\int e^x \sin(x)\,dx$.

A Useful Shortcut

Repeated application of integration by parts, as in Example 6, can quickly become tedious, even with the aid of a reduction formula. For integrals of the form $\int x^n \cdot f(x)\,dx$ where n is an integer, it is often possible to arrange the integration-by-parts ingredients in a table to allow much speedier computation.

Example 8. Evaluate $\int x^2 e^{3x}\,dx$.

Solution. Make a table (see margin) with two columns. In the second entry of the left column, start with $u = x^2$ and below it list the successive derivatives of x^2: $2x$, 2 and 0 (you can stop when you get to 0). In the right column start with $v' = e^{3x}$ and below it list successive antiderivatives of e^{3x}: $\frac{1}{3}e^{3x}$, $\frac{1}{9}e^{3x}$ and $\frac{1}{27}e^{3x}$. Now multiply the functions in each row and add the results, alternating signs:

		e^{3x}	$= v'$
$u =$	x^2	$\frac{1}{3}e^{3x}$	$(+)$
	$2x$	$\frac{1}{9}e^{3x}$	$(-)$
	2	$\frac{1}{27}e^{3x}$	$(+)$
	0	$\frac{1}{81}e^{3x}$	$(-)$

$$x^2 \cdot \frac{1}{3}e^{3x} - 2x \cdot \frac{1}{9}e^{3x} + 2 \cdot \frac{1}{27}e^{3x} - 0 \cdot \frac{1}{81}e^{3x}$$

We can stop here, because all of the remaining terms will be 0. Now add a constant and you have the result: $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$. ◀

Apply this new technique to the second integral in Example 6 to verify that you get the same result with the new method. Which method is faster?

This shortcut method only works when the chosen u in the initial integration-by-parts setup is x^n (so that taking several derivatives results in 0) and when the chosen dv is a function like e^{ax} or $\sin(bx)$ so that repeated antidifferentiation is fairly easy. But when this shortcut does apply, it can save you a great deal of time.

Practice 7. Evaluate $\int x^5 \cos(x)\,dx$ and $\int x^3 e^{-2x}\,dx$.

8.2 Problems

Problems 1–6 list one part (u or dv) needed for integration by parts. Find the other part (dv or u), calculate du and v, and apply the Integration by Parts formula to evaluate the integral.

1. $\int 12x \ln(x)\,dx, \quad u = \ln(x)$
2. $\int xe^{-x}\,dx, \quad u = x$
3. $\int x^4 \ln(x)\,dx, \quad dv = x^4\,dx$
4. $\int x \sec^2(3x)\,dx, \quad u = x$
5. $\int x \arctan(x)\,dx, \quad dv = x\,dx$
6. $\int x(5x+1)^{19}\,dx, \quad u = x$

In Problems 7–24, evaluate the integral.

7. $\int_0^1 \frac{x}{e^{3x}}\,dx$
8. $\int_0^1 10xe^{3x}\,dx$
9. $\int x \sec(x) \tan(x)\,dx$
10. $\int_0^{\pi} 5x \sin(2x)\,dx$
11. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 7x \cos(3x)\,dx$
12. $\int 6x \sin(x^2+1)\,dx$
13. $\int 12x \cos(3x^2)\,dx$
14. $\int x^2 \cos(x)\,dx$
15. $\int_1^3 \ln(2x+5)\,dx$
16. $\int x^3 \ln(5x)\,dx$
17. $\int_1^e (\ln(x))^2\,dx$
18. $\int_1^e \sqrt{x} \ln(x)\,dx$
19. $\int \arcsin(x)\,dx$
20. $\int x^2 e^{5x}\,dx$
21. $\int x \arctan(3x)\,dx$
22. $\int x \ln(x+1)\,dx$
23. $\int_1^2 \frac{\ln(x)}{x}\,dx$
24. $\int_1^2 \frac{\ln(x)}{x^2}\,dx$

25. Write $\sin^n(x) = \sin^{n-1}(x) \cdot \sin(x)$ and use integration by parts to obtain the following reduction formula for $\int \sin^n(x)\,dx$:
$$\frac{1}{n}\left[-\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx\right]$$
26. Write $\cos^n(x) = \cos^{n-1}(x) \cdot \cos(x)$ and use integration by parts to obtain the following reduction formula for $\int \cos^n(x)\,dx$:
$$\frac{1}{n}\left[\cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx\right]$$
27. Use integration by parts to obtain a reduction formula for $\int \sec^n(x)\,dx$.
28. Use integration by parts to obtain a reduction formula for $\int \tan^n(x)\,dx$.

In Problems 29–40, use a result from Problems 25–28 to evaluate the integral.

29. $\int \sin^3(x)\,dx$
30. $\int \sin^4(x)\,dx$
31. $\int \sin^5(x)\,dx$
32. $\int \cos^3(x)\,dx$
33. $\int \cos^4(x)\,dx$
34. $\int \cos^5(x)\,dx$
35. $\int \sec^3(x)\,dx$
36. $\int \sec^4(x)\,dx$
37. $\int \sec^5(x)\,dx$
38. $\int \sin^3(5x-2)\,dx$
39. $\int \cos^3(2x+3)\,dx$
40. $\int \sec^3(7x-1)\,dx$

In Problems 41–44, obtain a reduction formula using integration by parts.

41. $\int x^n e^{ax}\,dx$

42. $\int x^n \sin(ax)\,dx$

43. $\int x\,(\ln(x))^n\,dx$

44. $\int x^n \cos(ax)\,dx$

45. The integral $\int x(2x+5)^{19}\,dx$ can be evaluated with integration by parts or by substitution.
   (a) Evaluate the integral using integration by parts with $u = x$ and $dv = (2x+5)^{19}\,dx$.
   (b) Evaluate the integral using a change of variable with $w = 2x+5$.
   (c) Which method is easier?

46. The integral $\int \frac{x}{\sqrt{1+x}}\,dx$ can be evaluated with integration by parts or by substitution.
   (a) Evaluate the integral using integration by parts with $u = x$ and $dv = \frac{1}{\sqrt{1+x}}\,dx$.
   (b) Evaluate the integral using a change of variable with $w = 1+x$.
   (c) Which method is easier?

In Problems 47–68, evaluate the integral using any appropriate method.

47. $\int x\,(\ln(x))^2\,dx$

48. $\int x^2 \arctan(x)\,dx$

49. $\int_0^1 e^{-x}\sin(x)\,dx$

50. $\int_0^1 \frac{\cos(x)}{e^x}\,dx$

51. $\int \sin(\ln(x))\,dx$

52. $\int \cos(\ln(x))\,dx$

53. $\int \cos\left(\sqrt{x}\right)\,dx$

54. $\int \sin\left(\sqrt{x}\right)\,dx$

55. $\int e^{3x}\sin(x)\,dx$

56. $\int e^x \cos(3x)\,dx$

57. $\int_0^\infty xe^{-x}\,dx$

58. $\int_0^\infty x^2 e^{-3x}\,dx$

59. $\int_0^\infty e^{-x}\sin(x)\,dx$

60. $\int_0^\infty e^{-2x}\cos(3x)\,dx$

61. $\int x\sqrt{x+1}\,dx$

62. $\int x\sqrt{x^2+1}\,dx$

63. $\int x\cos(x^2)\,dx$

64. $\int x^2\sqrt{x^3+1}\,dx$

65. $\int x^2\cos(x)\,dx$

66. $\int x^3\sqrt{x^2+1}\,dx$

67. $\int x^3\sqrt[3]{x^2+1}\,dx$

68. $\int x^2\sin(x)\,dx$

In Problems 69–72, solve the initial value problem.

69. $y' = x\sin(x)$, $y(0) = 0$

70. $y' = xe^{7x}$, $y(0) = 1$

71. $y' = \frac{x}{e^{x+y}}$, $y(0) = 1$

72. $y' = x\sin(x)\cos^2(y)$, $y(0) = \frac{\pi}{4}$

73. Consider $\int_0^1 x\sin(x)\,dx$ and $\int_0^1 \sin(x)\,dx$.
   (a) Before evaluating the integrals, which do you think is larger? Why?
   (b) Evaluate both integrals. Was your prediction in part (a) correct?

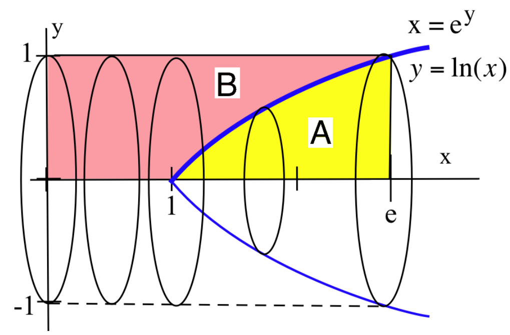
74. Consider $\int_0^\pi x\sin(x)\,dx$ and $\int_0^\pi \sin(x)\,dx$.
   (a) Before evaluating the integrals, which do you think is larger? Why?
   (b) Evaluate both integrals. Was your prediction in part (a) correct?

75. The figure below shows two regions, A and B. The volume swept out when region A is revolved about the x-axis is (using the disk method):

$$\int_{x=1}^{x=e} \pi\,(\ln(x))^2\,dx$$

and the volume swept out when region B is revolved about the x-xis is (using the tube method):

$$\int_{y=0}^{y=1} 2\pi y e^y\,dy$$



   (a) Before evaluating the integrals, which volume do you think is larger? Why?
   (b) Evaluate the integrals. Was your prediction in part (a) correct?

76. Refer to regions A and B from Problem 75.

    (a) Compute the volume of the solid generated when A is revolved around the y-axis.

    (b) Compute the volume of the solid generated when B is revolved around the y-axis.

77. Calculate the volume swept out when the region between the x-axis and the graph of $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the y-axis.

78. Calculate the volume swept out when the region between the x-axis and the graph of $y = \cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis.

79. Calculate the volume swept out when the region between the x-axis and the graph of $y = x\sin(x)$ for $0 \le x \le \pi$ is rotated about the y-axis.

80. Calculate the volume swept out when the region between the x-axis and the graph of $y = x\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis.

81. Determine if the area of the region between the graph of $y = xe^{-x}$ and the positive x-axis is finite. (If so, compute the area.)

82. Determine if the area of the region between the graph of $y = x^2e^{-x}$ and the positive x-axis is finite. (If so, compute the area.)

83. Determine if the volume of the solid obtained by revolving the region between the graph of $y = xe^{-x}$ and the positive x-axis about the x-axis is finite. (If so, compute the volume.)

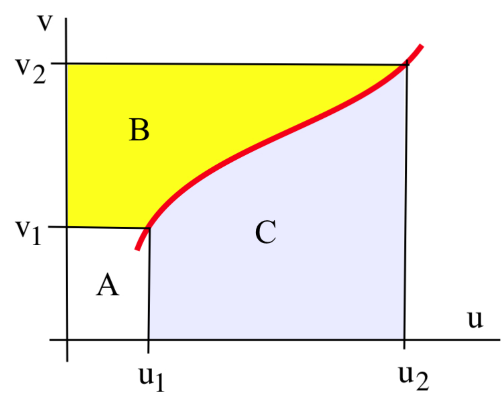
84. Determine if the volume of the solid obtained by revolving the region between the graph of $y = x^2e^{-x}$ and the positive x-axis about the x-axis is finite. (If so, compute the volume.)

85. Determine if the volume of the solid obtained by revolving the region between the graph of $y = xe^{-x}$ and the positive x-axis about the y-axis is finite. (If so, compute the volume.)

86. Determine if the volume of the solid obtained by revolving the region between the graph of $y = x^2e^{-x}$ and the positive x-axis about the y-axis is finite. (If so, compute the volume.)

87. We obtained the Integration by Parts formula analytically, starting with the Product Rule, but the formula also has a geometric interpretation. In the figure below, let D be the large rectangle formed by the regions A, B and C so that:

    (area of C) = (area of D) − (area of A) − (area of B)



    (a) Represent the area of the large rectangle D as a function of u_2 and v_2.

    (b) Represent the area of the small rectangle A as a function of u_1 and v_1.

    (c) Represent the area of region C as an integral with respect to the variable u.

    (d) Represent the area of region B as an integral with respect to the variable v.

    (e) Rewrite the area equation using the results of parts (a)–(d). This should look familiar.

88. Suppose f and f' are continuous and bounded on the interval $[0, 2\pi]$, meaning that $|f(x)| < M$ and $|f'(x)| < M$ when $0 \le x \le 2\pi$. The n-th Fourier Sine Coefficient of f is defined as:

$$S_n = \int_0^{2\pi} f(x)\sin(nx)\,dx$$

    (a) Use the Integration By Parts Formula with $u = f(x)$ and $dv = \sin(nx)\,dx$ to represent the formula for S_n in a different way.

    (b) Use the new representation of S_n from part (a) to determine what happens to the values of S_n when n is very large ($n \to \infty$). (Hint: $|f'(x)\cos(nx)| = |f'(x)| \cdot |\cos(nx)| < M \cdot 1$.)

    (c) What happens to the n-th Fourier Cosine Coefficients $C_n = \displaystyle\int_0^{2\pi} f(x)\cos(nx)\,dx$ as $n \to \infty$?

8.2 Practice Answers

1. $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$ and $dv = x\,dx \Rightarrow v = \frac{1}{2}x^2$, so:

$$\begin{aligned}\int x\cos(x)\,dx = \int \cos(x)\cdot x\,dx = \int u\,dv = u\cdot v - \int v\,du\\ = \cos(x)\cdot\frac{1}{2}x^2 - \int \frac{1}{2}x^2\left[-\sin(x)\right]\,dx\\ = \frac{1}{2}x^2\cos(x) + \int \frac{1}{2}x^2\sin(x)\,dx\end{aligned}$$

resulting in a new integral worse than the original integral.

2. (a) With $u = x$, $dv = \sin(x)\,dx$ so $du = dx$ and $v = -\cos(x)$:

$$\begin{aligned}\int x\sin(x)\,dx = \int u\,dv = u\cdot v - \int v\,du\\ = x\cdot(-\cos(x)) - \int\left[-\cos(x)\right]\,dx\\ = -x\cos(x) + \int\cos(x)\,dx = -x\cos(x) + \sin(x) + C\end{aligned}$$

(b) With $u = x$, $dv = e^{5x}\,dx$ so $du = dx$ and $v = \frac{1}{5}e^{5x}$:

$$\begin{aligned}\int xe^{5x}\,dx = \int u\,dv = u\cdot v - \int v\,du\\ = x\cdot\frac{1}{5}e^{5x} - \int\frac{1}{5}e^{5x}\,dx = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C\end{aligned}$$

3. Let $u = \ln(x)$ and $dv = dx$ so that $du = \frac{1}{x}\,dx$ and $v = x$:

$$\begin{aligned}\int \ln x\,dx = \int u\,dv = u\cdot v - \int v\,du\\ = \ln(x)\cdot x - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C\end{aligned}$$

Using this result:

$$\int_1^e \ln x\,dx = \left[x\ln(x) - x\right]_1^e = \left[e\ln(e) - e\right] - \left[1\ln(1) - 1\right] = 1$$

4. With $n = 2$: $\int x^2\ln(x)\,dx = \frac{x^3}{3}\left[\ln(x) - \frac{1}{3}\right] + C$

With $n = 0$: $\int \ln(x)\,dx = \frac{x^1}{1}\left[\ln(x) - \frac{1}{1}\right] + C = x\ln(x) - x + C$

5. Set $u = x^n$ and $dv = \sin(x)\,dx$ so $du = nx^{n-1}\,dx$ and $v = -\cos(x)$:

$$\begin{aligned}\int x^n\sin(x)\,dx = \int u\,dv = uv - \int v\,du\\ = -x^n\cos(x) - \int\left[-\cos(x)\right]\cdot nx^{n-1}\,dx\\ = -x^n\cos(x) + n\int x^{n-1}\cos(x)\,dx\end{aligned}$$

6. Proceeding as in Example 7, set $u = e^x$ so that $dv = \sin(x)\,dx$. Then $du = e^x\,dx$ and $v = -\cos(x)\,dx$, yielding:

$$\begin{aligned}\int e^x \sin(x)\,dx &= \int u\,dv = uv - \int v\,du \\ &= e^x \cdot (-\cos(x)) - \int (-\cos(x)) \cdot e^x\,dx \\ &= -e^x \cos(x) + \int e^x \cos(x)\,dx\end{aligned}$$

For this new integral, set $u = e^x$ so that $dv = \cos(x)\,dx$. Then $du = e^x\,dx$ and $v = \sin(x)\,dx$, yielding:

$$\int e^x \cos(x)\,dx = \int u\,dv = uv - \int v\,du = e^x \sin(x) - \int e^x \sin(x)\,dx$$

Combining these results we get:

$$\begin{aligned}\int e^x \sin(x)\,dx &= -e^x \cos(x) + \int e^x \cos(x)\,dx \\ &= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx\end{aligned}$$

and solving for the integral we started with yields:

$$2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) + K$$

so that:

$$\int e^x \sin(x)\,dx = \frac{1}{2}\left[-e^x \cos(x) + e^x \sin(x)\right] + C$$

7. For the first integral, let $u = x^5$ and take derivatives until you get 0:

$$x^5 \longrightarrow 5x^4 \longrightarrow 20x^3 \longrightarrow 60x^2 \longrightarrow 120x \longrightarrow 120 \longrightarrow 0$$

Put these in the left column of the margin table. Then set $v' = \cos(x)$ and start taking antiderivatives:

$$\cos(x) \longrightarrow \sin(x) \longrightarrow -\cos(x) \longrightarrow -\sin(x) \longrightarrow \cos(x)$$

and put these in the right column of the margin table. Now multiply the entries in each and add, using alternating signs:

		$\cos(x)$	$= v'$
$u =$	x^5	$\sin(x)$	$(+)$
	$5x^4$	$-\cos(x)$	$(-)$
	$20x^3$	$-\sin(x)$	$(+)$
	$60x^2$	$\cos(x)$	$(-)$
	$120x$	$\sin(x)$	$(+)$
	120	$-\cos(x)$	$(-)$
	0	$-\sin(x)$	$(+)$

$$\begin{aligned}\int x^5 \cos(x)\,dx =& x^5 \sin(x) - 5x^4\left[-\cos(x)\right] + 20x^3\left[-\sin(x)\right] \\ & - 60x^2 \cos(x) + 120x \sin(x) - 120\left[-\cos(x)\right] + C\end{aligned}$$

which we can rewrite as:

$$\left[x^5 - 20x^3 + 120x\right]\sin(x) + \left[5x^4 - 60x^2 + 120\right]\cos(x) + C$$

Using the same method with the second integral yields:

$$x^3\left[-\frac{1}{2}e^{-2x}\right] - 3x^2\left[\frac{1}{4}e^{-2x}\right] + 6x\left[-\frac{1}{8}e^{-2x}\right] - 6\left[\frac{1}{16}e^{-2x}\right] + C$$

so $\int x^3 e^{-2x}\,dx = -e^{-2x}\left[\frac{1}{2}x^3 + \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{8}\right] + C.$

8.3 Partial Fraction Decomposition

Rational functions (polynomials divided by polynomials) and their integrals play important roles in mathematics and applications, but if you look through an integral table (such as Appendix I) you will find very few formulas for antiderivatives of rational functions. Partly this is because the general formulas are rather complicated and have many special cases, and partly it is because they can all be reduced to just a few cases using the algebraic technique discussed in this section: Partial Fraction Decomposition.

In algebra you learned to add rational functions to get a single rational function (just as in arithmetic you learned how to add many fractions to get a single fraction). Partial Fraction Decomposition is a technique for reversing that procedure to "decompose" a single rational function into a sum of simpler rational functions. This allows us to turn the integral of a single rational function into the sum of integrals of simpler functions.

Example 1. Verify that the algebraic decomposition:

$$\frac{17x-35}{2x^2-5x} = \frac{7}{x} + \frac{3}{2x-5}$$

is true and use this fact to evaluate $\displaystyle\int \frac{17x-35}{2x^2-5x}\,dx$.

Solution. Working from the right side of the above equality, combine the two rational expressions by converting to a common denominator:

$$\frac{7}{x} + \frac{3}{2x-5} = \frac{7}{x}\cdot\frac{(2x-5)}{(2x-5)} + \frac{3}{(2x-5)}\cdot\frac{x}{x} = \frac{14x-35+3x}{2x^2-5x} = \frac{17x-35}{2x^2-5x}$$

which is the expression on the left side. This decomposition allows us to exchange the original integral for two much easier ones:

$$\int \frac{17x-35}{2x^2-5x}\,dx = \int \frac{7}{x}\,dx + \int \frac{3}{2x-5}\,dx = 7\ln(|x|) + \frac{3}{2}\ln(|2x-5|) + C$$

which can also be written as $\ln\left(|x|^7 \cdot |2x-5|^{\frac{3}{2}}\right) + C$. ◀

Practice 1. Verify that the algebraic decomposition:

$$\frac{7x-11}{3x^2-8x-3} = \frac{4}{3x+1} + \frac{1}{x-3}$$

is true and use this fact to evaluate $\displaystyle\int \frac{7x-11}{3x^2-8x-3}\,dx$.

Example 1 illustrates how to use a "decomposed" fraction to find an antiderivative of a rational function, but it does not show how to achieve this decomposition. The algebraic basis for the Partial Fraction Decomposition technique relies on the Fundamental Theorem of

Algebra, which guarantees that every polynomial with integer coefficients can be factored into a product of linear factors of the form $ax + b$ and irreducible quadratic factors of the form $ax^2 + bx + c$ (with $b^2 - 4ac < 0$). Unfortunately, the Fundamental Theorem of Algebra does not tell us *how* to find these factors, which typically will be more complicated than the examples in this section, but every polynomial has such factors. Before we apply the Partial Fraction Decomposition technique, the fraction must have the following form:

You might think that the Fundamental of Algebra would be easier to prove than the Fundamental Theorem of Calculus, because you studied algebra before calculus. Although we have already proved the Fundamental Theorem of Calculus in Chapter 4, a proof of the Fundamental Theorem of Algebra requires some higher-powered math.

- (degree of the numerator) < (degree of the denominator)
- The denominator has been factored into a product of linear factors and irreducible quadratic factors.

If the first assumption is not true, we can use polynomial division until we get a remainder with a smaller degree than the denominator. If the second assumption is not true, we simply cannot use the Partial Fraction Decomposition technique.

Now might be a good time to review polynomial division if you have not used it recently.

Example 2. Put each fraction into a form ready for Partial Fraction Decomposition:

(a) $\dfrac{2x^2 + 4x - 6}{x^2 - 2x}$ (b) $\dfrac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6}$ (c) $\dfrac{7x^2 + 12x - 12}{x^3 - 4x}$

Solution. (a) The degree of the numerator is not lower than the degree of the denominator, so we need to use polynomial division to rewrite the rational function:

$$\frac{2x^2 + 4x - 6}{x^2 - 2x} = 2 + \frac{8x - 6}{x^2 - 2x} = 2 + \frac{8x - 6}{x(x - 2)}$$

and conclude by factoring the denominator.

(b) The degree of the numerator is bigger than the degree of the denominator, so we need to use polynomial division to rewrite the rational function:

$$\frac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6} = 3x + \frac{9x + 8}{x^2 - x - 6} = 3x + \frac{9x + 8}{(x + 2)(x - 3)}$$

and conclude by factoring the denominator.

(c) Here the degree of the numerator is smaller than the degree of the denominator so we need only factor the denominator:

$$\frac{7x^2 + 12x - 12}{x^3 - 4x} = \frac{7x^2 + 12x - 12}{x(x + 2)(x - 2)}$$

and no polynomial division is required. ◀

Distinct Linear Factors

If you can factor the denominator into a product of distinct linear factors, then it turns out you can write the original fraction as the sum of fractions of the form $\frac{\text{number}}{\text{linear factor}}$. Your job is to find the numbers in the numerators, and that requires solving a system of equations.

We won't prove you can always do this, but you should be able to understand *why* this technique works by examining the solutions of Examples 3 and 4.

Example 3. Find values for A and B so that:

$$\frac{17x-35}{x(2x-5)} = \frac{A}{x} + \frac{B}{2x-5}$$

Solution. Multiply both sides of this equation by the common denominator $x(2x-5)$ to get:

$$17x - 35 = A(2x-5) + Bx = 2Ax - 5A + Bx = (2A+B)x - 5A$$

The leftmost and rightmost expressions in this equality are both linear functions, and we want them to be equal for all values of x, so the coefficients of x must be the same: $17 = 2A + B$. Similarly, the constant terms must be the same: $-5A = -35 \Rightarrow A = 7$. Putting this into the previous equation:

$$17 = 2A + B \Rightarrow 17 = 2(7) + B \Rightarrow B = 3$$

so we now know that:

$$\frac{17x-35}{x(2x-5)} = \frac{7}{x} + \frac{3}{2x-5}$$

which agrees with what we saw in Example 1. ◀

Practice 2. Find values of A and B so that:

$$\frac{6x-7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

In general, there will be one unknown coefficient for each distinct linear factor of the denominator. If the number of distinct linear factors is large, we would need to solve a large system of equations for the unknowns. For any situation involving only distinct linear factors, however, a useful shortcut exists.

Example 4. Find values for A, B and C so that:

$$\frac{2x^2+7x+9}{x(x+1)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}$$

Solution. Multiplying both sides of the above equation by the common denominator $x(x+1)(x+3)$ yields:

$$2x^2 + 7x + 9 = A(x+1)(x+3) + Bx(x+3) + Cx(x+1)$$

At this point, you could combine the expressions on the right side of this new equation into one big quadratic polynomial and compare its coefficients to those of the quadratic polynomial on the left side, resulting in three equations in three unknowns. Or you could plug in some well-chosen values of x to avoid much of that algebra. If $x = 0$:

$$2 \cdot 0^2 + 7 \cdot 0 + 9 = A(0+1)(0+3) + B \cdot 0 \cdot (0+3) + C \cdot 0 \cdot (0+1)$$

so $9 = 3A \Rightarrow A = 3$. Similarly, if $x = -1$:

$$2(-1)^2 + 7(-1) + 9 = A(0)(2) + B(-1)(2) + C(-1)(0) \Rightarrow 4 = -2B$$

so $B = -2$. And if $x = -3$:

$$2(-3)^2 + 7(-3) + 9 = A(-2)(0) + B(-3)(0) + C(-3)(-2) \Rightarrow 6 = 6C$$

so $C = 1$. We now know that:

$$\frac{2x^2 + 7x + 9}{x(x+1)(x+3)} = \frac{3}{x} - \frac{2}{x+1} + \frac{1}{x+3}$$

and can easily integrate each of these three simpler fractions. ◀

Practice 3. Evaluate the integral $\displaystyle\int \frac{2x^2 + 7x + 9}{x(x+1)(x+3)}\, dx$.

Practice 4. Apply the method of Example 4 to the Partial Fraction Decomposition in Example 3.

For fractions whose denominators contain irreducible quadratic factors or repeated factors, the form of the decomposition becomes more complicated—and with it, the algebra required to find the values of the constants. We will not discuss why the following suggestions work to decompose more general rational functions, except to note that they provide enough (but not too many) unknown coefficients.

Distinct Irreducible Quadratic Factors

If the factored denominator includes a distinct irreducible quadratic factor, then the Partial Fraction Decomposition sum contains a fraction of the form:

$$\frac{\text{linear polynomial}}{\text{irreducible quadratic factor}} \quad \text{or} \quad \frac{Ax + B}{x^2 + px + q}$$

where we typically need to solve a system of equations to find the values of the unknown coefficients A and B, given p and q with $p^2 - 4q < 0$.

Example 5. Find values for A, B and C so that:

$$\frac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x}$$

Solution. Multiply the equation on both sides by the common denominator $(x^2+2x+5)x$ to get:

$$x^2+3x-15=(Ax+B)x+C(x^2+2x+5)$$

Put $x=0$ into the equation to get:

$$-15=(B)(0)+C(5)\Rightarrow C=-3$$

Unfortunately, there no other numbers we can use for x to make terms on the right side vanish, but we can plug in any other "nice" number we want. The "nicest" numbers after 0 are 1 and -1:

$$\begin{aligned} x=1: &\quad -11=(A+B)-3(8) &&\Rightarrow\ A+B=13 \\ x=-1: &\quad -17=(-A+B)(-1)-3(4) &&\Rightarrow\ A-B=-5 \end{aligned}$$

We now have two equations in two unknowns. Adding these equations yields $2A=8\Rightarrow A=4$ and subtracting the second equation from the first yields $2B=18\Rightarrow B=9$. This tells us that:

$$\frac{x^2+3x-15}{(x^2+2x+5)x}=\frac{4x+9}{x^2+2x+5}-\frac{3}{x}$$

The original function is now in a form that is ready for integration. ◀

Practice 5. Evaluate the integral $\displaystyle\int\frac{x^2+3x-15}{(x^2+2x+5)x}\,dx$.

In general, there are two unknown coefficients for each distinct irreducible quadratic factor in the denominator.

Example 6. Decompose the rational function $\dfrac{6x^3+36x^2+50x+53}{(x^2+4)(x^2+4x+5)}$.

Solution. The degree of the denominator (4) is bigger than the degree of the numerator (3) and fortunately the denominator has already been factored into two irreducible quadratics. We can write:

The denominator in unfactored form is:

$$x^4+4x^3+9x^2+16x+20$$

Would you be able to factor this into two irreducible quadratic factors if it had not already been factored for you?

$$\frac{6x^3+36x^2+50x+53}{(x^2+4)(x^2+4x+5)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+4x+5}$$

Multiplying by the common denominator $(x^2+4)(x^2+4x+5)$ yields:

$$\begin{aligned} 6x^3+36x^2+50x+53 &= (Ax+B)(x^2+4x+5)+(Cx+D)(x^2+4) \\ &= Ax^3+4Ax^2+5Ax+Bx^2+4Bx+5B+Cx^3+4Cx+Dx^2+4D \\ &= (A+C)x^3+(4A+B+D)x^2+(5A+4B+4C)x+(5B+4D) \end{aligned}$$

To simplify the algebra, note that:

$$C=6-A \text{ and } D=\frac{1}{4}(53-5B)$$

then substitute these expressions into the second and third equations to reduce a system of four equations in four unknowns to a system of two equations in two unknowns.

Compare coefficients between the first and last polynomials to see that $A+C=6$, $4A+B+D=36$, $5A+4B+4C=50$ and $5B+4D=53$. After much algebra (see margin note), this system of four equations in four unknowns reduces to $A=6$, $B=5$, $C=0$ and $D=7$. ◀

Repeated Factors

If the factored denominator contains a linear factor raised to a power (greater than one), the decomposition requires one unknown coefficient for each power of the linear factor. For example:

$$\frac{\text{something}}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Similarly, if the factored denominator contains an irreducible quadratic factor raised to a power (greater than one), then the decomposition requires one linear term (with two unknown coefficients) for each power of the irreducible quadratic. For example:

$$\frac{\text{something}}{x^2(x^2+9)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} + \frac{Ex+F}{(x^2+9)^2} + \frac{Gx+H}{(x^2+9)^3}$$

This leads to a system of 8 equations with 8 unknowns.

Example 7. Evaluate $\displaystyle\int \frac{-4x^2+5x+3}{x^3-2x^2+x}\,dx$.

Solution. Before we can integrate, we need to decompose the integrand into simpler rational functions. The degree of the denominator (3) is bigger than the degree of the numerator (2) but we do need to factor:

$$x^3 - 2x^2 + x = x(x^2-2x+1) = x(x-1)^2$$

so that we can write:

$$\frac{-4x^2+5x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiply this equation by the common denominator $x(x-1)^2$ to get:

$$-4x^2+5x+3 = A(x-1)^2 + Bx(x-1) + Cx$$

Inserting the values $x=0$ or $x=1$ yields:

$$\begin{aligned} x=0: &\quad 3 = (A)(-1)^2 + B(0)(-1) + C(0) &&\Rightarrow\; A=3 \\ x=1: &\quad 4 = A(0) + B(1)(0) + C(1) &&\Rightarrow\; C=4 \end{aligned}$$

Using $A=3$ and $C=4$ with $x=-1$ results in:

$$-6 = 3(-2)^2 + B(-1)(-2) - 4 \;\Rightarrow\; 2B = -14 \;\Rightarrow\; B = -7$$

We can now integrate:

$$\begin{aligned} \int \frac{-4x^2+5x+3}{x^3-2x^2+x}\,dx &= \int \left[\frac{3}{x} - \frac{7}{x-1} + \frac{4}{(x-1)^2}\right] dx \\ &= 3\ln(|x|) - 7\ln(|x-1|) - \frac{4}{x-1} + C \end{aligned}$$

which can also be written $\ln\left(\dfrac{|x|^3}{|x-1|^7}\right) - \dfrac{4}{x-1} + C$. ◀

Practice 6. Evaluate $\int \frac{2x^2+27x+85}{(x+5)^2}\,dx$.

Be careful: Partial Fraction Decomposition only works with *rational* functions, although you may be able to turn another type of integrand into a rational function using substitution. For example, we can write:

$$\int \sec(\theta)\,d\theta = \int \frac{1}{\cos(\theta)}\,d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\,d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\,d\theta$$

and substitute $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ to convert this to:

$$\int \sec(\theta)\,d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\,d\theta = \int \frac{1}{1-u^2}\,du$$

Problem 41 asks you to complete this integration and resubstitute to obtain the antiderivative pattern for $\sec(\theta)$.

which is now a job for partial fractions.

Additional Applications

The primary use of the partial fraction technique in this course is to convert rational functions into a form that makes them easier to integrate, but this algebraic technique can also be used to simplify the differentiation of certain rational functions. (In a later math course, you will also use partial fractions when computing certain inverse Laplace transforms.) The next Example illustrates the use of partial fractions to make a differentiation problem easier.

Example 8. For $f(x) = \frac{2x+13}{x^2+x-2}$, calculate $f'(x)$, $f''(x)$ and $f'''(x)$.

Solution. You already know how to calculate these derivatives using the Quotient Rule, but that process becomes rather tedious for the second and third derivatives of this function. Instead, we can use the partial fraction technique to rewrite f as:

$$f(x) = \frac{5}{x-1} - \frac{3}{x+2} = 5(x-1)^{-1} - 3(x+2)^{-1}$$

Computing the derivatives is now quite straightforward:

$$\begin{aligned} f'(x) &= -5(x-1)^{-2} + 3(x+2)^{-2} \\ f''(x) &= 10(x-1)^{-3} - 6(x+2)^{-3} \\ f'''(x) &= -30(x-1)^{-4} + 18(x+2)^{-4} \end{aligned}$$

You will appreciate the value of this shortcut if you attempt to compute $f'''(x)$ using the Quotient Rule. ◀

Practice 7. Use a partial fraction decomposition of $g(x) = \frac{9x+1}{x^2-2x-3}$ to calculate $g'(x)$, $g''(x)$ and $g^{(4)}(x)$.

The rational functions in the Examples and Problems in this section have been carefully constructed to allow you to easily factor denominators and to decompose fractions with a minimum of effort. In practice, factoring higher-order polynomials can require numerical methods (like Newton's Method) to approximate roots and decomposing rational functions whose denominators include repeated linear irreducible quadratic factors can lead to systems of many equations in many unknowns that benefit from the tools available in linear algebra. As a result, computers are best suited to handle the partial fraction decomposition of more complicated rational functions.

8.3 Problems

In Problems 1–32, decompose the integrand and then evaluate the integral.

1. $\displaystyle\int \frac{7x+2}{x(x+1)}\,dx$
2. $\displaystyle\int \frac{7x+9}{(x+3)(x-1)}\,dx$
3. $\displaystyle\int \frac{11x+25}{x^2+9x+8}\,dx$
4. $\displaystyle\int \frac{3x+7}{x^2-1}\,dx$
5. $\displaystyle\int \frac{2x^2+15x+25}{x^2+5x}\,dx$
6. $\displaystyle\int \frac{3x^3+3x^2}{x^2+x-2}\,dx$
7. $\displaystyle\int \frac{6x^2+9x-15}{x(x+5)(x-1)}\,dx$
8. $\displaystyle\int \frac{6x^2-x-1}{x^3-x}\,dx$
9. $\displaystyle\int \frac{8x^2-x+3}{x^3+x}\,dx$
10. $\displaystyle\int \frac{9x^2+13x+15}{x^3+2x^2-3x}\,dx$
11. $\displaystyle\int \frac{11x^2+23x+6}{x^2(x+2)}\,dx$
12. $\displaystyle\int \frac{6x^2+14x-9}{x(x+3)^2}\,dx$
13. $\displaystyle\int \frac{3x+13}{(x+2)(x-5)}\,dx$
14. $\displaystyle\int \frac{2x+11}{(x-7)(x-2)}\,dx$
15. $\displaystyle\int_2^5 \frac{2}{x^2-1}\,dx$
16. $\displaystyle\int_1^3 \frac{5x^2+5x+3}{x^3+x}\,dx$
17. $\displaystyle\int \frac{2x^2+5x-3}{x^2-1}\,dx$
18. $\displaystyle\int \frac{2x^2+19x+22}{x^2+x-12}\,dx$
19. $\displaystyle\int \frac{3x^2+19x+24}{x^2+6x+5}\,dx$
20. $\displaystyle\int \frac{7x^2+8x-2}{x^2+2x}\,dx$
21. $\displaystyle\int \frac{3x^2-1}{x^3+x}\,dx$
22. $\displaystyle\int \frac{x^4+5x^3+x-15}{x^2+5x}\,dx$
23. $\displaystyle\int \frac{x^3+3x^2-4x+30}{x^2+3x-10}\,dx$
24. $\displaystyle\int \frac{7x^3+x^2+7x+10}{x^4+2x^3}\,dx$
25. $\displaystyle\int \frac{12x^2+19x-6}{x^3+3x^2}\,dx$
26. $\displaystyle\int \frac{2x+5}{(x+1)^2}\,dx$
27. $\displaystyle\int \frac{7x^2+3x+7}{x^3+x}\,dx$
28. $\displaystyle\int \frac{7x^2-4x+4}{x^3+1}\,dx$
29. $\displaystyle\int_2^\infty \frac{2}{x^2-1}\,dx$
30. $\displaystyle\int_2^\infty \frac{7x+2}{6x^2-13x+16}\,dx$
31. $\displaystyle\int \frac{6x^2+5x+61}{(x-1)(x^2+4x+13)}\,dx$
32. $\displaystyle\int \frac{x-84}{(x+5)(x^2-6x+34)}\,dx$

Integrals can be very sensitive to small changes in the integrand. In 33–34, notice how similar functions require vastly different integration methods.

33. (a) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ (b) $\displaystyle\int \frac{1}{x^2+2x+1}\,dx$ (c) $\displaystyle\int \frac{1}{x^2+2x+0}\,dx$
34. (a) $\displaystyle\int \frac{1}{x^2-6x+8}\,dx$ (b) $\displaystyle\int \frac{1}{x^2-6x+9}\,dx$ (c) $\displaystyle\int \frac{1}{x^2-6x+10}\,dx$

In Problems 35–40, use a partial fraction decomposition to compute the first and second derivatives of the given function.

35. $f(x) = \dfrac{7x+2}{x(x+1)}$
36. $F(x) = \dfrac{7x+9}{(x+3)(x-1)}$
37. $g(x) = \dfrac{11x+25}{x^2+9x+8}$
38. $G(x) = \dfrac{3x+7}{x^2-1}$
39. $h(x) = \dfrac{2x^2+15x+25}{x^2+5x}$
40. $H(x) = \dfrac{9x^2+13x+15}{x^3+2x^2-3x}$

41. Obtain a formula for $\int \sec(\theta)\, d\theta$ by writing:

$$\int \frac{1}{\cos(\theta)}\, d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\, d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\, d\theta$$

and using the substitution $u = \sin(\theta)$ to turn this integrand into a rational function.

42. Obtain a formula for $\int \csc(\theta)\, d\theta$ by writing:

$$\int \frac{1}{\sin(\theta)}\, d\theta = \int \frac{\sin(\theta)}{\sin^2(\theta)}\, d\theta = \int \frac{\sin(\theta)}{1-\cos^2(\theta)}\, d\theta$$

and using the substitution $u = \cos(\theta)$ to turn this integrand into a rational function.

Logistic Growth

The following two applications involve a type of differential equation that can be solved by separating variables and then using a partial fraction decomposition to calculate the antiderivatives. The same type of differential equation is also used to model the spread of rumors and diseases, as well as the growth of some populations and chemical reactions.

The growth rate of many different types of populations depends not only on the number of individuals currently in the population (leading to exponential growth) but also on a "carrying capacity" of the environment. (As a population grows, it might deplete a food source, slowing the growth of the population.) If x is the size of a population at time t and the growth rate of x is proportional only to the size of the population, we get the differential equation:

$$\frac{dx}{dt} = kx$$

which we investigated in Section 6.4. If we want to create a model in which the growth slows as the size of the population nears the carrying capacity M, we can multiply this model for $\frac{dx}{dt}$ by factor $\left(1 - \frac{x}{M}\right)$:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right)$$

When x is small (relative to M), this new factor is close to 1 so that:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right) \approx kx$$

leading to growth that is "almost exponential." When x gets close to M, this new factor is close to 0:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right) \approx kx\left(1 - \frac{M}{M}\right) = 0$$

so that the growth of x slows down and x is (roughly) constant. We call this differential equation the **logistic equation** and its solution a **logistic function**.

43. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 5$.
    (a) Create a direction field for the ODE.
    (b) Solve the corresponding logistic IVP.
    (c) Graph the population $x(t)$ for $0 \le t \le 20$.
    (d) When will the population be 20? 50? 90? 100?
    (e) What is the population after a "long" time? (Find the limit of x, as $t \to \infty$.)
    (f) Explain the shape of the graph in (c) in the context of a population of bacteria.
    (g) When is the growth rate largest?
    (h) What is the population when the growth rate is largest?
    (i) What would happen if $x(0) > 100$?

44. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 150$.
    (a) Solve the corresponding logistic IVP.
    (b) Graph the population $x(t)$ for $0 \le t \le 20$.
    (c) When will the population be 120? 110? 100?
    (d) What is the population after a "long" time?
    (e) Explain the shape of the graph in (b).

45. Let k and M be positive constants, and assume the initial population is $x(0) = x_0$.
    (a) Solve the corresponding logistic IVP.
    (b) What is the population after a "long" time?
    (c) When is the growth rate largest?
    (d) What is the population at that time?

Chemical Reactions

In certain chemical reactions, a new material X is formed from materials A and B, and the rate at which X forms is proportional to the product of the amount of A and the amount of B remaining. Let x represent the amount of material X present at time t, and assume that the reaction begins with a grams of A, b grams of B and no material X (so that $x(0) = 0$). Then the rate of formation of material X can be described by the differential equation:

$$\frac{dx}{dt} = k(a - x)(b - x)$$

46. Solve the IVP given above for x if $k = 1$ and the reaction begins with
    (a) 7 grams of A and 5 grams of B.
    (b) 6 grams of A and 6 grams of B.

47. Solve the IVP given above for x if $k = 1$ and the reaction begins with
    (a) a grams of A and b grams of B with $a \neq b$.
    (b) c grams of A and c grams of B (for $c > 0$).

8.3 Practice Answers

1. To verify the identity:

$$\frac{4}{(3x+1)} \cdot \frac{(x-3)}{(x-3)} + \frac{1}{(x-3)} \cdot \frac{(3x+1)}{(3x+1)} = \frac{(4x-12)+(3x+1)}{(3x+1)(x-3)} = \frac{7x-11}{3x^2-8x-3}$$

To evaluate the integral:

$$\int \frac{7x-11}{3x^2-8x-3}\,dx = \int \left[\frac{4}{(3x+1)} + \frac{1}{(x-3)}\right] dx = \frac{4}{3}\ln(|3x+1|) + \ln(|x-3|) + C$$

2. Multiply both sides of the equality by the common denominator $(x+3)(x-2)$ to get:

$$6x-7 = A(x-2)+B(x+3) = Ax-2A+Bx+3B = (A+B)x+(-2A+3B)$$

so $A+B=6$ and $-2A+3B=-7$. Solving this system of equations yields $A=5$ and $B=1$.

3. Using the result of Example 4:

$$\int \left[\frac{3}{x} - \frac{2}{x+1} + \frac{1}{x+3}\right] dx = 3\ln(|x|) - 2\ln(|x+1|) + \ln(|x+3|) + C$$

4. Multiply each side of the equation by the common denominator $x(2x-5)$ to get:

$$\frac{17x-35}{x(2x-5)} = \frac{A}{x} + \frac{B}{2x-5} \;\Rightarrow\; 17x-35 = A(2x-5)+Bx$$

Now put $x=0$ and $x=\frac{5}{2}$ into this new equation to get:

$$x=0: \quad -35 = -5A+0 \;\Rightarrow\; A=7$$
$$x=\frac{5}{2}: \quad \frac{15}{2} = A(0) + B\left(\frac{5}{2}\right) \;\Rightarrow\; B=3$$

5. Using the result of Example 5:

$$\begin{aligned}
\int \left[\frac{4x+9}{x^2+2x+5} - \frac{3}{x}\right] dx &= \int \left[\frac{4x+8+1}{(x+1)^2+1} - \frac{3}{x}\right] dx \\
&= \int \left[\frac{2(2x+2)}{x^2+2x+5} + \frac{5}{(x+1)^2+4} - \frac{3}{x}\right] dx \\
&= 2\ln\left(x^2+2x+5\right) + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) - 3\ln(|x|) + C
\end{aligned}$$

6. Because the degree of the numerator equals the degree of the denominator, we must use polynomial division to rewrite the integrand as:

$$\frac{2x^2+27x+85}{x^2+10x+25} = 2 + \frac{7x+35}{x^2+10x+25} = 2 + \frac{7(x+5)}{(x+5)^2} = 2 + \frac{7}{x+5}$$

This integrand does not require partial fraction decomposition:

$$\int \frac{2x^2+27x+85}{(x+5)^2}\,dx = \int \left[2 + \frac{7}{x+5}\right] dx = 2x + 5\ln(|x+5|) + C$$

7. First write:

$$g(x) = \frac{9x+1}{x^2-2x-3} = \frac{9x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

and then multiply through the last equality by the common denominator $(x-3)(x+1)$ to get:

$$9x+1 = A(x+1) + B(x-3)$$

Now put $x=3$ and $x=-1$ into this new equation to get:

$$\begin{aligned} x = 3: &\quad 28 = 4A + B(0) \;\Rightarrow\; A = 7 \\ x = -1: &\quad -8 = A(0) + B(-4) \;\Rightarrow\; B = 2 \end{aligned}$$

We can now rewrite $g(x)$ as $g(x) = 7(x-3)^{-1} + 2(x+1)^{-1}$ so that:

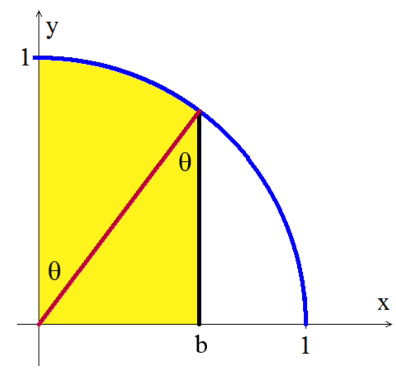
$$\begin{aligned} g'(x) &= -7(x-3)^{-2} - 2(x+1)^{-2} \\ g''(x) &= 14(x-3)^{-3} + 4(x+1)^{-3} \\ g'''(x) &= -42(x-3)^{-4} - 12(x+1)^{-4} \\ g^{(4)}(x) &= 168(x-3)^{-5} + 48(x+1)^{-5} \end{aligned}$$

8.4 Trigonometric Substitution

In the previous section, you learned how to decompose rational expressions into simpler forms that were easier to integrate. This section investigates a method for transforming integrands involving radical expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ using a specialized change of variable.

While we know that $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x)+C$, and while we can compute $\int \frac{x}{\sqrt{1-x^2}}\,dx$ and $\int x\sqrt{1-x^2}\,dx$ without much difficulty using the substitution $u = 1-x^2$, the simpler-looking integral $\int \sqrt{1-x^2}\,dx$ poses a greater challenge. It does, however, have a geometric interpretation that allows us to find an antiderivative.

Consider $\int_0^b \sqrt{1-x^2}\,dx$ for any b with $0 < b < 1$. This definite integral represents the area of the region under the curve $y = \sqrt{1-x^2}$ between the y-axis (where $x = 0$) and the vertical line $x = b$ (see margin). We can split this region into two pieces (as in the margin figure): a sector of a circle and a triangle.



The triangle has base b and height $\sqrt{1-b^2}$, so its area is $\frac{1}{2}b\cdot\sqrt{1-b^2}$. The area of a sector of a circle with radius r and central angle θ is $\frac{\theta}{2\pi}\cdot\pi r^2 = \frac{1}{2}r^2\theta$. But $r = 1$ and θ is also the angle of the upper vertex of the triangle, so:

$$\sin(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{1} = b \;\Rightarrow\; \theta = \arcsin(b)$$

which tells us:

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2}b\cdot\sqrt{1-b^2} + \frac{1}{2}\arcsin(b)$$

and we can easily deduce that:

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\cdot\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$

If $b = 1$, the integral should give the area of a quarter-circle of radius 1, and we can then verify that:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{2}\cdot 1\cdot 0 + \frac{1}{2}\arcsin(1) = 0 + \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

The appearance of $\arcsin(x)$ in the antiderivative of $\sqrt{1-x^2}$ suggests a somewhat unusual substitution. Because $\arcsin(x)$ represents an angle, we can call it θ and write:

$$\arcsin(x) = \theta \;\Rightarrow\; x = \sin(\theta) \;\Rightarrow\; dx = \cos(\theta)\,d\theta$$

If we make the substitution $x = \sin(\theta)$, the integral becomes:

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)\,d\theta = \int \sqrt{\cos^2(\theta)}\cdot\cos(\theta)\,d\theta$$
$$= \int \cos(\theta)\cdot\cos(\theta)\,d\theta = \int \cos^2(\theta)\,d\theta$$

Ordinarily we would write:

$$\sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

but $\theta = \arcsin(x)$ and the range of arcsine is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\cos(\theta) \geq 0$.

This is an integral we already know how to work out:

$$\int \cos^2(\theta)\,d\theta = \int \left[\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right]d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$
$$= \frac{1}{2}\theta + \frac{1}{4}\cdot 2\sin(\theta)\cos(\theta) + C = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$$

Now we must convert the variable θ back to x. We started out with $\theta = \arcsin(x) \Rightarrow x = \sin(\theta)$ so we just need to rewrite $\cos(\theta)$ in terms of x. The Pythagorean identity tells us that:

$$\cos^2(\theta) + \sin^2(\theta) = 1 \ \Rightarrow \cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2$$

Ordinarily we would write:

$$\cos(\theta) = \pm\sqrt{1-x^2}$$

but from the discussion in the preceding margin note we know that $\cos(\theta) \geq 0$.

so $\cos(\theta) = \sqrt{1-x^2}$. Therefore:

$$\frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C = \frac{1}{2}\arcsin(x) + \frac{1}{2}\cdot x \cdot \sqrt{1-x^2} + C$$

which agrees with our original geometric result.

Another Change of Variable

The preceding discussion suggests a new type of substitution: instead of setting u = function of x, replace x with a (trigonometric) function of θ. Each trigonometric substitution will involve four major steps:

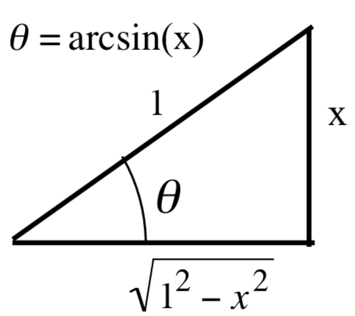
1. Choose a substitution to make x = a trigonometric function of θ.
2. Rewrite the original integral in terms of θ and $d\theta$.
3. Find an antiderivative of the new integrand.
4. Rewrite this antiderivative in terms of the original variable x.

The discussion that follows examines how these steps play out in three situations. The first step requires you to make a decision. The other steps follow from that decision. For most students, the key to success with the Trigonometric Substitution technique is to **think triangles**.

Step 1: Choosing the substitution

The Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$ played an important role in our first application of the trigonometric substitution technique. The familiar Pythagorean Theorem can help guide you to a correct choice of substitution so that a trigonometric identity will always come to the rescue when converting from θ back to x in Step 4. Thinking of a right triangle, we can state the Pythagorean Theorem as:

$$(\text{side})^2 + (\text{other side})^2 = (\text{hypotenuse})^2$$
$$\Rightarrow \ (\text{other side})^2 = (\text{hypotenuse})^2 - (\text{side})^2$$



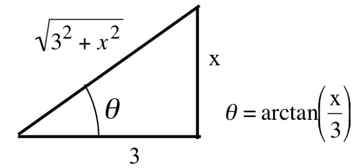
A right triangle representing our initial substitution $x = \sin(\theta)$ appears in the margin. Here x is the length of the side opposite θ and 1 is the length of the hypotenuse, so that the "other side" has length $\sqrt{1-x^2}$, the radical expression that appeared in the original integral.

We now investigate the trigonometric substitutions for three representative patterns: $3^2 + x^2$, $3^2 - x^2$ and $x^2 - 3^2$.

There is nothing special about the number 3 in these examples: we would have used 5 or π or $\sqrt{17}$.

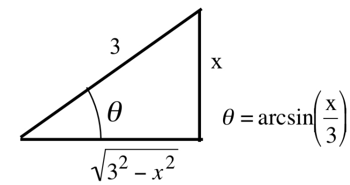
- The pattern $\mathbf{3^2 + x^2}$ matches the Pythagorean pattern if 3 and x are sides of a right triangle (see margin). Then:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3} \Rightarrow x = 3\tan(\theta)$$



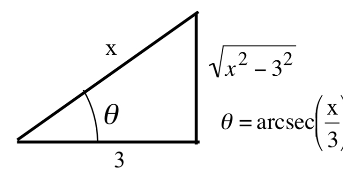
- The pattern $\mathbf{3^2 - x^2}$ matches the Pythagorean pattern if 3 is the hypotenuse and x is a side of a right triangle (see margin). Then:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{3} \Rightarrow x = 3\sin(\theta)$$



- The pattern $\mathbf{x^2 - 3^2}$ matches the Pythagorean pattern if x is the hypotenuse and 3 is a side of a right triangle (see margin). Then:

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{x}{3} \Rightarrow x = 3\sec(\theta)$$



Step 2: Rewrite the original integral in terms of θ and $d\theta$

Once you have made the choice for the substitution, several things follow automatically: you can easily calculate dx, you can solve for θ, and you can rewrite the original pattern of interest as a function of θ.

- With the pattern $\mathbf{3^2 + x^2}$, differentiation yields:

$$x = 3\tan(\theta) \Rightarrow dx = 3\sec^2(\theta)\,d\theta$$

while solving the substitution equation for θ yields:

$$x = 3\tan(\theta) \Rightarrow \tan(\theta) = \frac{x}{3} \Rightarrow \theta = \arctan\left(\frac{x}{3}\right)$$

Here $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ because that is the range of the arctangent function. This means that $\sec(\theta) > 0$ always holds, so that $\sqrt{\sec^2(\theta)} = \sec(\theta)$.

and the original pattern becomes:

$$3^2 + x^2 = 3^2 + 3^2\tan^2(\theta) = 3^2\left[1 + \tan^2(\theta)\right] = 3^2\sec^2(\theta)$$

- With the pattern $\mathbf{3^2 - x^2}$, differentiation yields:

$$x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$$

$$x = 3\sin(\theta) \Rightarrow \sin(\theta) = \frac{x}{3} \Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$$

$$3^2 - x^2 = 3^2 - 3^2\sin^2(\theta) = 3^2\left[1 - \sin^2(\theta)\right] = 3^2\cos^2(\theta)$$

Here $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ because that is the range of the arcsine function. This means that $\cos(\theta) \ge 0$ always holds, so that $\sqrt{\cos^2(\theta)} = \cos(\theta)$.

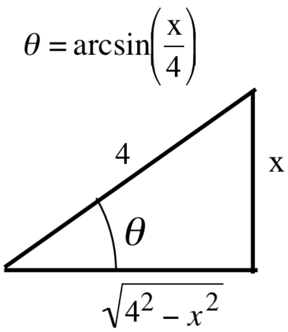
- With the pattern $\mathbf{x^2 - 3^2}$, differentiation yields:

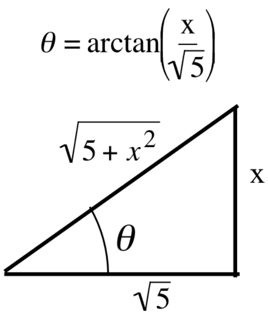
$$x = 3\sec(\theta) \Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$$

$$x = 3\sec(\theta) \Rightarrow \sec(\theta) = \frac{x}{3} \Rightarrow \theta = \text{arcsec}\left(\frac{x}{3}\right)$$

$$x^2 - 3^2 = 3^2\sec^2(\theta) - 3^2 = 3^2\left[\sec^2(\theta) - 1\right] = 3^2\tan^2(\theta)$$

Here $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta < \pi$ because that is the range of the arcsecant function. We will want $\tan(\theta) \ge 0$ to avoid absolute values when simplifying $\sqrt{\tan^2(\theta)}$; to do so, we will need to assume that $x \ge 3$ so that $0 \le \theta < \frac{\pi}{2}$.





Example 1. For the patterns $16 - x^2$ and $5 + x^2$, select an appropriate substitution for x, calculate dx and θ, and use the substitution to simplify the pattern.

Solution. The pattern $16 - x^2$ matches the Pythagorean pattern if 4 is a hypotenuse and x is the side of a right triangle. Then:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{4} \Rightarrow x = 4\sin(\theta) \Rightarrow dx = 4\cos(\theta)\,d\theta$$

while $\theta = \arcsin\left(\frac{x}{4}\right)$ and $16 - x^2$ becomes:

$$16 - [4\sin(\theta)]^2 = 16 - 16\sin^2(\theta) = 16\left[1 - \sin^2(\theta)\right] = 16\cos^2(\theta)$$

The pattern $5 + x^2$ matches the Pythagorean pattern if x and $\sqrt{5}$ are the sides of a right triangle. Then:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{5}} \Rightarrow x = \sqrt{5}\tan(\theta) \Rightarrow dx = \sqrt{5}\sec^2(\theta)\,d\theta$$

while $\theta = \arctan\left(\frac{x}{\sqrt{5}}\right)$ and $5 + x^2$ becomes:

$$5 + \left[\sqrt{5}\tan(\theta)\right]^2 = 5 + 5\tan^2(\theta) = 5\left[1 + \tan^2(\theta)\right] = 5\sec^2(\theta)$$

Drawing a right triangle for each pattern (as in the margin) will help you visualize the appropriate substitution. ◀

Practice 1. For the patterns $25 + x^2$ and $x^2 - 13$, decide on the appropriate substitution for x, calculate dx and θ, and use the substitution to simplify the pattern.

Example 2. Use $x = 5\tan(\theta)$ to rewrite the integral $\int \frac{1}{\sqrt{25 + x^2}}\,dx$.

Solution. Differentiating $x = 5\tan(\theta)$ yields $dx = 5\sec^2(\theta)\,d\theta$ and:

$$25 + x^2 = 25 + 25\tan^2(\theta) = 25\left[1 + \tan^2(\theta)\right] = 25\sec^2(\theta)$$

so that $\sqrt{25 + x^2} = \sqrt{25\sec^2(\theta)} = 5\sec(\theta)$ and the integral becomes:

$$\int \frac{1}{\sqrt{25 + x^2}}\,dx = \int \frac{1}{5\sec(\theta)} \cdot 5\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta$$

which is an integral we already know how to evaluate. ◀

Practice 2. Use $x = 5\sin(\theta)$ to rewrite the integral $\int \frac{1}{\sqrt{25 - x^2}}\,dx$.

Step 3: Find an antiderivative of the new integrand

After changing the variable, the new integrand typically involves products of powers of trigonometric functions and we can use any of our previous methods (another change of variable, integration by parts, a trigonometric identity or integral tables) to find an antiderivative.

Step 4: Rewrite this antiderivative in terms of the original variable

Once we have found an antiderivative, usually involving a trigonometric function of θ, we can replace θ with the appropriate inverse trigonometric function of x and simplify. Because the antiderivatives commonly contain trigonometric functions, we frequently need to simplify a trigonometric function of an inverse trigonometric function, and at this stage it is often *very* helpful to refer back to the right triangle we used at the beginning of the substitution process.

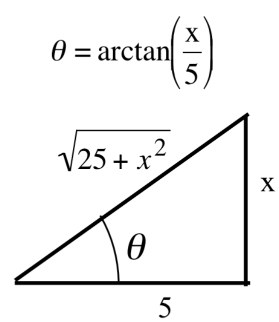
Example 3. Evaluate $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,dx$.

Solution. In Example 2 we used the substitution $x = 5\tan(\theta)$ to convert this integral to:

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \int \sec(\theta)\,d\theta = \ln\left(|\sec(\theta)+\tan(\theta)|\right) + C$$

We now need to rewrite $\sec(\theta)$ and $\tan(\theta)$ in this antiderivative in terms of x using the fact that $\theta = \arctan\left(\frac{x}{5}\right)$:

$$\ln\left(\left|\sec\left(\arctan\left(\frac{x}{5}\right)\right) + \tan\left(\arctan\left(\frac{x}{5}\right)\right)\right|\right) + C$$



Referring to the right triangle for this substitution (see margin):

$$\sec\left(\arctan\left(\frac{x}{5}\right)\right) = \frac{\sqrt{25+x^2}}{5} \quad\text{and}\quad \tan\left(\arctan\left(\frac{x}{5}\right)\right) = \frac{x}{5}$$

so, putting all of these pieces together:

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \ln\left(\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right) + C$$

Where did the absolute value signs go? Because:

$$-\frac{x}{5} \le \left|\frac{x}{5}\right| = \frac{\sqrt{x^2}}{5} < \frac{\sqrt{x^2+25}}{5}$$

we know that:

$$\frac{\sqrt{x^2+25}}{5} + \frac{x}{5} > 0$$

As always, we can check that this is indeed an antiderivative of the original integrand by differentiating. ◀

Practice 3. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$.

Variations on a Theme

While we will typically apply trigonometric substitution to integrands involving radicals of the form $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ or $\sqrt{x^2-a^2}$, we can adapt this technique for more general integrands by incorporating other methods, such as u-substitution and completing the square.

Example 4. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+2x+26}}\,dx$.

Solution. The polynomial inside the radical in this integrand is an irreducible quadratic so — just as we did with rational functions — we can complete the square:

$$x^2 + 2x + 26 = (x+1)^2 + 25$$

and next use the substitution $u = x + 1 \Rightarrow du = dx$:

$$\int \frac{1}{\sqrt{x^2+2x+26}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+5^2}}\,dx = \int \frac{1}{\sqrt{u^2+5^2}}\,du$$

The integrand is now ready for a trigonometric substitution. In fact, this resembles the integral in Example 3, so:

$$\begin{aligned}\int \frac{1}{\sqrt{u^2+5^2}}\,du &= \ln\left(\frac{\sqrt{25+u^2}}{5} + \frac{u}{5}\right) + C \\ &= \ln\left(\sqrt{25+u^2} + u\right) + C - \ln(5) \\ &= \ln\left(\sqrt{25+(x+1)^2} + (x+1)\right) + K\end{aligned}$$

Here we employ the logarithmic identity:

$$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$$

We could have made a similar simplification in Example 3.

or $\ln\left(x+1+\sqrt{x^2+2x+26}\right) + K$. ◀

Wrap-up

When using trig substitution, remember to **think triangles**. The first and last steps (choosing the substitution and writing the answer in terms of x) are much easier if you have drawn the appropriate triangle for the problem. Of course, you also need to practice the method.

8.4 Problems

In Problems 1–8, determine the appropriate substitution for the given integrand.

1. $\sqrt{49-x^2}$
2. $\sqrt{x^2-36}$
3. $\left(81+x^2\right)^{\frac{5}{2}}$
4. $\sqrt{8+x^2}$
5. $\sqrt{x^2-7}$
6. $\sqrt{99-x^2}$
7. $\dfrac{1}{x\sqrt{100-x^2}}$
8. $\dfrac{1}{x^2\sqrt{x^2-100}}$

In Problems 9–14, make the given substitution and simplify the result, then calculate dx.

9. $x = 3 \cdot \sin(\theta)$ in $\dfrac{1}{\sqrt{9-x^2}}$
10. $x = 3 \cdot \tan(\theta)$ in $\dfrac{1}{\sqrt{x^2+9}}$
11. $x = 3 \cdot \sec(\theta)$ in $\dfrac{1}{\sqrt{x^2-9}}$
12. $x = 6 \cdot \sin(\theta)$ in $\dfrac{1}{36-x^2}$

13. $x = \sqrt{2} \cdot \tan(\theta)$ in $\dfrac{1}{\sqrt{2+x^2}}$

14. $x = \sec(\theta)$ in $\dfrac{1}{\sqrt{x^2-1}}$

In Problems 13–18, (a) solve for θ as a function of x, (b) replace θ in $f(\theta)$ with you result from part (a), and (c) simplify.

15. $x = 3 \cdot \sin(\theta)$, $f(\theta) = \cos(\theta) \cdot \tan(\theta)$

16. $x = 3 \cdot \tan(\theta)$, $f(\theta) = \sin(\theta) \cdot \tan(\theta)$

17. $x = 3 \cdot \sec(\theta)$, $f(\theta) = \sqrt{1+\sin^2(\theta)}$

18. $x = 5 \cdot \sin(\theta)$, $f(\theta) = \dfrac{\cos(\theta)}{1+\sec(\theta)}$

19. $x = 5 \cdot \tan(\theta)$, $f(\theta) = \dfrac{\cos^2(\theta)}{1+\cot(\theta)}$

20. $x = 5 \cdot \sec(\theta)$, $f(\theta) = \cos(\theta) + 7 \cdot \tan^2(\theta)$

In Problems 21–44, evaluate the integral. (More than one method works for some of the integrals.)

21. $\displaystyle\int \frac{1}{x\sqrt{9-x^2}}\,dx$

22. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

23. $\displaystyle\int \frac{1}{\sqrt{x^2+49}}\,dx$

24. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

25. $\displaystyle\int \sqrt{36-x^2}\,dx$

26. $\displaystyle\int \sqrt{1-36x^2}\,dx$

27. $\displaystyle\int \frac{1}{\sqrt{36+x^2}}\,dx$

28. $\displaystyle\int \frac{1}{x\sqrt{25-x^2}}\,dx$

29. $\displaystyle\int \frac{x^2}{\sqrt{49-x^2}}\,dx$

30. $\displaystyle\int \frac{\sqrt{25-x^2}}{x^2}\,dx$

31. $\displaystyle\int \frac{x}{\sqrt{25-x^2}}\,dx$

32. $\displaystyle\int \frac{1}{x^2+49}\,dx$

33. $\displaystyle\int \frac{x}{x^2+49}\,dx$

34. $\displaystyle\int \frac{1}{49x^2+25}\,dx$

35. $\displaystyle\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\,dx$

36. $\displaystyle\int \frac{1}{(4x^2-1)^{\frac{3}{2}}}\,dx$

37. $\displaystyle\int \frac{5}{2x\sqrt{x^2-25}}\,dx$

38. $\displaystyle\int \frac{1}{x\sqrt{3-x^2}}\,dx$

39. $\displaystyle\int \frac{1}{25-x^2}\,dx$

40. $\displaystyle\int \frac{1}{a^2+x^2}\,dx$

41. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\,dx$

42. $\displaystyle\int \frac{1}{x\sqrt{a^2+x^2}}\,dx$

43. $\displaystyle\int \frac{1}{x^2\sqrt{a^2+x^2}}\,dx$

44. $\displaystyle\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx$

In Problems 45–50, complete the square and make an appropriate substitution (as necessary), then evaluate the integral.

45. $\displaystyle\int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$

46. $\displaystyle\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$

47. $\displaystyle\int \frac{1}{x^2+10x+29}\,dx$

48. $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$

49. $\displaystyle\int \frac{1}{\sqrt{x^2+4x+3}}\,dx$

50. $\displaystyle\int \frac{1}{\sqrt{x^2-6x-16}}\,dx$

51. The integral $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$ arises when finding antiderivatives using partial fractions.
   (a) Evaluate this integral using an appropriate trigonometric substitution.
   (b) Now evaluate it using integration by parts.
   (c) Which method is easier?

52. Evaluate $\displaystyle\int \frac{1}{(x^2-8x+25)^2}\,dx$.

53. Evaluate $\displaystyle\int \frac{8x}{(x^2+25)^2}\,dx$.

8.4 Practice Answers

1. For the pattern $25 + x^2$, use $x = 5 \cdot \tan(\theta) \Rightarrow dx = 5 \cdot \sec^2(\theta)\, d\theta$, while $\theta = \arctan\left(\frac{x}{5}\right)$ and:

$$25 + x^2 = 25 + 25\tan^2(\theta) = 25\left[1 + \tan^2(\theta)\right] = 25\sec^2(\theta)$$

For $x^2 - 13$, use $x = \sqrt{13}\sec(\theta) \Rightarrow dx = \sqrt{13}\sec(\theta) \cdot \tan(\theta)\, d\theta$, while $\theta = \operatorname{arcsec}\left(\frac{x}{\sqrt{13}}\right)$ and:

$$x^2 - 13 = 13\sec^2(\theta) - 13 = 13\left[\sec^2(\theta) - 1\right] = 13\tan^2(\theta)$$

2. $x = 5\sin(\theta) \Rightarrow dx = 5\cos(\theta)\, d\theta$, while $\theta = \arcsin\left(\frac{x}{5}\right)$ so:

$$25 - x^2 = 25\left[1 - \sin^2(\theta)\right] = 25\cos^2(\theta)$$

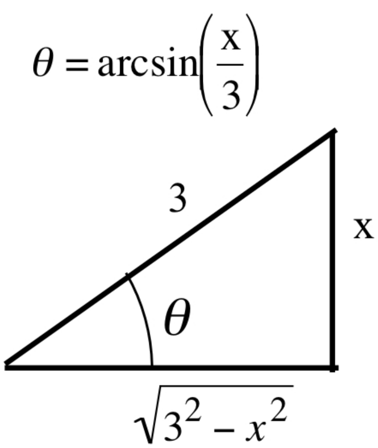
and the integral becomes:

$$\int \frac{1}{\sqrt{25 - x^2}}\, dx = \int \frac{1}{\sqrt{25\cos^2(\theta)}} \cdot 5\cos(\theta)\, d\theta = \frac{5}{5}\int \frac{\cos(\theta)}{\cos(\theta)}\, d\theta$$
$$= \int 1\, d\theta = \theta + C = \arcsin\left(\frac{x}{5}\right) + C$$

3. Use $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\, d\theta$ while $\theta = \arcsin\left(\frac{x}{2}\right)$ and:

$$9 - x^2 = 9\left[1 - \sin^2(\theta)\right] = 9\cos^2(\theta)$$

Then the integral becomes:



$$\int \frac{1}{x^2\sqrt{9 - x^2}}\, dx = \int \frac{1}{9\sin^2(\theta)\sqrt{9\cos^2(\theta)}} \cdot 3\cos(\theta)\, d\theta$$
$$= \int \frac{3\cos(\theta)}{27\sin^2(\theta)\cos(\theta)}\, d\theta = \frac{1}{9}\int \csc^2(\theta)\, d\theta$$
$$= -\frac{1}{9}\cot(\theta) + C = -\frac{1}{9}\cot\left(\arcsin\left(\frac{x}{2}\right)\right) + C$$
$$= -\frac{1}{9} \cdot \frac{\sqrt{9 - x^2}}{x} + C$$

8.5 Integrals of Trigonometric Functions

In the previous section, we learned how to turn integrands involving various radical and rational expressions containing the variable x into functions consisting of products of powers of trigonometric functions of θ. An overwhelming number of combinations of trigonometric functions can appear in these integrals, but fortunately most fall into a few general patterns—and most can be integrated using reduction formulas and integral tables. This section examines some of these patterns and illustrates how to obtain some of their integrals.

Integrals of functions of this type also arise in other mathematical applications, such as Fourier series.

Products of $\sin(ax)$ *and* $\cos(bx)$

We can handle the integrals $\int \sin(ax) \cdot \sin(bx)\, dx$, $\int \cos(ax) \cdot \cos(bx)\, dx$ and $\int \sin(ax) \cdot \cos(bx)\, dx$ by referring to the trigonometric identities for sums and differences of sine and cosine:

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\
\sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B) \\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\
\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)
\end{aligned}$$

By adding or subtracting pairs of identities, we can write products such as $\sin(ax)\cos(bx)$ as a sum or difference of single sines or cosines. For example, by adding the first two identities, we get:

$$\begin{aligned}
& 2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B) \\
\Rightarrow \quad & \sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]
\end{aligned}$$

Using this last identity (for $a \neq b$):

$$\begin{aligned}
\int \sin(ax)\cos(bx)\, dx &= \int \frac{1}{2}\Big[\sin\left((a+b)x\right) + \sin\left((a-b)x\right)\Big]\, dx \\
&= \frac{1}{2}\left[-\frac{\cos\left((a+b)x\right)}{a+b} - \frac{\cos\left((a-b)x\right)}{a-b}\right] + C
\end{aligned}$$

The other integrals of products of sine and cosine follow similarly.

If $a \neq b$, then:

$$\int \sin(ax)\sin(bx)\, dx = \frac{1}{2}\left[\frac{\sin\left((a-b)x\right)}{a-b} - \frac{\sin\left((a+b)x\right)}{a+b}\right] + C$$

$$\int \cos(ax)\cos(bx)\, dx = \frac{1}{2}\left[\frac{\sin\left((a-b)x\right)}{a-b} + \frac{\sin\left((a+b)x\right)}{a+b}\right] + C$$

$$\int \sin(ax)\cos(bx)\, dx = -\frac{1}{2}\left[\frac{\cos\left((a-b)x\right)}{a-b} + \frac{\cos\left((a+b)x\right)}{a+b}\right] + C$$

If $a = b$, we have already developed the relevant integral patterns:

$$\boxed{\begin{aligned} \int \sin^2(ax)\,dx &= \frac{x}{2} - \frac{\sin(2ax)}{4a} + C = \frac{x}{2} - \frac{\sin(ax)\cdot\cos(ax)}{2a} + C \\ \int \cos^2(ax)\,dx &= \frac{x}{2} + \frac{\sin(2ax)}{4a} + C = \frac{x}{2} + \frac{\sin(ax)\cdot\cos(ax)}{2a} + C \\ \int \sin(ax)\cos(ax)\,dx &= \frac{\sin^2(ax)}{2a} + C = \frac{1-\cos(2ax)}{4a} + C \end{aligned}}$$

The first and second of these integral formulas follow from the identities $\sin^2(ax) = \frac{1}{2} - \frac{1}{2}\cos^2(2ax)$ and $\cos^2(ax) = \frac{1}{2} + \frac{1}{2}\cos^2(2ax)$. The third can be obtained by changing the variable to $u = \sin(ax)$.

Powers of Sine and Cosine Alone: $\int \sin^n(x)\,dx$ *or* $\int \cos^n(x)\,dx$

See Problems 25 and 26 from Section 8.2.

We can find antiderviatives of $\sin^n(x)$ or $\cos^n(x)$ using integration by parts or reduction formulas that we obtained using integration by parts. For small values of n we can also find the antiderivatives directly.

For **even powers** of sine or cosine, we can reduce the exponent by repeatedly applying the identities $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ and $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$.

Example 1. Evaluate $\int \sin^4(x)\,dx$.

Solution. Applying the identity for $\sin^2(\theta)$, we can write $\sin^4(x)$ as:

$$\left[\sin^2(x)\right]^2 = \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]^2 = \frac{1}{4}\left[1 - 2\cos(2x) + \cos^2(2x)\right]$$

and integrating gives:

$$\begin{aligned} \int \sin^4(x)\,dx &= \frac{1}{4}\int \left[1 - 2\cos(2x) + \cos^2(2x)\right]\,dx \\ &= \frac{1}{4}\left[x - \sin(2x) + \frac{x}{2} + \frac{1}{8}\sin(4x)\right] + C \\ &= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C \end{aligned}$$

using the formula for $\int \cos^2(2u)\,du$. ◀

Practice 1. Evaluate $\int \cos^4(x)\,dx$.

For **odd powers** of sine or cosine we can split off one factor of sine or cosine and rewrite the remaining even power using the identities $\sin^2(\theta) = 1 - \cos^2(\theta)$ or $\cos^2(\theta) = 1 - \sin^2(\theta)$, then integrate by changing the variable.

Example 2. Evaluate $\int \sin^5(x)\,dx$.

Solution. First split off one power of sine, writing:

$$\sin^5(x) = \sin^4(x) \cdot \sin(x) = \left[\sin^2(x)\right]^2 \cdot \sin(x) = \left[1 - \cos^2(x)\right]^2 \sin(x)$$

and then integrate, using the substitution $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^5(x)\,dx &= \int \left[1-\cos^2(x)\right]^2 \sin(x)\,dx = -\int \left[1-u^2\right]^2 du \\ &= -\int \left[1 - 2u^2 + u^4\right] du = -\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + C \\ &= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C\end{aligned}$$

The reduction formula obtained in Problem 25 of Section 8.2 yields:

$$\int \sin^5(x)\,dx = \frac{1}{5}\sin^4(x)\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) - \frac{8}{15}\cos(x) + K$$

which looks nothing like the result above, but these functions (aside from the different constants of integration) are in fact equal. ◀

You should be able to *see* this by graphing the two functions, and *prove* this using trig identities.

Practice 2. Evaluate $\int \cos^5(x)\,dx$.

Patterns for $\int \sin^m(x)\cos^n(x)\,dx$

For integrands of the form $\sin^m(x)\cos^n(x)$, if the **exponent of sine is odd**, you can split off one factor of $\sin(x)$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to rewrite the remaining even power of sine in terms of cosine, then change the variable using $u = \cos(x)$.

Example 3. Evaluate $\int \sin^3(x)\cos^6(x)\,dx$.

Solution. First split off a power of sine, writing:

$$\sin^3(x)\cos^6(x) = \sin(x)\sin^2(x)\cos^6(x) = \sin(x)\left[1-\cos^2(x)\right]\cos^6(x)$$

and then use the substitution $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^3(x)\cos^6(x) &= \int \sin(x)\left[1-\cos^2(x)\right]\cos^6(x)\,dx \\ &= \int -\left[1-u^2\right]u^6\,du = \int \left[u^8 - u^6\right] du \\ &= \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cos^9(x) - \frac{1}{7}\cos^7(x) + C\end{aligned}$$

You can verify this is the correct antiderivative by differentiating the result and comparing it to the original integrand. ◀

You may need to use some trig identities.

Practice 3. Evaluate $\int \sin^3(x)\cos^4(x)\,dx$.

If the **exponent of cosine** is odd, split off one $\cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to rewrite the remaining even power of cosine in terms of sine. Then use the change of variable $u = \sin(x)$.

If **both exponents are even**, use the identities $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ to rewrite the integral in terms of powers of $\cos(2x)$, then proceed by integrating even powers of cosine.

Powers of Secant or Tangent Alone

You integrate any power of $\sec(x)$ and $\tan(x)$ by knowing that:

$$\int \sec(x)\,dx = \ln\left(|\sec(x) + \tan(x)|\right) + C$$

$$\text{and} \quad \int \tan(x)\,dx = \ln\left(|\sec(x)|\right) + C$$

and using the reduction formulas:

See Problems 27 and 28 in Section 8.2.

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\cdot\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\text{and} \quad \int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

Example 4. Evaluate $\int \sec^3(x)\,dx$.

Solution. Using the first reduction formula with $n = 3$:

$$\begin{aligned}\int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\int \sec(x)\,dx \\ &= \frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\ln\left(|\sec(x) + \tan(x)|\right) + C\end{aligned}$$

You could also write $\sec^3(x) = \sec(x)\cdot\sec^2(x)$ and work out this integral directly using integration by parts. ◀

Practice 4. Evaluate $\int \tan^3(x)\,dx$ and $\int \sec^5(x)\,dx$.

Patterns for $\int \sec^m(x)\tan^n(x)\,dx$

The patterns for evaluating $\int \sec^m(x)\tan^n(x)\,dx$ resemble those for $\int \sin^m(x)\cos^n(x)\,dx$: we treat the even and odd powers differently and we use the identities $\tan^2(\theta) = \sec^2(\theta) - 1$ and $\sec^2(\theta) = \tan^2(\theta) + 1$.

If the **exponent of secant is even**, factor off $\sec^2(x)$, replace the other even powers (if any) of secant using $\sec^2(x) = \tan^2(x) + 1$, and then make the change of variable $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$.

If the **exponent of tangent is odd**, factor off $\sec(x)\tan(x)$, replace any remaining even powers of tangent using $\tan^2(x) = \sec^2(x) - 1$, and then make the change of variable $u = \sec(x) \Rightarrow du = \sec(x)\tan(x)\,dx$.

If the **exponent of secant is odd and the exponent of tangent is even**, replace the even powers of tangent using $\tan^2(x) = \sec^2(x) - 1$. Then the integral contains only powers of secant, and you can use the strategy for integrating powers of secant alone.

Example 5. Evaluate $\int \sec(x)\tan^2(x)\,dx$.

Solution. Because the exponent of secant is odd and and the exponent of tangent is even, we can use the last method mentioned above and replace $\tan^2(x)$ with $\sec^2(x) - 1$. Then:

$$\begin{aligned}\int \sec(x)\tan^2(x)\,dx &= \int \sec(x)\left[\sec^2(x)-1\right]\,dx = \int \sec^3(x)\,dx - \int \sec(x)\,dx \\ &= \frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\ln\left(|\sec(x)+\tan(x)|\right) - \ln\left(|\sec(x)+\tan(x)|\right) + C \\ &= \frac{1}{2}\sec(x)\cdot\tan(x) - \frac{1}{2}\ln\left(|\sec(x)+\tan(x)|\right) + C\end{aligned}$$

where we used the result of Example 4 for $\int sec^3(x)\,dx$. ◀

Practice 5. Evaluate $\int \sec^4(x)\tan^2(x)\,dx$.

Wrap-Up

Even if you use integral tables (or computers) for most of your future work, it is important to realize that most of the integral patterns for products of powers of trigonometric functions can be obtained using some basic trigonometric identities and the techniques we have discussed in this and earlier sections.

8.5 Problems

In Problems 1–36, evaluate the integral. (More than one method works for some of the integrals.)

1. $\int \sin^2(3x)\,dx$
2. $\int \cos^2(5x)\,dx$
3. $\int e^x \sin(e^x)\cos(e^x)\,dx$
4. $\int \frac{1}{x}\sin^2\left(\ln(x)\right)\,dx$
5. $\int_0^\pi \sin^4(3x)\,dx$
6. $\int_0^\pi \cos^4(5x)\,dx$
7. $\int_0^\pi \cos^3(5x)\,dx$
8. $\int_0^\pi \sin^3(7x)\,dx$
9. $\int \sin(7x)\cos(7x)\,dx$
10. $\int \sin(7x)\cos^2(7x)\,dx$
11. $\int \sin(7x)\cos^3(7x)\,dx$
12. $\int \sin^2(\pi x)\cos(\pi x)\,dx$
13. $\int \sin^2(3x)\cos^2(3x)\,dx$
14. $\int \sin^2(\pi x)\cos^3(\pi x)\,dx$
15. $\int \sin^5(x)\cos^2(x)\,dx$
16. $\int \sin^2(x)\cos^5(x)\,dx$
17. $\int \sec^4(4x)\,dx$
18. $\int \tan^4(4x)\,dx$
19. $\int \tan^5(4x)\,dx$
20. $\int \sec^3(4x)\,dx$
21. $\int \sec^2(5x)\tan(5x)\,dx$
22. $\int \sec^2(5x)\tan^2(5x)\,dx$
23. $\int \sec^3(5x)\tan(5x)\,dx$
24. $\int \sec^3(5x)\tan^2(5x)\,dx$

25. $\int \sec^4(\theta)\tan(\theta)\,d\theta$

26. $\int \sec^4(\theta)\tan^2(\theta)\,d\theta$

27. $\int \sec^4(\theta)\tan^4(\theta)\,d\theta$

28. $\int \sec^4(\theta)\tan^{2015}(\theta)\,d\theta$

29. $\int \frac{\sin^2(x)}{\cos^2(x)}\,dx$

30. $\int \frac{1}{\cos^2(x)}\,dx$

31. $\int \cos^4(\theta)\tan^4(\theta)\,d\theta$

32. $\int \cos^4(\theta)\tan^2(\theta)\,d\theta$

33. $\int \sin(x)\cos(3x)\,dx$

34. $\int \sin(7x)\cos(3x)\,dx$

35. $\int \sin(x)\sin(3x)\,dx$

36. $\int \cos(7x)\cos(3x)\,dx$

37. Show that if n is a positive, **odd** integer, then:

$$\int_0^{2\pi} \sin^n(x)\,dx = 0$$

38. Using integral tables or reduction formulas, it is straightforward to show that:

$$\int_0^{2\pi} \sin^2(x)\,dx = \pi$$
$$\int_0^{2\pi} \sin^4(x)\,dx = \frac{3}{4}\pi$$
$$\int_0^{2\pi} \sin^6(x)\,dx = \frac{3}{4}\cdot\frac{5}{6}\pi$$

Evaluate $\displaystyle\int_0^{2\pi} \sin^8(x)\,dx$, then make a prediction about the value of $\int_0^{2\pi} \sin^{10}(x)\,dx$ and evaluate that integral.

The definite integrals of various combinations of sine and cosine on the interval $[0, 2\pi]$ exhibit a number of interesting patterns. For now, these are simply curiosities and a source of additional practice problems, but the patterns are very important as the foundation for an applied topic, Fourier series, which you may encounter in more advanced courses. Problems 39–41 ask you to show that the definite integral on $[0, 2\pi]$ of $\sin(mx)$ multiplied by almost any other $\sin(nx)$ or $\cos(nx)$ is 0. The only nonzero value comes when $\sin(mx)$ is multiplied by itself.

39. Show that if m and n are integers with $m \neq n$, then:

$$\int_0^{2\pi} \sin(mx)\cdot\sin(nx)\,dx = 0$$

40. Show that if m and n are integers, then:

$$\int_0^{2\pi} \sin(mx)\cdot\cos(nx)\,dx = 0$$

(Consider $m = n$ and $m \neq n$ separately.)

41. Show that if $m \neq 0$ is an integer, then:

$$\int_0^{2\pi} \sin(mx)\cdot\sin(mx)\,dx = \pi$$

Problems 42–47, concern the following function $P(x)$, a **trigonometric polynomial**:

$$P(x) = 5\sin(x) + 7\cos(x) - 4\sin(2x) + 8\cos(2x) - 2\sin(3x)$$

In Problems 42–45, use the *results* of Problems 39–41 to quickly evaluate each integral.

42. $a_1 = \frac{1}{\pi}\int_0^{2\pi} \sin(1x)\cdot P(x)\,dx$

43. $a_2 = \frac{1}{\pi}\int_0^{2\pi} \sin(2x)\cdot P(x)\,dx$

44. $a_3 = \frac{1}{\pi}\int_0^{2\pi} \sin(3x)\cdot P(x)\,dx$

45. $a_4 = \frac{1}{\pi}\int_0^{2\pi} \sin(4x)\cdot P(x)\,dx$

46. Describe how the values of a_k in Problems 42–45 are related to the coefficients of $P(x)$, then make up your own trigonometric polynomial $Q(x)$ and see if your description holds for the a_k values calculated from $Q(x)$.

47. Just by knowing the a_k values we can "rebuild" part of P(x). Find a similar method for getting the coefficients of the cosine terms of $P(x)$: $b_k = ??$

8.5 Practice Answers

1. Using $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$, we can write $\cos^4(x)$ as:

$$\left[\cos^2(x)\right]^2 = \left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right]^2 = \frac{1}{4}\left[1 + 2\cos(2x) + \cos^2(2x)\right]$$

and integrating gives:

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\int \left[1 + 2\cos(2x) + \cos^2(2x)\right]\,dx \\ &= \frac{1}{4}\left[x + \sin(2x) + \frac{x}{2} + \frac{1}{8}\sin(4x)\right] + C \\ &= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\end{aligned}$$

2. First split off one power of sine, writing:

$$\cos^5(x) = \cos^4(x)\cdot\cos(x) = \left[\cos^2(x)\right]^2\cdot\cos(x) = \left[1 - \sin^2(x)\right]^2\cos(x)$$

and then integrate, using $u = \sin(x) \Rightarrow du = \cos(x)\,dx$:

$$\begin{aligned}\int \cos^5(x)\,dx = \int \left[1 - \sin^2(x)\right]^2\cos(x)\,dx &= \int \left[1 - u^2\right]^2\,du \\ = \int \left[1 - 2u^2 + u^4\right]\,du &= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + C \\ = \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C\end{aligned}$$

3. First split off a power of sine, writing:

$$\sin^3(x)\cos^4(x) = \sin(x)\sin^2(x)\cos^4(x) = \sin(x)\left[1 - \cos^2(x)\right]\cos^4(x)$$

and then use the substitution $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^3(x)\cos^4(x) &= \int \sin(x)\left[1 - \cos^2(x)\right]\cos^4(x)\,dx \\ &= \int -\left[1 - u^2\right]u^4\,du = \int \left[u^6 - u^4\right]\,du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C\end{aligned}$$

4. For the first integral, write:

$$\tan^3(x) = \tan(x)\cdot\tan^2(x) = \tan(x)\left[\sec^2(x) - 1\right]$$

so that the integral becomes:

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \left[\tan(x)\sec^2(x) - \tan(x)\right]\,dx \\ &= \frac{1}{2}\tan^2(x) - \ln\left(|\sec(x)|\right) + C \\ &= \frac{1}{2}\tan^2(x) + \ln\left(|\cos(x)|\right) + C\end{aligned}$$

For the second integral, use the reduction formula (twice):

$$\begin{aligned}\int \sec^5(x)\,dx &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx \\ &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\left[\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx\right] \\ &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) \\ &\qquad\qquad + \frac{3}{8}\ln\left(|\sec(x)+\tan(x)|\right) + C\end{aligned}$$

5. First write:

$$\begin{aligned}\sec^4(x)\tan^2(x) &= \sec^2(x)\cdot\sec^2(x)\cdot\tan^2(x) \\ &= \sec^2(x)\left[1+\tan^2(x)\right]\tan^2(x)\end{aligned}$$

and then use the substitution $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$:

$$\begin{aligned}\int \sec^4(x)\tan^2(x) &= \int \left[1+u^2\right]u^2\,du = \int \left[u^2+u^4\right]\,du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3(x) + \frac{1}{5}\tan^5(x) + C\end{aligned}$$

8.6 Integration Tactics

This section, currently in preparation, will be included in the official first printing of this edition. It will present a review of the integrations tactics presented earlier in the chapter, outline strategies for selecting an appropriate integration method to attempt to find an antiderivative of a given integrand, explore some more advanced (optional) integration techniques, and include integrands involving hyperbolic and inverse hyperbolic functions.

For the problems starting on the next page, use any integration technique you have learned so far to evaluate the given integrals.

8.6 Problems

1. $\int \sqrt{1-x}\,dx$

2. $\int (1+2x)^{2017}\,dx$

3. $\int x\sqrt{a^2-x^2}\,dx$

4. $\int x\sqrt{1-x}\,dx$

5. $\int \sqrt{a+bx}\,dx$

6. $\int \frac{1}{1+4x^2}\,dx$

7. $\int \frac{x}{1-x^4}\,dx$

8. $\int \frac{\sec^2(\theta)}{1+\tan(\theta)}\,d\theta$

9. $\int \frac{x^4+x^3+1}{x^3}\,dx$

10. $\int \frac{y^2}{\sqrt{1-y^3}}\,dy$

11. $\int \frac{\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{3}{2}}}{x^{\frac{1}{3}}}\,dx$

12. $\int \frac{4x}{(a^2-x^2)^2}\,dx$

13. $\int \left(\sqrt{a}-\sqrt{x}\right)^2\,dx$

14. $\int \frac{3}{1-4x}\,dx$

15. $\int \frac{x^2}{\sqrt{1+x^3}}\,dx$

16. $\int \frac{2x-1}{2x+3}\,dx$

17. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

18. $\frac{1}{4}\int \left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)^2\,dx$

19. $\int \frac{x^3-x^2+2x-1}{1+x^2}\,dx$

20. $\int \frac{\sec^2(\varphi)}{\sqrt{1+\tan(\varphi)}}\,d\varphi$

21. $\int \frac{x^3-5x+7}{3x-4}\,dx$

22. $\int \frac{1}{e^t}\,dt$

23. $\int \frac{\sin(2\theta)}{a+b\cos(2\theta)}\,d\theta$

24. $\int \frac{1}{1+\sin(\varphi)}\,d\varphi$

25. $\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

26. $\int \frac{1}{4x^2+9}\,dx$

27. $\int \frac{1}{1-2x+2x^2}\,dx$

28. $\int \frac{x+1}{x^2+4}\,dx$

29. $\int \frac{x^2+x}{x^2+1}\,dx$

30. $\int \frac{e^y}{\sqrt{1-e^{2y}}}\,dy$

31. $\int \frac{e^{2t}}{\sqrt{1-e^{2t}}}\,dt$

32. $\int \frac{\sin(3\alpha)}{1+\cos^2(3\alpha)}\,d\alpha$

33. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

34. $\int \frac{1}{x\left[1+(\ln(x))^2\right]}\,dx$

35. $\int \frac{x}{(1+x^2)^2}\,dx$

36. $\int x^{2017}\ln(x)\,dx$

37. $\int x^2\arcsin(x)\,dx$

38. $\int t\tan^2(t)\,dt$

39. $\int x^3\sqrt{a^2-x^2}\,dx$

40. $\int x^3\sqrt{a^2-x^4}\,dx$

41. $\int \cos(\theta)\cdot\ln\left(\sin(\theta)\right)\,d\theta$

42. $\int \frac{x^2}{(9-x^2)^{\frac{3}{2}}}\,dx$

43. $\int \frac{x^2}{(1+x^2)^2}\,dx$

44. $\int \frac{y^3}{\sqrt{25-y^2}}\,dy$

45. $\int \frac{\sqrt{x}}{1+x}\,dx$

46. $\displaystyle\int \frac{1}{1-\sqrt{x}}\,dx$

47. $\displaystyle\int \frac{x+3}{\sqrt{1+2x}}\,dx$

48. $\displaystyle\int \frac{x}{(x+1)^2}\,dx$

49. $\displaystyle\int e^{2x}\sin\left(e^x\right)\,dx$

50. $\displaystyle\int x^3 e^{x^2}\,dx$

51. $\displaystyle\int \frac{x^3}{\left(a^2+x^2\right)^2}\,dx$

52. $\displaystyle\int \cos\left(\sqrt{t}\right)\,dt$

53. $\displaystyle\int x^2\sqrt{1-x}\,dx$

54. $\displaystyle\int \frac{\sec^2(\theta)\tan^2(\theta)}{\sqrt{1+\tan(\theta)}}\,d\theta$

55. $\displaystyle\int \frac{x^3-x}{2+3x}\,dx$

56. $\displaystyle\int x\cos(3x)\,dx$

57. $\displaystyle\int \left[1+\cos\left(\frac{\theta}{2}\right)\right]^3\sin\left(\frac{\theta}{2}\right)\,d\theta$

58. $\displaystyle\int \sin^2(2\varphi)\cos(2\varphi)\,d\varphi$

59. $\displaystyle\int \frac{1-e^{-x}}{1+e^{-x}}\,dx$

60. $\displaystyle\int \frac{1}{(1-x)^4}\,dx$

61. $\displaystyle\int \frac{x^2}{(1-x)^4}\,dx$

62. $\displaystyle\int \frac{1}{x^2+8x+20}\,dx$

63. $\displaystyle\int \cot(\theta)\cdot\ln\left(\sin(\theta)\right)\,d\theta$

64. $\displaystyle\int \frac{1}{x\left[\ln(x)\right]^2}\,dx$

65. $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

66. $\displaystyle\int e^{2x}\sqrt{1-e^{2x}}\,dx$

67. $\displaystyle\int \cos\left(\frac{x}{2}\right)\,dx$

68. $\displaystyle\int e^{\tan(\theta)}\sec^2(\theta)\,d\theta$

69. $\displaystyle\int \frac{x^3+x^2+x}{x^2+9}\,dx$

70. $\displaystyle\int \left(a^{\frac{2}{3}}-y^{\frac{2}{3}}\right)^3\,dy$

71. $\displaystyle\int \frac{x+x^3}{\sqrt{4-x^4}}\,dx$

72. $\displaystyle\int \frac{x^3}{\left(1-x^2\right)^2}\,dx$

73. $\displaystyle\int \theta\sec^2(\theta)\,d\theta$

74. $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx$

75. $\displaystyle\int e^{\sqrt{x}}\,dx$

76. $\displaystyle\int_0^a y\left(b-\frac{b}{a}y\right)^3\,dy$

77. $\displaystyle\int \frac{x-x^2}{1+x^2}\,dx$

78. $\displaystyle\int e^{-x}\left(1+e^{-x}\right)^3\,dx$

79. $\displaystyle\left(1-x^3\right)^2\,dx$

80. $\displaystyle\int x\,(a-x)^{\frac{5}{2}}\,dx$

81. $\displaystyle\int \frac{1}{x^2-6x+10}\,dx$

82. $\displaystyle\int \frac{x^3}{1+x^8}\,dx$

83. $\displaystyle\int \frac{e^x}{1-3e^x}\,dx$

84. $\displaystyle\int \frac{\sqrt{1+\ln(x)}}{x}\,dx$

85. $\displaystyle\int \left[\ln(x)\right]^2\,dx$

86. $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

87. $\displaystyle\int x^5 e^{-x^3}\,dx$

88. $\displaystyle\int \frac{x^2}{x^2-4}\,dx$

89. $\displaystyle\int \frac{2x^2+x-1}{x^3+x^2-4x-4}\,dx$

90. $\displaystyle\int \frac{x}{1-x^4}\,dx$

91. $\displaystyle\int \frac{x^3}{x^2+3x+2}\,dx$

92. $\displaystyle\int \frac{x^4}{x^3+2x^2-x-2}\,dx$

93. $\displaystyle\int \frac{e^{3t}}{1-e^{2t}}\,dt$

94. $\displaystyle\int \frac{x^3-1}{x(x+1)^3}\,dx$

95. $\displaystyle\int \frac{1}{x^3-x^2}\,dx$

96. $\displaystyle\int \frac{x^3-1}{x(x-2)^2}\,dx$

97. $\displaystyle\int \frac{1}{(x^2+x)(x-1)^2}\,dx$

98. $\displaystyle\int \frac{1}{e^{2y}-2e^y}\,dy$

99. $\displaystyle\int \frac{x^2}{x^4+12x^3+52x^2+96x+64}\,dx$

100. $\displaystyle\int \frac{x^2+4x+10}{x^3+2x^2+5x}\,dx$

101. $\displaystyle\int \frac{x^2}{x^2-4x+5}\,dx$

102. $\displaystyle\int \frac{1}{x^3+4x^2+8x}\,dx$

103. $\displaystyle\int \frac{x^2}{25-x^4}\,dx$

104. $\displaystyle\int \frac{x}{1-x^8}\,dx$

105. $\displaystyle\int \frac{x}{x^2+x+1}\,dx$

106. $\displaystyle\int \frac{x}{x^3+x^2+4x+4}\,dx$

107. $\displaystyle\int \frac{x^3}{\left(1+x^2\right)^2}\,dx$

108. $\displaystyle\int \frac{x}{x^2-2x+2}\,dx$

109. $\displaystyle\int \frac{1}{\tan(\theta)-\cot(\theta)}\,d\theta$

110. $\displaystyle\int \frac{x^2-2}{1+6x-x^3}\,dx$

111. $\displaystyle\int \frac{x^4+1}{\left(1+x^2\right)^2}\,dx$

112. $\displaystyle\int_0^8 \frac{1}{1+\sqrt[3]{x}}\,dx$

113. $\displaystyle\int_0^{\ln(2)} \sqrt{e^t-1}\,dt$

114. $\displaystyle\int_0^4 \frac{1}{1+\sqrt{x}}\,dx$

115. $\displaystyle\int_2^3 \frac{x}{1-x^4}\,dx$

116. $\displaystyle\int \sqrt{25-9x^2}\,dx$

117. $\displaystyle\int \frac{1}{\sqrt{10x-x^2}}\,dx$

118. $\displaystyle\int \frac{x^2}{\sqrt{4x-x^2}}\,dx$

119. $\displaystyle\int \frac{1}{(9x^2-36x+32)^{\frac{3}{2}}}\,dx$

120. $\displaystyle\int \frac{\left(1-x^2\right)^{\frac{3}{2}}}{x^6}\,dx$

121. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(2\varphi)\cos(5\varphi)\,d\varphi$

122. $\displaystyle\int \frac{\cos^3(\theta)}{\sin^4(\theta)}\,d\theta$

123. $\displaystyle\int \sin(\varphi)\sin(3\varphi)\sin(5\varphi)\,d\varphi$

124. $\displaystyle\int \tan(x)\sqrt{\sec(x)}\,dx$

125. $\displaystyle\int x\cosh(x)\,dx$

126. $\displaystyle\int x^2\sinh(x)\,dx$

127. $\displaystyle\int e^x\sinh(x)\,dx$

128. $\displaystyle\int \tanh(x)\,dx$

129. $\displaystyle\int \coth(x)\,dx$

130. $\displaystyle\int \operatorname{sech}(x)\,dx$

131. $\displaystyle\int \operatorname{argtanh}(x)\,dx$

132. $\displaystyle\int \operatorname{argsinh}(x)\,dx$

133. $\displaystyle\int x\operatorname{argsinh}(x)\,dx$

8.7 MacLaurin Polynomials

In this chapter you have learned to find antiderivatives of a wide variety of elementary functions, but many more such functions fail to have an antiderivative that can be expressed in terms of other elementary functions. By this point, you should easily be able to find antiderviatives for $\sin(x)$, $x \cdot \sin(x)$, $x \cdot \sin(x^2)$ or $x \cdot e^{-x^2}$, but no matter how hard you try, you won't be able to find antiderivatives for $\sin(x^2)$, $\sin(x^3)$ or e^{-x^2}.

An **elementary function** is a function that can be expressed using a finite number of compositions or combinations of exponential functions, logarithms, trigonometric and inverse trig functions, polynomials and constants, using sums, products and exponentiation.

In Section 2.8, you used linear approximations (tangent lines) to approximate values of complicated functions. In this section, we will investigate how to use polynomials — slightly more complicated, but still relatively "nice" functions — to approximate integrands such as $\sin(x^2)$ and then approximate values of definite integrals such as $\int_0^1 \sin(x^2)\,dx$. This approach will motivate the exploration of many of the concepts to follow in the next two chapters.

Proving that you can't find elementary antiderviatives of integrands such as $\sin(x^2)$ turns out to be quite complicated.

Polynomials

Consider any (non-vertical) line: if you know its y-intercept, b, and its slope, m, you can write down an equation of the line: $y = b + mx$. If we write $y = f(x)$, then $f(0) = b + m \cdot 0 = b$ and $f'(0) = m$, so an equation of *any* linear function is completely determined by its value at $x = 0$ and the value of its first derivative at $x = 0$. It turns out that if you know the values of any polynomial $P(x)$ and all of its derivatives at $x = 0$, you can use those values to find a formula for $P(x)$.

Example 1. If $P(x)$ is a cubic polynomial with $P(0) = 7$, $P'(0) = 5$, $P''(0) = 16$ and $P'''(0) = 18$, find a formula for $P(x)$.

Because $P(x)$ is a cubic, its derivatives of order four and higher are all 0.

Solution. Because $P(x)$ is a cubic polynomial, we can write it as:

$$P(x) = A + Bx + Cx^2 + Dx^3$$

for some numbers A, B, C and D. We know that $P(0) = 7$, and substituting $x = 0$ into the formula above tells us that $P(0) = A$, so $A = 7$. We also know that $P'(0) = 5$, while:

$$P'(x) = B + 2Cx + 3Dx^2 \;\Rightarrow\; P'(0) = B$$

so $B = 5$. Similarly, we know that $P''(0) = 16$ while:

$$P''(x) = 2C + 3 \cdot 2Dx \;\Rightarrow\; P''(0) = 2C$$

so $2C = 16 \Rightarrow C = \frac{16}{2} = 8$. Finally, we know that $P'''(0) = 18$ while:

$$P'''(x) = 3 \cdot 2 \cdot D \;\Rightarrow P'''(0) = 6D \;\Rightarrow\; 6D = 18 \;\Rightarrow D = 3$$

Therefore $P(x) = 7 + 5x + 8x^2 + 3x^3$. (You should verify that this cubic polynomial and its derivatives have the values specified above.) ◀

Practice 1. If $P(x)$ is a fourth-degree polynomial with $P(0) =$ ˘3, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$ and $P^{(4)}(0) = 24$, find a formula for $P(x)$.

Now consider a general polynomial of order 5 with coefficients a_0, a_1, a_2, a_3, a_4, a_5:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Observe that $P(0) = a_0$ and then differentiate to get:

$$P'(x) = a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + 4 \cdot a_4 x^3 + 5 \cdot a_5 x^4$$

Putting $x = 0$ into this new equation tells us that $P'(0) = a_1$. Differentiating again yields:

$$P''(x) = 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + 5 \cdot 4 \cdot a_5 x^3$$

so that $P''(0) = 2 \cdot a_2$. Differentiating again:

$$P'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x + 5 \cdot 4 \cdot 3 \cdot a_5 x^2$$

so that $P'''(0) = 3 \cdot 2 \cdot a_3$. Continuing this process, $P^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot a_4$ and $P^{(5)}(0) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot a_5$. In general, we can write $P^{(k)}(0) = k! \cdot a_k$, where:

We call this expression $k!$ "k **factorial**," defined for any positive integer k.

$$k! = k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1$$

and $k = 1, 2, 3, 4$ or 5. If we define $0! = 1$, then the equation $P^{(k)}(0) = k! \cdot a_k$ holds for $k = 0$ as well (the 0-th derivative of a function is just the function itself). And for any integer $k \geq 6$, $a_k = 0$ and $P^{(k)}(x) = 0$, so for any integer $k \geq 0$ we have:

$$P^{(k)}(0) = k! \cdot a_k \;\Rightarrow\; a_k = \frac{P^{(k)}(0)}{k!}$$

Practice 2. If $P(x)$ is a fourth-degree polynomial with $P(0) =$ ˘3, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$ and $P^{(4)}(0) = 24$, find a formula for $P(x)$ using the above formula, then compare with your answer to Practice 1.

There was nothing special about the degree $n = 5$ of the polynomial in the preceding discussion. The formula $a_k = \dfrac{P^{(k)}(0)}{k!}$ holds for *any* polynomial of the form:

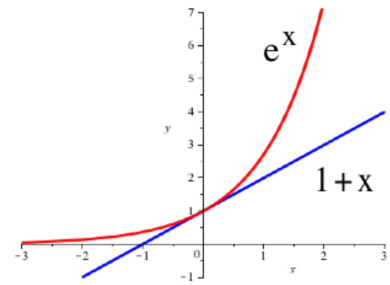
$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

And there was nothing special about the "base point" $x = 0$: we can develop similar formulas if we know the values of a function and all of its derivatives at $x = 1$ or $x = -3$ or $x = \sqrt{7}$ (but we'll stick with $x = 0$ for now, to keep things simple).

Using Polynomials to Approximate Functions

Given any function $f(x)$, we know that a tangent-line approximation to this function at $x = a$ is:

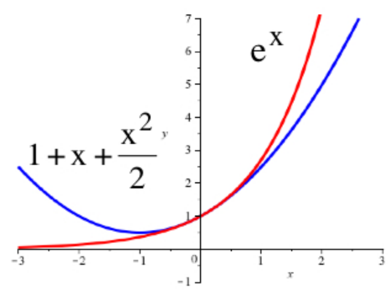
$$L(x) = f(a) + f'(a) \cdot (x - a)$$



If $a = 0$, this becomes $L(x) = f(0) + f'(0) \cdot x$. For $f(x) = e^x$, $f(0) = 1$ and $f'(x) = e^x \Rightarrow f'(0) = 1$, so $L(x) = 1 + x$ (see margin for a graph comparing $L(x)$ and $f(x)$). For values of x very close to 0, we can see that $L(x) = 1 + x \approx e^x = f(x)$ is a decent approximation, but for values of x not close to 0, we need a better approximation.

For the linear approximation $L(x)$, its 0-th derivative agrees with the 0-th derivative of $f(x)$ at $x = 0$: $L(0) = 1 = e^0 = f(0)$. Likewise, the first derivatives agree at $x = 0$: $L'(0) = 1 = e^0 = f'(0)$. But the second derivatives do not agree: $L''(0) = 0 \neq 1 = e^0 = f''(0)$. Can we find a reasonably simple function whose 0-th, first and second derivatives at $x = 0$ match those of $f(x) = e^x$? The next simplest function after a linear function is a quadratic function, so let's try $Q(x) = A + Bx + Cx^2$, for which $Q'(x) = B + 2Cx$ and $Q''(x) = 2C$. We need:

$$\begin{aligned} Q(0) = f(0) &\Rightarrow A = e^0 = 1 \\ Q'(0) = f'(0) &\Rightarrow B = e^0 = 1 \\ Q''(0) = f''(0) &\Rightarrow 2C = e^0 = 1 \Rightarrow C = \frac{1}{2} \end{aligned}$$



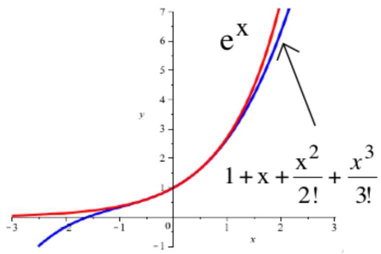
so $Q(x) = 1 + x + \frac{1}{2}x^2$ (see margin for a graph of $Q(x)$ and $f(x)$). While $L(x)$ did a decent job of approximating $f(x) = e^x$ on the interval $[-0.2, 0.2]$, our quadratic approximation $Q(x)$ appears to do a nice job on the interval $[-1, 1]$, but could we do better?

A cubic polynomial is not much more complicated than a quadratic, so let's look for something of the form $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. We want derivatives 0 through 3 of our polynomial to match derivatives 0 through 3 of the function $f(x) = e^x$. We know that:

$$f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x \Rightarrow f'''(x) = e^x$$

so that $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$ and $f'''(0) = 1$. We therefore need $P(0) = 1$, $P'(0) = 1$, $P''(0) = 1$ and $P'''(0) = 1$. From our work earlier in this section we know this requires that $a_k = \dfrac{P^{(k)}(0)}{k!}$ for $k = 0$, 1, 2 and 3. Thus:

$$\begin{aligned} a_0 &= \frac{P(0)}{0!} = \frac{1}{1} = 1 \\ a_1 &= \frac{P'(0)}{1!} = \frac{1}{1} = 1 \end{aligned}$$



$$a_2 = \frac{P''(0)}{2!} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$
$$a_3 = \frac{P'''(0)}{3!} = \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{6}$$

which tells us that $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ (see margin for a graphs of $P(x)$ and $f(x)$). This polynomial appears to approximate $f(x) = e^x$ quite well on an even bigger interval.

Could we do even better? You may notice a pattern in our work above. If we used a fourth-order polynomial, then we would also need:

$$a_4 = \frac{P^{(4)}(0)}{4!} = \frac{f^{(4)}(0)}{4!} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{24}$$

so $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ would approximate e^x even better. In general, for any positive integer n, we would have:

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n$$

Named after Scottish mathematician Colin MacLaurin (1698–1746).

We call this a **MacLaurin polynomial** of order n for $f(x) = e^x$.

Example 2. Find a MacLaurin polynomial with three nonzero terms for $f(x) = \sin(x)$.

Solution. We want to find a polynomial of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

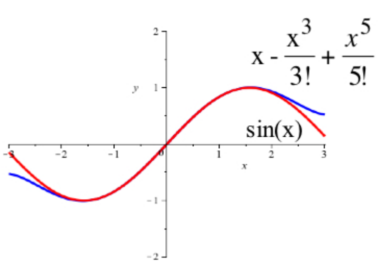
so that $P^{(k)}(0) = f^{(k)}(0)$ for three nonzero values. We know that:

$$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x)$$
$$\Rightarrow f'''(x) = -\cos(x) \Rightarrow f^{(4)}(x) = \sin(x)$$

and that this pattern then repeats. This tells us that:

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 1$$

so that $P(0) = f(0) = 0$, $P'(0) = f'(0) = 1$, $P''(0) = f''(0) = 0$, $P'''(0) = f'''(0) = -1$, $P^{(4)}(0) = f^{(4)}(0) = 0$ and $P^{(5)}(0) = f^{(5)}(0) = 1$. Using the formula $a_k = \frac{P^{(k)}(0)}{k!}$ yields:



$$P(x) = \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

so that $P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ should do the job. (See the margin for a graph of this polynomial compared to $\sin(x)$.) ◀

Practice 3. Find a MacLaurin polynomial with five nonzero terms for $f(x) = \sin(x)$.

Practice 4. Find a MacLaurin polynomial with three nonzero terms for $f(x) = \cos(x)$.

Applications of MacLaurin Polynomials

In Section 7.6, we developed a concrete definition of the exponential function $\exp(x) = e^x$, but this definition did not provide us with a useful way to compute values of e^x (other than $e^0 = 1$). To estimate the value of $e = e^1$ using the definition of e^x, we would need to use the numerical techniques of Section 4.9 to evaluate $L(b) = \int_1^b \frac{1}{t}\,dt$ for various values of b until we found a value for which $L(b) \approx 1$.

In this section, however, we have found polynomials that closely approximate e^x, so we can evaluate one of these polynomials at $x = 1$ to approximate e:

$$e = e^1 \approx 1 + 1 + \frac{1}{2!} \cdot 1^2 + \frac{1}{3!} \cdot 1^3 + \frac{1}{4!} \cdot 1^4 + \frac{1}{5!} \cdot 1^5 + \frac{1}{6!} \cdot 1^6 \approx 2.178$$

Using higher-degree MacLaurin polynomials for e^x will result in even better approximations of e.

Practice 5. Use a MacLaurin polynomial to approximate $\dfrac{1}{e}$ and $\sqrt{e}$.

Practice 6. Use a MacLaurin polynomial to approximate $\sin(1)$ and compare you approximation to what your calculator reports for $\sin(1)$.

In Section 4.9, we learned various numerical integration techniques to approximate values of definite integrals. With enough computing power available, techniques such as the Trapezoidal Rule and Simpson's Rule allow you to approximate values of definite integrals such as $\displaystyle\int_0^1 e^{-x^2}\,dx$ or $\displaystyle\int_0^1 \sin(x^2)\,dx$ (for which it is impossible to use the Fundamental Theorem of Calculus unless we can think of an antiderivative of the integrand). Unfortunately, these approximation methods require you (or a computer) to evaluate the integrand at many different values of x. Now that we know how to approximate transcendental functions with polynomials, we might first approximate a complicated integrand with a "nice" polynomial and then integrate the polynomial instead of the transcendental function.

Example 3. Approximate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. We don't know how to find an antiderivatve for e^{-x^2}, so we can't use the Fundamental Theorem of Calculus. But we already know a MacLaurin polynomial for e^u:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5$$

into which we can substitute $u = -x^2$ to get:

$$e^{-x^2} \approx 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}$$

and use this polynomial to approximate the integrand:

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx \int_0^1 \left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}\right] dx \\ &= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 - \frac{1}{1320}x^{11}\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} \approx 0.7467\end{aligned}$$

The Trapezoidal Rule with $n = 25$ yields a similar approximation, but requires many more computations. ◀

Practice 7. Approximate the value of $\displaystyle\int_0^1 \sin(x^2)\,dx$.

How Good Are These Approximations?

Section 4.9 included (without proof) error bounds for the Trapezoidal Rule and Simpson's Rule that provided guarantees for how closely the results of these numerical methods approximated the exact values of the integrals we were trying to compute. We can — and will — state (and prove) a similar error bound for the MacLaurin polynomial approximations we have learned how to use in this section, but we will defer that discussion until Chapter 10.

8.7 Problems

In Problems 1–6, $P(x) = Ax + B$ is a linear polynomial, with the values of $P(0)$ and $P'(0)$ given. Find A and B and then write a formula for $P(x)$.

1. $P(0) = 5$, $P'(0) = 3$
2. $P(0) = -2$, $P'(0) = 7$
3. $P(0) = 4$, $P'(0) = -1$
4. $P(0) = 8$, $P'(0) = 5$
5. $P(0) = 4$, $P'(0) = 0$
6. $P(0) = -3$, $P'(0) = -2$
7. If $P(x) = A + Bx$, how are the values of A and B related to the values of $P(0)$ and $P'(0)$?

In 8–13, $P(x) = A + Bx + Cx^2$ is a quadratic polynomial, with values of $P(0)$, $P'(0)$ and $P''(0)$ given. Find A, B and C, then write a formula for $P(x)$.

8. $P(0) = 5$, $P'(0) = 3$, $P''(0) = 4$
9. $P(0) = -2$, $P'(0) = 7$, $P''(0) = 6$
10. $P(0) = 4$, $P'(0) = -1$, $P''(0) = -2$
11. $P(0) = 8$, $P'(0) = 5$, $P''(0) = 10$
12. $P(0) = 4$, $P'(0) = 0$, $P''(0) = -4$
13. $P(0) = -3$, $P'(0) = -2$, $P''(0) = 4$
14. If $P(x) = A + Bx + Cx^2$, how are the values of A, B and C related to $P(0)$, $P'(0)$ and $P''(0)$?

In Problems 15–20, $P(x) = A + Bx + Cx^2 + Dx^3$ is a cubic polynomial, with values of $P(0)$, $P'(0)$, $P''(0)$ and $P'''(0)$ given. Find the values of A, B, C and D, and then write a formula for $P(x)$.

15. $P(0) = 5$, $P'(0) = 3$, $P''(0) = 4$, $P'''(0) = 6$
16. $P(0) = -2$, $P'(0) = 7$, $P''(0) = 6$, $P'''(0) = 18$
17. $P(0) = 4$, $P'(0) = -1$, $P''(0) = -2$, $P'''(0) = -12$
18. $P(0) = 8$, $P'(0) = 5$, $P''(0) = 10$, $P'''(0) = 12$
19. $P(0) = 4$, $P'(0) = 0$, $P''(0) = -4$, $P'''(0) = 36$
20. $P(0) = -3$, $P'(0) = -2$, $P''(0) = 4$, $P'''(0) = 36$

21. If $P(x) = A + Bx + Cx^2 + Dx^3$, how are the values of A, B, C and D related to the values of $P(0)$, $P'(0)$, $P''(0)$ and $P'''(0)$?

In Problems 22–28, fill in the table below for $f(x)$ and $P(x)$, then graph $f(x)$ and $P(x)$ for $-2 \le x \le 2$.

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x) \rvert$
0.0			
0.1			
0.2			
0.3			
1.0			
2.0			

22. $f(x) = \sin(x)$, $P(x) = x - \dfrac{1}{2 \cdot 3}x^3$
23. $f(x) = \sin(x)$, $P(x) = x - \dfrac{1}{2 \cdot 3}x^3 + \dfrac{1}{2 \cdot 3 \cdot 4 \cdot 5}x^5$
24. $f(x) = \cos(x)$, $P(x) = 1 - \dfrac{1}{2}x^2$
25. $f(x) = \cos(x)$, $P(x) = 1 - \dfrac{1}{2}x^2 + \dfrac{1}{2 \cdot 3 \cdot 4}x^4$
26. $f(x) = e^x$, $P(x) = 1 + x + \dfrac{1}{2}x^2$
27. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3$
28. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4}x^4$

In 29–38, find a MacLaurin polynomial with four nonzero terms that approximates the given function.

29. e^{2x}
30. $\sin(2x)$
31. $\dfrac{1}{1-x}$
32. $\dfrac{1}{1+x}$
33. $\ln(1+x)$
34. $\sqrt{1+x}$
35. $\cos\left(x^2\right)$
36. e^{-x^2}
37. $x^3 \cdot \sin\left(x^2\right)$
38. $x \cdot e^{-x^2}$

In 39–42, use a MacLaurin polynomial with four nonzero terms to approximate the value of the definite integral.

39. $\displaystyle\int_0^1 \sin\left(x^3\right)\,dx$
40. $\displaystyle\int_0^{\frac{1}{2}} \cos\left(x^2\right)\,dx$
41. $\displaystyle\int_0^{\frac{1}{2}} e^{-x^3}\,dx$
42. $\displaystyle\int_0^1 x \cdot \sin\left(x^3\right)\,dx$

8.7 Practice Answers

1. With $P(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, $P'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 \Rightarrow P''(x) = 2C + 6Dx + 12Ex^2 \Rightarrow P'''(x) = 6D + 24Ex \Rightarrow P^{(4)}(x) = 24E$ so $3 = P(0) = A \Rightarrow A = 3$, $4 = P'(0) = B \Rightarrow B = 4$, $10 = P''(0) = 2C \Rightarrow C = 5$, $12 = P'''(0) = 6D \Rightarrow D = 2$ and $24 = P^{(4)}(0) = 24E \Rightarrow E = 1$, hence:

$$P(x) = 3 + 4x + 5x^2 + 2x^3 + x^4$$

2. With $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, $a_0 = \dfrac{P(0)}{0!} = \dfrac{3}{1} = 3$, $a_1 = \dfrac{P'(0)}{1!} = \dfrac{4}{1} = 4$, $a_2 = \dfrac{P''(0)}{2!} = \dfrac{10}{2} = 5$, $a_3 = \dfrac{P'''(0)}{3!} = \dfrac{12}{6} = 2$ and $a_4 = \dfrac{P^{(4)}(0)}{4!} = \dfrac{24}{24} = 1$ so:

$$P(x) = 3 + 4x + 5x^2 + 2x^3 + x^4$$

which agrees with the result of Practice 1.

3. Continuing with the differentiation process from Example 2:

$$f^{(5)}(x) = \cos(x) \Rightarrow f^{(6)}(x) = -\sin(x) \Rightarrow f^{(7)}(x) = -\cos(x)$$
$$\Rightarrow f^{(8)}(x) = \sin(x) \Rightarrow f^{(9)}(x) = \cos(x)$$

so that $f^{(6)}(0) = 0$, $f^{(7)}(0) = -1$, $f^{(8)}(0) = 0$ and $f^{(9)}(0) = 1$. Hence $a_6 = \frac{0}{6!} = 0$, $a_7 = \frac{-1}{7!} = -\frac{1}{5040}$, $a_8 = \frac{0}{8!} = 0$ and $a_9 = \frac{1}{9!} = 362880$, yielding the polynomial:

$$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9$$

4. Differentiating $f(x) = \cos(x)$ yields $f'(x) = -\sin(x) \Rightarrow f''(x) = -\cos(x) \Rightarrow f'''(x) = \sin(x) \Rightarrow f^{(4)}(x) = \cos(x)$ so that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$ and $f^{(4)}(0) = 1$, hence $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{-1}{2!} = -\frac{1}{2}$, $a_3 = 0$ and $a_4 = \frac{1}{4!} = \frac{1}{24}$. A MacLaurin polynomial is thus:

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

Notice that differentiating the result of Example 2 yields the same result.

5. Using the MacLaurin polynomial for e^x from the discussion preceding Practice 5 with $x = -1$ yields:

$$\frac{1}{e} = e^{-1} \approx 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!} + \frac{(-1)^6}{6!}$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{53}{144} \approx 0.368$$

while using $x = \frac{1}{2}$ approximates $\sqrt{e} = e^{\frac{1}{2}}$ by:

$$1 + \left(\frac{1}{2}\right) + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 + \frac{1}{5!}\left(\frac{1}{2}\right)^5 + \frac{1}{6!}\left(\frac{1}{2}\right)^6$$
$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} + \frac{1}{46080} \approx 1.647872$$

6. Using the result of Practice 3:

$$\sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \frac{1}{362880} \approx 0.8414710097$$

which agrees to five decimal places with 0.8414709848.

7. Substituting $u = x^2$ into the MacLaurin polynomial from Practice 3 yields:

$$\sin(x^2) \approx x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} - \frac{1}{5040}x^{14} + \frac{1}{362880}x^{18}$$

Integrating this polynomial from $x = 0$ to $x = 1$ yields:

$$\left[\frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11} - \frac{1}{75600}x^{15} + \frac{1}{6894720}x^{19}\right]_0^1$$

which evaluates to (approximately) 0.3102683028.

A
Answers

Important Note about Precision of Answers: In many of the problems in this book you are required to read information from a graph and to calculate with that information. You should take reasonable care to read the graphs as accurately as you can (a small straightedge is helpful), but even skilled and careful people make slightly different readings of the same graph. That is simply one of the drawbacks of graphical information. When answers are given to graphical problems, the answers should be viewed as the best approximations we could make, and they usually include the word "approximately" or the symbol "$\approx$" meaning "approximately equal to." Your answers should be close to the given answers, but you should not be concerned if they differ a little. (Yes those are vague terms, but it is all we can say when dealing with graphical information.)

Section 4.0

1. (a) $(10)(12)+(8)(4)=152$
   (b) $(10)(20)-(3)(8)=176$

3. $bh+\frac{1}{2}b(H-h)=bh+\frac{1}{2}bH-\frac{1}{2}bh=b\left(\frac{h+H}{2}\right)$

5. (a) $1\cdot 3+1\cdot 2=5$ (b) area <5

7. Answers will vary from 5 to 13.

9. $A(1)=1$, $A(2)=2.5$, $A(3)=4.5$, $A(4)=6$, $A(5)=7$

11. $C(1)=1.5$, $C(2)=4$, $C(3)=7.5$
    $C(x)$ is sum of rectangular and triangular areas:
$$C(x)=x+\frac{1}{2}x\cdot x=x+\frac{1}{2}x^2$$

13. $(20)(30)+\frac{1}{2}(10)(30)=600+150=750$ feet

15. (a) A: 20 seconds to stop; B: 40 seconds to stop
    (b) A: $\frac{1}{2}(20)(80)=800$ ft; B: $\frac{1}{2}(40)(40)=800$ ft

17. miles, \$, ft^3, kilowatt-hours, people, square meals

Section 4.1

1. $2^2+3^2+4^2=29$

3. $(1+1)^2+(1+2)^2+(1+3)^2=29$

5. $\cos(0)+\cos(\pi)+\cos(2\pi)+\cos(3\pi)+\cos(4\pi)+\cos(5\pi)=1+(-1)+1+(-1)+1+(-1)=0$

7. $\sum_{k=3}^{94}k$ 9. $\sum_{k=3}^{12}k^2$ 11. $\sum_{k=1}^{7}k\cdot 2^k$

13. (a) $(1+2)+(2+2)+(3+2)=3+4+5=12$
    (b) $(1+2+3)+(2+2+2)=12$

15. (a) $5\cdot 1+5\cdot 2+5\cdot 3=5+10+15=30$
    (b) $5\cdot(1+2+3)=5\cdot 6=30$

17. (a) $1^2+2^2+3^2=1+4+9=14$
    (b) $(1+2+3)^2=6^2=36$

19. $f(0)+f(1)+f(2)+f(3)=0^2+1^2+2^2+3^2=14$

21. $2\cdot f(0)+2\cdot f(1)+2\cdot f(2)+2\cdot f(3)=2\cdot 0+2\cdot 1+2\cdot 4+2\cdot 9=28$

23. $g(1)+g(2)+g(3)=3+6+9=18$

25. $g^2(1)+g^2(2)+g^2(3)=3^2+6^2+9^2=126$

27. $h(2)+h(3)+h(4)=\frac{2}{2}+\frac{2}{3}+\frac{2}{4}=\frac{13}{6}$

29. $(1)(2)+(4)(1)+(9)\left(\frac{2}{3}\right)=12$

31. $(1^2-0^2)+(2^2-1^2)+(3^2-2^2)+(4^2-3^2)+\cdots+(7^2-6^2)=7^2-0^2=49$

33. $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)=1-\frac{1}{6}=\frac{5}{6}$

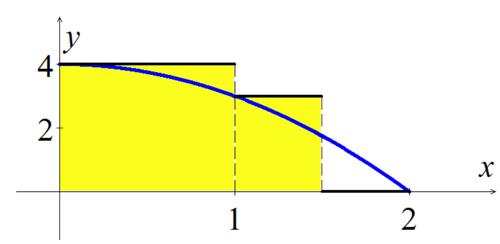
35. $\left(\sqrt{1}-\sqrt{0}\right)+\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)+\cdots+\left(\sqrt{9}-\sqrt{8}\right)=3-0=3$

37. (a) $[2,3]$, $[3,4.5]$, $[4.5,6]$, $[6,7]$ (b) 1, 1.5, 1.5, 1
    (c) mesh $=1.5$ (d) $1+1.5+1.5+1=5$

39. (a) $[-3,-1]$, $[-1,0]$, $[0,1.5]$, $[1.5,2]$
    (b) 2, 1, 1.5, 0.5 (c) mesh $=2$
    (d) $2+1+1.5+0.5=5$

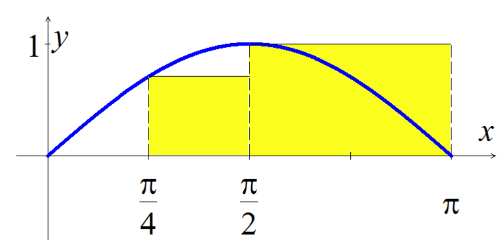
41. (a) $[3,3.8]$, $[3.8,4.5]$, $[4.5,5.2]$, $[5.2,7]$
    (b) 0.8, 0.7, 0.7, 1.8 (c) mesh $=1.8$
    (d) $0.8+0.7+0.7+1.8=4$

43. $\Delta x_1+\Delta x_2+\cdots+\Delta x_n=(x_1-x_0)+(x_2-x_1)+(x_3-x_2)+\cdots+(x_n-x_{n-1})=x_n-x_0$

45. (a) 

$f(0)(1)+f(1)(0.5)+f(2)(0.5)=(4)(1)+(3)(0.5)+(0)(0.5)=5.5$

(b) $f(1)(1)+f(1.5)(0.5)+f(1.5)(0.5)=(3)(1)+(1.75)(0.5)+(1.75)(0.5)=4.75$

47. (a) 

$\left(\frac{\pi}{4}\right)(0)+\left(\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{\pi}{2}\right)(1)\approx 2.13$

(b) $\left(\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{\pi}{4}\right)(1)+\left(\frac{\pi}{2}\right)(0)\approx 1.34$

49. (a) $(2)(1)+(5)(2)+(17)(1) \le \text{RS} \le (5)(1)+(17)(2)+(26)(1) \;\Rightarrow\; 29 \le \text{RS} \le 65$

(b) $(2)(1)+(5)(1)+(10)(1)+(17)(1) \le \text{RS} \le (5)(1)+(10)(1)+(17)(1)+(26)(1) \;\Rightarrow\; 34 \le \text{RS} \le 58$

(c) $2(0.5)+3.25(0.5)+5(1)+10(1)+17(1) \le \text{RS} \le 3.25(0.5)+5(0.5)+10(1)+17(1)+26(1)$
$\Rightarrow 34.625 \le \text{RS} \le 57.125$

51. (a) $(0)\left(\frac{\pi}{2}\right)+(0)\left(\frac{\pi}{2}\right) \le \text{RS} \le (1)\left(\frac{\pi}{2}\right)+(1)\left(\frac{\pi}{2}\right) \;\Rightarrow\; 0 \le \text{RS} \le \pi$

(b) $0\left(\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)+0\left(\frac{\pi}{2}\right) \le \text{RS} \le \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)+1\left(\frac{\pi}{4}\right)+1\left(\frac{\pi}{2}\right) \Rightarrow\; 0.56 \le \text{RS} \le 2.91$

(c) $0\left(\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)+0\left(\frac{\pi}{4}\right) \le \text{RS} \le \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}\right)+1\left(\frac{\pi}{4}\right)+1\left(\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) \Rightarrow\; 1.11 \le \text{RS} \le 2.68$

53. (a) $|7.402-7.362| = 0.04$ (b) $|7.390-7.372| = 0.018$

55. $|\text{error}| = (\text{base})(\text{height}) = \frac{4-2}{50}(65-9) = \frac{56}{25} = 2.24$

57. (a) $\frac{100(101)}{2} = 5050$

(b)
$$\begin{array}{rccccccccccc} S = & 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ +\,S = & 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 2S = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & = 100(101) & = 10100 \;\Rightarrow\; S = 5050 \end{array}$$

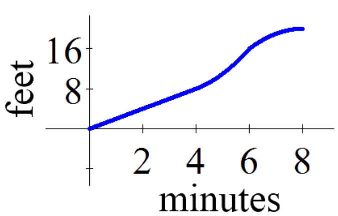
59. $10+11+12+\cdots+20 = (1+2+3+\cdots+20)-(1+2+3+\cdots+9) = \frac{20(21)}{2}-\frac{9(10)}{2} = 210-45 = 165$

61. $\sum_{k=1} 10\left(k^3+k\right) = \sum_{k=1} 10\,k^3 + \sum_{k=1} 10\,k = \left[\frac{10(11)}{2}\right]^2 + \left[\frac{10(11)}{2}\right] = (55)^2+55 = 3080$

Section 4.2

1. $\int_0^4 [2+3x]\,dx$ 3. $\int_2^5 \cos(5x)\,dx$ 5. $\int_1^5 x^3\,dx$ 7. $\int_{0.5}^2 x\cdot\sin(x)\,dx$

9. $\int_1^3 \ln(x)\,dx$ 11. $\int_1^3 2x\,dx = 8$ 13. $\int_{-1}^0 |x|\,dx = \frac{1}{2}$ 15. $\int_0^4 \left[3-\frac{x}{2}\right]\,dx = 8$



17. (a) 3 (b) -1 (c) 6 (d) 8 (e) 7

19. (a) See margin figure. (b) 24 ft (c) 24 feet from starting point

21. meters 23. ft^3 25. gram-meters 27. ft/sec

29. $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $m_k = \frac{2}{n}(k-1)$, $M_k = \frac{2}{n}k$, so $f(m_k) = \left[\frac{2}{n}(k-1)\right]^3$ and $f(M_k) = \left[\frac{2}{n}k\right]^3$

(a)
$$\begin{aligned} \text{LS} &= \sum_{k=1}^{n} f(m_k)\,\Delta x = \sum_{k=1}^{n}\left[\frac{2}{n}(k-1)\right]^3 \Delta x = \frac{2}{n}\cdot\frac{8}{n^3}\left[\sum_{k=1}^{n} k^3 - 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right] \\ &= \frac{16}{n^4}\left[\left(\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{3}{12}n^2\right) - 3\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{2}{12}n\right) + 3\left(\frac{1}{2}n^2+\frac{1}{2}n\right) - n\right] \\ &= \frac{16}{n^4}\left[\frac{1}{4}n^4-\frac{1}{2}n^3+\frac{1}{4}n^2\right] = 4-\frac{8}{n}+\frac{4}{n^2} \longrightarrow 4 \end{aligned}$$

(b)
$$\text{US} = \sum_{k=1}^{n} f(M_k)\,\Delta x = \sum_{k=1}^{n}\left[\frac{2}{n}(k)\right]^3\frac{2}{n} = \frac{16}{n^4}\left[\sum_{k=1}^{n} k^3\right] = \frac{16}{n^4}\left[\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{3}{12}n^2\right] = 4+\frac{8}{n}+\frac{4}{n^2} \longrightarrow 4$$

Section 4.3

1. 5   3. 0   5. 3   7. 0   9. -5

11. -5 13. 0   15. $4.5+5=9.5$

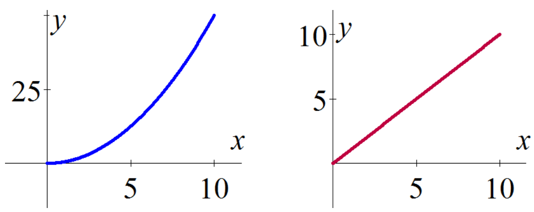
17. $10+3=13$ 19. $5+2=7$

21. 1   23. -1 25. 2   27. 1   29. 1

31. (a) $8\cdot 6=48$ (b) 24   33. (a) 32 (b) $8^2=64$

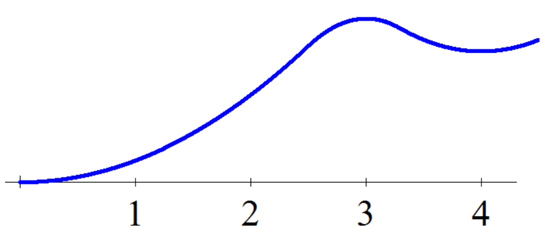
35. 8   37. 2.5   39. 3   41. 7

43. (a) $y=A(x)=\frac{1}{2}x^2$ (below left)

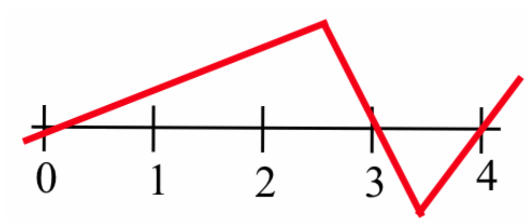


(b) $y=A'(x)=x$ (above right)

45. (a)



(b)



47. (a) f is continuous on $[1,4]$ (b) f is not differentiable on $[1,4]$ (not differentiable at $x\approx 2.5$ and $x\approx 3.3$) (c) f is integrable on $[1,4]$

49. (a) f is not continuous on $[1,4]$ (not continuous at $x=2$) (b) f is not differentiable on $[1,4]$ (not differentiable at $x=2$) (c) f is integrable on $[1,4]$

51. $\int_1^4 v(t)\,dt=\int_1^2 v(t)\,dt+\int_2^4 v(t)\,dt$
$=35+50=85$ miles

53. (a) The rectangle associated with the interval containing $x=2$ has width w and height 5, so its area is $5w$; all of the other rectangles have the same height (7) and the sum of their widths is $(4-1)-w=3-w$, so their total area is $7(3-w)$: $\text{RS}=5w+7(3-w)$

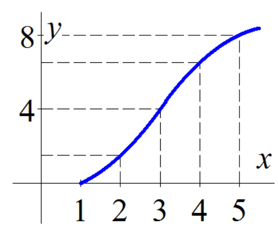
(b) Because $0<w\le\|\mathcal{P}\|$, as $\|\mathcal{P}\|\to 0\Rightarrow w\to 0$ so $\lim\limits_{\|\mathcal{P}\|\to 0}\text{RS}=\lim\limits_{w\to 0}[5w+7(3-w)]=21$

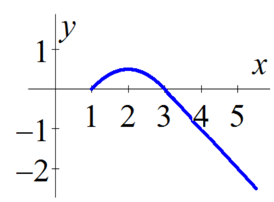
(c) $\int_1^4 g(x)\,dx=\lim\limits_{\|\mathcal{P}\|\to 0}\text{RS}=21$
$\int_1^4 7\,dx=7(4-1)=7(3)=21$

(d) A (very) similar argument shows that redefining any constant function f at a single point does not alter the value of $\int_a^b f(x)\,dx$. A (somewhat) similar argument can extend this result to all integrable functions.

Section 4.4

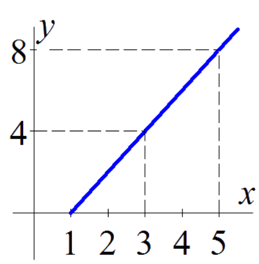
1. (a) See figure below left.
(b) $A(1)=0$, $A(2)=1.5$, $A(3)=4$, $A(4)=6.5$
(c) $A'(1)=1$, $A'(2)=2$, $A'(3)=3$, $A'(4)=2$

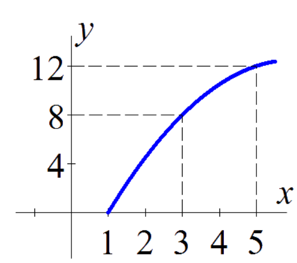




3. (a) See figure above right.
(b) $A(1)=0$, $A(2)=0.5$, $A(3)=0$, $A(4)=-1$
(c) $A'(1)=1$, $A'(2)=0$, $A'(3)=-1$, $A'(4)=-1$

5. (a) See figure below left.
(b) $A(1)=0$, $A(2)=2$, $A(3)=4$, $A(4)=6$
(c) $A'(1)=2$, $A'(2)=2$, $A'(3)=2$, $A'(4)=2$





7. (a) See figure above right.

(b) $A(1) = 0$, $A(2) = 4.5$, $A(3) = 8$, $A(4) = 10.5$
(c) $A'(1) = 5$, $A'(2) = 4$, $A'(3) = 3$, $A'(4) = 2$

9. (a) $x^2\Big|_0^3 = 9$ (b) $x^2\Big|_1^3 = 8$ (c) $x^2\Big|_0^1 = 1$

11. (a) $2x^3\Big|_1^3 = 52$ (b) $2x^3\Big|_1^2 = 14$ (c) $2x^3\Big|_0^3 = 54$

13. (a) $x^4\Big|_0^3 = 81$ (b) $x^4\Big|_1^3 = 80$ (c) $x^4\Big|_0^1 = 1$

15. (a) $x^3\Big|_{-3}^3 = 54$ (b) $x^3\Big|_{-3}^0 = 27$ (c) $x^3\Big|_0^3 = 27$

17. (a) $x^3\Big|_0^2 = 8$ (b) $x^3\Big|_1^3 = 26$ (c) $x^3\Big|_3^1 = -26$

19. (a) $\displaystyle\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100$ ft
(b) $50 = \displaystyle\int_0^T 2t\,dt = T^2 \Rightarrow T = \sqrt{50} \approx 7.07$ sec

21. (a) $\displaystyle\int_0^{10} 4t^3\,dt = t^4\Big|_0^{10} = 10000$ ft
(b) $5000 = \displaystyle\int_0^T 4t^3\,dt = T^4 \Rightarrow T = \approx 8.41$ sec

23. (a) $75 - 3t^2 = 0 \Rightarrow t = 5$ sec
(b) $\displaystyle\int_0^5 \left[75 - 3t^2\right]\,dt = 75t - t^3\Big|_0^5 = 250$ ft
(c) $125 = \displaystyle\int_0^T \left[75 - 3t^2\right]\,dt = 75T - T^3$, so use a graph of $y = x^3 - 75x + 125$ (or Newton's method) to solve for $T \approx 1.74$ sec

25. The total area is $\displaystyle\int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}\cdot 27 = 9$.
(a) $\dfrac{1}{2}\cdot 9 = \dfrac{1}{3}x^3\Big|_0^T = \dfrac{1}{3}T^3 \Rightarrow T = \sqrt[3]{\dfrac{27}{2}} \approx 2.38$
(b) $\dfrac{1}{3}\cdot 9 = \displaystyle\int_0^T x^2\,dx = \frac{1}{3}T^3 \Rightarrow T = \sqrt[3]{9} \approx 2.08$
$\dfrac{2}{3}\cdot 9 = \displaystyle\int_0^T x^2\,dx = \frac{1}{3}T^3 \Rightarrow T = \sqrt[3]{18} \approx 2.62$

Section 4.5

1. (a) $A(x) = x^3 \Rightarrow A'(x) = 3x^2$ so $A'(1) = 3$, $A'(2) = 12$ and $A'(3) = 27$
(b) $A'(x) = \mathbf{D}\left[\int_0^x 3t^2\,dt\right] = 3x^2$ so $A'(1) = 3$, $A'(2) = 12$ and $A'(3) = 27$

3. $A'(x) = 2x$ so $A'(1) = 2$, $A'(2) = 4$, $A'(3) = 6$

5. $A'(x) = 2x$ so $A'(1) = 2$, $A'(2) = 4$, $A'(3) = 6$

7. $A'(1) \approx 0.84$, $A'(2) \approx 0.91$, $A'(3) \approx 0.14$

9. $A'(x) = f(x)$, $A'(1) = 2$, $A'(2) = 1$, $A'(3) = 2$

11. $A'(x) = f(x)$, $A'(1) = 1$, $A'(2) = 2$, $A'(3) = 2$

13. $F(1) - F(0) = 6 - 5 = 1$

15. $F(3) - F(1) = 9 - \frac{1}{3} = \frac{26}{3}$

17. $F(5) - F(1) \approx 1.61 - 0 = 1.61$

19. $F(3) - F\left(\frac{1}{2}\right) \approx 1.10 - (-0.69) = 1.79$

21. $F\left(\frac{\pi}{2}\right) - F(0) = 1 - 0 = 1$

23. $F(1) - F(0) \approx 0.67 - 0 = 0.67$

25. $F(7) - F(1) = \frac{2}{3}(7)^{\frac{3}{2}} - \frac{2}{3} \approx 11.68$

27. $F(9) - F(1) = 3 - 1 = 2$

29. $F(3) - F(-2) \approx 20.09 - 0.14 = 19.95$

31. $F\left(\frac{\pi}{4}\right) - F(0) = 1 - 0 = 1$

33. $F(3) - F(0) = \frac{2}{3}(10)^{\frac{3}{2}} - \frac{2}{3} \approx 20.42$

35. $F(x) = \frac{1}{3}x^3 \Rightarrow F(2) - F(-1) = \frac{8}{3} - \left(-\frac{1}{3}\right) = 3$

37. $F(x) = \ln(x) \Rightarrow F(e) - F(1) = 1 - 0 = 1$

39. $F(x) = \frac{2}{3}x^{\frac{3}{2}} \Rightarrow F(100) - F(25) = \frac{2000}{3} - \frac{250}{3} = \frac{1750}{3}$

41. $F(x) = -\frac{1}{x} \Rightarrow F(10) - F(1) = -0.1 - (-1) = 0.9$

43. $F(x) = e^x \Rightarrow F(1) - F(0) = e - 1 \approx 1.718$

45. $F(x) = \tan(x) \Rightarrow F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right) = 1 - \frac{1}{\sqrt{3}} \approx 0.423$

47. $\int_3^3 f(x)\,dx = 0$ for any integrable f

49. $\displaystyle\int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi = -(-1) - (-1) = 2$

51. $\displaystyle\int_0^{3.5} \lfloor x \rfloor\,dx = 0 + 1 + 2 + \frac{1}{2}(3) = 4.5$

53. $\displaystyle\int_0^3 (x-2)^2\,dx = \int_0^3 (x^2 - 4x + 4)\,dx =$
$\dfrac{1}{3}x^3 - 2x^2 + 4x\Big|_0^3 = 3$

55. $\mathbf{D}\left(A(3x)\right) = 3\tan(3x)$, $\mathbf{D}\left(A(x^2)\right) = 2x\tan(x^2)$, $\mathbf{D}\left(A(\sin(x))\right) = \cos(x)\tan(\sin(x))$

57. $\sqrt{1+5x}(5)$

59. $\sqrt{1+\sin(x)}\cdot\cos(x)$

61. $\left[3(1-2x)^2 + 2\right](-2)$

63. $-\cos(3x)$

65. $\tan(x^2)\cdot 2x - \tan(x)$

67. $5\ln(x)\cos(3\ln(x))\cdot\frac{1}{x}$

Section 4.6

1. $\dfrac{1}{4}x^4\Big|_1^2 = \dfrac{15}{4} \neq \dfrac{7}{2} = \left[\dfrac{1}{3}x^3\Big|_1^2\right]\left[\dfrac{1}{2}x^2\Big|_1^2\right]$

3. $\dfrac{1}{4} \neq \dfrac{1}{3} \cdot \dfrac{1}{2}$

5. $\dfrac{1}{3}\sin(3x) + C$

7. $-\cos(2 + e^x) + C$

9. $\tan(\sin(x)) + C$

11. $\dfrac{5}{2}\ln|3 + 2x| + C$

13. $-\dfrac{1}{3}\cos(1 + x^3) + C$

15. $\dfrac{1}{4}\sin(4x) + C$

17. $\frac{1}{48}\left(5 + x^4\right)^{12} + C$

19. $\ln\left|2 + x^3\right| + C$

21. $\dfrac{1}{2}(\ln(x))^2 + C$

23. $\frac{1}{24}(1 + 3x)^8 + C$

25. $\sec(e^x) + C$

27. $\dfrac{1}{3}\sin(3x)\Big|_0^{\frac{\pi}{2}} = -\dfrac{1}{3}$

29. $-\cos(2 + e^x)\Big|_0^1 \approx -0.996$

31. $\dfrac{1}{18}\left(1 + x^3\right)^6\Big|_{-1}^1 = \dfrac{32}{9}$

33. $\dfrac{5}{2}\ln|3 + 2x|\Big|_0^2 = \dfrac{5}{2}\ln\left(\dfrac{7}{3}\right)$

35. $-\dfrac{1}{3}\left(1 - x^2\right)^{\frac{3}{2}}\Big|_0^1 = \dfrac{1}{3}$

37. $\dfrac{2}{9}(1 + 3x)^{\frac{3}{2}}\Big|_0^1 = \dfrac{14}{9}$

39. $\frac{1}{2}x - \frac{1}{20}\sin(10x) + C$

41. $\frac{1}{4}\sin(2x) + C$

43. $\frac{1}{2}x - \frac{1}{4}\sin(2x)\Big|_0^{\pi} = \frac{\pi}{2}$

45. $\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + C$

47. $\frac{1}{2}e^{2x} + 2e^x + x + C$

49. $\frac{1}{6}x^6 + \frac{1}{4}x^4 + \frac{5}{3}x^3 + 5x + C$

51. $\frac{1}{2}e^{2x} + \frac{1}{4}e^{4x} + C$

53. $\frac{2}{7}x^{\frac{7}{2}} + \frac{6}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + C$

55. $3x - 3\ln|x + 1| + C$

57. $\frac{1}{2}x^2 - x + C$

59. $x^2 - 11x + 7\ln|x - 1| + C$

61. $x + 3\ln|x - 1| + C$

63. $\frac{2}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$

65. area of semicircle with radius 1 $= \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

67. area of semicircle with radius 3 $= \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$

69. $(2)(2) + \frac{1}{2}\pi(1)^2 = 4 + \frac{\pi}{2}$

Section 4.7

1. Answers will vary between 11 (using left endpoints) and 6 (using right endpoints).

3. Between 4 (left endpoints) and 6 (right).

5. Using left-hand widths:

$$(40)\,[0 + 70 + 55 + 90 + 130 + 115] = 18400 \text{ ft}^2$$

Right-hand widths (70, 55,...) and average widths ($\frac{70}{2}$, $\frac{125}{2}$, ...) yield the same result.

7. $\displaystyle\int_{-1}^{2}\left[(x^2 + 3) - 1\right]\,dx = 9$

9. $\displaystyle\int_0^1\left[x - x^2\right]\,dx + \int_1^2\left[x^2 - x\right]\,dx = 1$

11. $\displaystyle\int_1^e\left[x - \frac{1}{x}\right]\,dx = \frac{1}{2}e^2 - \frac{3}{2}$

13. $\displaystyle\int_0^{\frac{\pi}{4}}\left[(x + 1) - \cos(x)\right]\,dx = \frac{1}{32}\pi^2 + \frac{1}{4}\pi - \frac{\sqrt{2}}{2}$

15. $\displaystyle\int_0^2\left[e^x - x\right]\,dx = e^2 - 3$

17. $\displaystyle\int_0^1\left[3 - \sqrt{1 - x^2}\right]\,dx = 3 - \frac{\pi}{4}$

19. Using $\mathcal{P} = \{0.5, 1.5, 2.5, 3.5, 4.5\}$, so that $\Delta x = 1$, and $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 4$:

$$\frac{1}{4.5 - 0.5}\int_{0.5}^{4.5} f(x)\,dx \approx \frac{1}{4}\,[6 + 6 + 4 + 3]\,(1) = \frac{19}{4}$$

21. With $\mathcal{P} = \{1.5, 2.5, 3.5\}$, $\Delta x = 1$, $c_1 = 2$, $c_2 = 3$:

$$\frac{1}{3.5 - 1.5}\int_{0.5}^{3.5} f(x)\,dx \approx \frac{1}{2}\,[6 + 4]\,(1) = \frac{10}{2} = 5$$

23. $\displaystyle\frac{1}{2 - 0}\int_0^2 f(x)\,dx = \frac{2}{2} = 1$

25. $\displaystyle\frac{1}{6 - 1}\int_1^6 f(x)\,dx = \frac{11}{5}$

27. $\displaystyle\frac{1}{4 - 0}\int_0^4\left[2x + 1\right]\,dx = 5$

29. $\displaystyle\frac{1}{3 - 1}\int_1^3 x^2\,dx = \frac{13}{3}$

31. $\displaystyle\frac{1}{\pi - 0}\int_0^{\pi}\sin(x)\,dx = \frac{2}{\pi}$

33. $C = 1$: $\overline{f} = \frac{2}{3}$; $C = 9$: $\overline{f} = 2$; $C = 81$: $\overline{f} = 6$; $C = 100$: $\overline{f} = \frac{20}{3}$. In general, $\overline{f} = \frac{2}{3}\sqrt{C}$.

35. (a) About 180 miles.
    (b) About 36 mph.

37. (a) 1,950 foot-pounds (b) 1,312.5 foot-pounds

39. (a) 1,200 ft-lbs (b) 600 ft-lbs (c) 400 foot-lbs

41. 1,275 foot-pounds

Section 4.8

1. $\dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)+C$

3. $x^2+\dfrac{2}{5}\arctan\left(\dfrac{x}{5}\right)+C$

5. $\dfrac{1}{3}\ln\left|\dfrac{x+3}{x-3}\right|+C$

7. $\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{x}{\sqrt{3}}\right)+C$

9. $e^x+\dfrac{7}{\sqrt{2}}\arctan\left(\dfrac{x}{\sqrt{2}}\right)+C$

11. $3\arcsin\left(\dfrac{x}{\sqrt{5}}\right)+C$

13. $\dfrac{1}{10}\arctan\left(\dfrac{5x}{2}\right)+C$

15. $\dfrac{5}{2}\arcsin(2x)+C$

17. $\dfrac{2}{3}\ln\left|3x+\sqrt{1+9x^2}\right|+C$

19. $(x+1)\ln(x+1)-x+K$

21. $\dfrac{3}{10}(5x^2+7)\left[\ln\left(5x^2+7\right)-1\right]+C$

23. $\sin(x)\left[\ln|\sin(x)|-1\right]+C$

25. $\dfrac{x}{2}\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C$

27. $\dfrac{x}{2}\sqrt{x^2+16}+8\ln\left|x+\sqrt{x^2+16}\right|+C$

29. $8+\dfrac{2}{5}\left[\arctan\left(\dfrac{3}{5}\right)-\arctan\left(\dfrac{1}{5}\right)\right]$

31. $\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{x}{\sqrt{3}}\right)\Big|_{-1}^{1}=\dfrac{\pi}{3\sqrt{3}}$

33. $3\left[\arcsin\left(\dfrac{2}{\sqrt{5}}\right)-\arcsin\left(\dfrac{1}{\sqrt{5}}\right)\right]$

35. $\dfrac{5}{2}\arcsin(2x)\Big|_{0}^{0.1}=\dfrac{5}{2}\arcsin(0.2)$

37. $7\ln(7)-6$

39. $3\ln(3)-2\ln(2)-1$

41. $3\sqrt{18}+\dfrac{9}{2}\ln\left(\dfrac{3+\sqrt{18}}{-3+\sqrt{18}}\right)$

43. $-\dfrac{1}{3}\sin^2(x)\cos(x)-\dfrac{2}{3}\cos(x)+C$

45. $\dfrac{1}{5}\cos^4(x)\sin(x)+\dfrac{4}{5}\displaystyle\int\cos^3(x)\,dx$

47. $x^2\sin(x)+2x\cos(x)-2\sin(x)+C$

49. average of $\sin(x)=\frac{2}{\pi}>\frac{1}{2}=$ average of $\sin^2(x)$

51. $C=e$: $\frac{1}{e-1}$; $C=10$: $\frac{1}{9}\left[10\ln(10)-9\right]$;
$C=100$: $\frac{1}{99}\left[100\ln(100)-99\right]$;
$C=1000$: $\frac{1}{999}\left[1000\ln(1000)-999\right]$

53. (a) $e^2-e\approx 4.67$ (b) $e^2\approx 7.39$ (c) $2e^2-e\approx 12.06$

55. $2\arctan(C)\approx 1.57, 2.94, 3.04, 3.07, 3.09$

Section 4.9

1. $T_4=\frac{1}{2}\left[2.1+2(3.8)+2(0.3)+2(-0.9)+2.2\right]=5.35$
$S_4=\frac{1}{3}\left[2.1+4(3.8)+2(0.3)+4(-0.9)+2.2\right]=5.5$

3. $T_8=7.35$, $S_8=\dfrac{22}{3}\approx 7.3333$

5. (a) $T_4=4$ (b) $S_4=4$ (c) 4

7. (a) $T_4=0.75$ (b) $S_4=\frac{2}{3}\approx 0.67$ (c) $\dfrac{2}{3}$

9. (a) $T_4=1.896118898$ (b) $S_4=2.004559755$ (c) 2

11. (a) $T_6=1.088534906$ (b) $S_6=1.090560447$

13. (a) $T_6=3.815780054$ (b) $S_6=3.826350295$

15. (a) $T_6=0.8159928163$ (b) $S_6=0.8120491229$

17. (a) $f(x)=x\Rightarrow f''(x)=0\Rightarrow B_2=0$, so the error bound is 0 (the Trapezoidal approximation is exact) (b) $f^{(4)}(x)=0\Rightarrow B_4=0$, so the error bound is 0 (the Simpson's Rule approximation is exact) (c) $n=1$ (d) $n=2$ (must be an even integer)

19. (a) $f(x)=x^3\Rightarrow f''(x)=6x$ so when $|x|\le 1$, $|f''(x)|\le 6$; taking $B_2=6$, $|\text{error}|\le\frac{2^3\cdot 6}{12\cdot 4^2}=0.25$ (b) $f^{(4)}(x)=0\Rightarrow B_4=0$, so error bound is 0 (c) $\frac{2^3\cdot 6}{12\cdot n^2}\le 0.001\Rightarrow n^2\ge 4000\Rightarrow n\ge 63.25$, so take $n=64$ (d) $n=2$

21. (a) $f''(x)=-\sin(x)\Rightarrow|f''(x)|\le 1\Rightarrow B_2=1$, so $|\text{error}|\le\frac{\pi^3\cdot 1}{12\cdot 4^2}\approx 0.1612$ (b) $f^{(4)}(x)=\sin(x)\Rightarrow\left|f^{(4)}(x)\right|\le 1=B_4$, so $|\text{error}|\le\frac{\pi^5\cdot 1}{180\cdot 4^4}\approx 0.0066$ (c) $\frac{\pi^3\cdot 1}{12\cdot n^2}\le 0.001\Rightarrow n^2\ge\frac{1000\pi^3}{12}\Rightarrow n\ge 50.83$, so take $n=51$ (d) $\frac{\pi^5\cdot 1}{180\cdot n^4}\le 0.001\Rightarrow n^4\ge\frac{1000\pi^3}{180}\Rightarrow n\ge 6.42$, so take $n=8$

23. $S_6=\frac{30}{3}[50+4(62)+2(92)+4(86)+2(74)$
$+4(50)+40]=12140\text{ ft}^2$

25. area: $S_6\approx 37166.7\text{ ft}^2$
volume: $(37166.7)(22)=817{,}667\text{ ft}^3$

27. distance $\approx T_{10}=4{,}010$ ft

29. On your own.

31. On your own.

33. (a) $L_4=3.5$ (b) $R_4=4.5$ (c) $M_4=4$ (d) 4

35. (a) $L_4=0.75$ (b) $R_4=0.75$ (c) $M_4=0.625$ (d) $\frac{2}{3}$

37. (a) 1.8961 (b) 1.8961 (c) 2.0523 (d) 2

39. On your own.

41. $S_{10}=6.12572$; $S_{40}=6.12573$

43. $S_{10}=22.1035$; $S_{40}=22.1035$

Section 5.1

1. $(8)(6)(1)+(6)(4)(2)+(3)(3)(1)=105$
3. $\pi(4)^2(0.5)+\pi(3)^2(1.0)+\pi(1)^2(2.0)=19\pi$
5. $(9)(0.2)+(6)(0.2)+(2)(0.2)=3.4$
7. $\int_0^3 (5-x)^2\,dx=39$
9. $\int_0^4 \frac{1}{2}(x+1)\sqrt{x}\,dx=\frac{136}{15}$
11. $\int_0^4 \pi\left[\frac{4-x}{2}\right]^2\,dx=\frac{16\pi}{3}$
13. If $A(y)$ is the cross-sectional area of object A at height y above the base and $B(y)$ is the cross-sectional area of object B at height y above its base, then we know that $A(y)=B(y)$ for all values of y, so $\text{Vol}(A)=\int_a^b A(y)\,dy=\int_a^b B(y)\,dy=$ Vol (B): the volumes are equal.
15. $\int_1^4 \left[\frac{1}{x}\right]^2\,dx=\frac{3}{4}$
17. $\int_0^4 \pi\left[3-\sqrt{x}\right]^2\,dx=12\pi$
19. $\int_0^\pi [\sin(x)]^2\,dx=\frac{\pi}{2}$
21. $\int_0^{\frac{\pi}{2}} \frac{1}{2}\cdot\pi\left[\frac{1}{2}\cos(x)\right]^2\,dx=\frac{\pi^2}{32}$
23. (a) $\int_{-1}^{\frac{3}{2}} \left[(3+x-x^2)-x^2\right]^2\,dx=\frac{625}{48}\approx 13.021$
    (b) $\int_{-1}^{\frac{3}{2}} \frac{1}{2}\cdot\pi\left[\frac{1}{2}(3+x-2x^2)\right]^2\,dx=\frac{625\pi}{384}\approx 5.1$
    (c) $\int_{-1}^{\frac{3}{2}} 2\left[3+x-2x^2\right]^2\,dx=\frac{625}{24}\approx 26.0$
    (d) $\int_{-1}^{\frac{3}{2}} \frac{1}{2}\left[\frac{3+x-2x^2}{\sqrt{2}}\right]^2\,dx=\frac{625}{192}\approx 3.3$
25. $\int_0^8 \left[\sqrt{8-y}\right]^2\,dy=32$
27. $\int_1^4 \frac{\pi}{2}\left[\frac{\sqrt{8-y}-2}{2}\right]^2\,dy=\frac{\pi}{48}\left[299-112\sqrt{7}\right]$
29. (a) H^2L (b) $\int_0^L \left(\frac{H}{L}x\right)^2\,dx=\frac{1}{3}H^2L$ (c) $\frac{1}{3}$
31. (a) BL (b) $\int_0^L \frac{B}{L^2}x^2\,dx=\frac{1}{3}BL$ (c) $\frac{1}{3}$

Section 5.2

1. $\int_0^5 \pi x^2\,dx=\frac{125\pi}{3}$
3. $\int_0^{\frac{\pi}{3}} \pi\cos^2(x)\,dx=\frac{\pi}{2}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]$
   (There are actually *two* such regions.)
5. $\int_0^7 \pi\left[\sqrt{7-x}\right]^2\,dx=\frac{49\pi}{2}$
7. $\int_0^{\sqrt{5}} \pi\left[5-x^2\right]^2\,dx=\frac{40\pi\sqrt{5}}{3}$
9. $\int_0^{121} \pi\left[\sqrt{121-x}\right]^2\,dx=\frac{14641\pi}{2}$
11. $\int_0^5 \pi\left[\sqrt{\frac{225-9x^2}{25}}\right]^2\,dx=30\pi$
13. $\int_0^1 \pi\left[x^2-\left(x^4\right)^2\right]\,dx=\frac{2\pi}{9}$
15. $\int_0^1 \pi\left[(\sqrt[4]{y})^2-(\sqrt{y})^2\right]\,dy=\frac{\pi}{6}$
17. $\int_0^1 \pi\left[\left(x^2\right)^2-\left(x^3\right)^2\right]\,dx=\frac{2\pi}{35}$
19. $\int_0^{\frac{\pi}{3}} \pi\left[\sec^2(x)-\cos^2(x)\right]\,dx=\pi\left[\sqrt{3}-\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right]$
21. $\int_0^1 \pi\left[(x-(-4))^2-\left(x^4-(-4)\right)^2\right]\,dx=\frac{118\pi}{45}$
23. $\int_0^1 \pi\left[(3-y)^2-(3-\sqrt[4]{y})^2\right]\,dy=\frac{22\pi}{15}$
25. $\int_0^1 \pi\left[(3-\sin(x))^2-(3-x)^2\right]\,dx$
    $=\pi\left[6\cos(1)-\frac{1}{4}\sin(2)-\frac{17}{6}\right]$
27. $\int_0^1 \pi\left[\left(y^2-(-2)\right)^2-\left(y^3-(-2)\right)^2\right]\,dy=\frac{41\pi}{105}$
29. $\int_0^1 \pi\left[(2-\sqrt{x})^2-(2-\sqrt[3]{x})^2\right]\,dx=\frac{7\pi}{30}$
31. $\int_{-2}^2 \pi\left[\sqrt{4-x^2}\right]^2\,dx=\frac{32\pi}{3}=\frac{4}{3}\pi(2)^3$
33. $\int_{-5}^5 \pi\left[3\sqrt{1-\frac{x^2}{25}}\right]^2\,dx=60\pi$
35. $\int_1^2 \pi\left[\left(x^2\right)^2-(1)^2\right]\,dx=\frac{26\pi}{5}$
37. $\int_0^\pi \pi\left|\frac{1}{2}-\sin(x)\right|^2\,dx=\pi\left[\frac{3\pi}{4}-2\right]$

39. (a) $\int_1^{10} \frac{1}{x}\,dx = \ln(10)$; $\int_1^{100} \frac{1}{x}\,dx = \ln(100)$; $\int_1^{M} \frac{1}{x}\,dx = \ln(M)$; $\lim_{M\to\infty} \ln(M) = \infty$

(b) $\int_1^{10} \frac{\pi}{x^2}\,dx = \frac{9}{10}\pi$; $\int_1^{100} \frac{\pi}{x^2}\,dx = \frac{99}{100}\pi$; $\int_1^{M} \frac{\pi}{x^2}\,dx = \left[1-\frac{1}{M}\right]\pi$; $\lim_{M\to\infty} \left[1-\frac{1}{M}\right]\pi = \pi$

Section 5.3

1. $\sqrt{15^2+20^2} + \sqrt{(-24)^2+18^2} + \sqrt{12^2+(-12)^2} \approx 71.97$ ft

3. $\sqrt{(1-0)^2+(2-1)^2} + \sqrt{(2-1)^2+(4-2)^2} \approx 3.65$

5. (a) $\sqrt{(2-0)^2+(5-1)^2} = 2\sqrt{5}$ (b) $\int_0^2 \sqrt{1+[2]^2}\,dx = 2\sqrt{5}$

7. (a) $\sqrt{(5-2)^2+(-5-1)^2} = 3\sqrt{5}$ (b) $\int_0^3 \sqrt{[1]^2+[-2]^2}\,dx = 3\sqrt{5}$

9. $y' = x^{\frac{1}{2}} = \sqrt{x}$, so $L = \int_0^4 \sqrt{1+\left[\sqrt{x}\right]^2}\,dx = \int_0^4 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^4 = \frac{2}{3}\left[5\sqrt{5}-1\right]$

11. $y' = x^2 - \frac{1}{4}x^{-2} \Rightarrow 1+[y']^2 = 1+x^4-\frac{1}{2}+\frac{1}{16}x^{-4} = x^4+\frac{1}{2}+\frac{1}{16}x^{-4} = \left(x^2+\frac{1}{4}x^{-2}\right)^2$, so
$L = \int_1^5 \sqrt{1+[y']^2}\,dx = \int_1^5 \left[x^2+\frac{1}{4}x^{-2}\right]dx = \left[\frac{1}{3}x^3-\frac{1}{4}x^{-1}\right]_1^5 = \left[\frac{125}{3}-\frac{1}{20}\right]-\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{623}{15} \approx 41.53$

13. $y' = x^4 - \frac{1}{4}x^{-4} \Rightarrow 1+[y']^2 = 1+x^8-\frac{1}{2}+\frac{1}{16}x^{-8} = x^8+\frac{1}{2}+\frac{1}{16}x^{-8} = \left(x^4+\frac{1}{4}x^{-4}\right)^2$, so
$L = \int_1^5 \sqrt{1+[y']^2}\,dx = \int_1^5 \left[x^4+\frac{1}{4}x^{-4}\right]dx = \left[\frac{1}{5}x^5-\frac{1}{12}x^{-3}\right]_1^5 = \left[625-\frac{1}{1500}\right]-\left[\frac{1}{5}-\frac{1}{12}\right] \approx 624.88$

15. $L = \int_0^1 \sqrt{1+[2x]^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \approx 1.4789$ (using technology)

17. $L = \int_1^9 \sqrt{1+\left[\frac{1}{2\sqrt{x}}\right]^2}\,dx = \int_1^9 \sqrt{1+\frac{1}{4x}}\,dx \approx 8.2681$

19. $L = \int_0^{\frac{\pi}{4}} \sqrt{1+[\cos(x)]^2}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{1+\cos^2(x)}\,dx \approx 1.058$; $L = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1+\cos^2(x)}\,dx \approx 0.852$

21. $\int_0^{2\pi} \sqrt{[-5\sin(t)]^2+[2\cos(t)]^2}\,dt = \int_0^{2\pi} \sqrt{25\sin^2(t)+4\cos^2(t)}\,dt \approx 23.018$

23. $\int_{10}^{20} \sqrt{[-t\cdot\sin(t)+\cos(t)]^2+[t\cdot\cos(t)+\sin(t)]^2}\,dt = \int_{10}^{20} \sqrt{1+t^2}\,dt \approx 150.346$

24. $\int_0^{2\pi} \sqrt{[R(1-\cos(t))]^2+[R\sin(t)]^2}\,dt = R\int_0^{2\pi} \sqrt{2(1-\cos(t))}\,dt \approx 8R$

25. $\frac{5280\text{ ft}}{2\pi\,\frac{\text{ft}}{\text{rev}}} \cdot 8\,\frac{\text{ft}}{\text{rev}} \approx 6723\text{ feet} \approx 1.27\text{ miles}$

27. $L = \int_0^4 \sqrt{1+[2x]^2}\,dx = \int_0^4 \sqrt{1+4x^2}\,dx \approx 16.8186$ so we need a such that $\int_0^a \sqrt{1+4x^2}\,dx \approx \frac{1}{2}(16.8186) = 8.4093$ and b and c such that $\int_0^b \sqrt{1+4x^2}\,dx \approx \frac{1}{3}(16.8186) = 5.6062$ and $\int_0^c \sqrt{1+4x^2}\,dx \approx \frac{2}{3}(16.8186) = 11.2124$. Guessing and checking using technology yields $a \approx 2.77$, $b \approx 2.22$ and $c \approx 3.23$.

29. $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2x^{\frac{1}{2}}}$

31. (a) A: $2\pi(3)(4) = 24\pi$; B: $2\pi(1)(2) = 4\pi$
    (b) A: $2\pi(5)(4) = 40\pi$; B: $2\pi(8)(2) = 32\pi$

33. (a) A: $2\pi(3)(3) = 18\pi$; B: $2\pi(4)(5) = 40\pi$
    (b) A: $2\pi(2)(3) = 12\pi$; B: $2\pi(6.5)(5) = 65\pi$

35. $\theta = \frac{\pi}{2} = 90°$ results in the midpoint of the line segment being furthest from the x-axis.

37. $\int_0^1 2\pi x\sqrt{1+[6x^2]^2}\,dx \approx 10.207$

39. $\int_0^1 2\pi\left(2x^2\right)\sqrt{1+[4x]^2}\,dx \approx 13.306$

41. $\int_0^2 2\pi x^3\sqrt{1+[3x^2]^2}\,dx = \frac{\pi}{27}\left[145\sqrt{145}-1\right]$

43. $\int_0^2 2\pi x\sqrt{1+[2x]^2}\,dx = \frac{\pi}{4}\left[17\sqrt{17}-1\right]$

45. Rotate $y = \sqrt{R^2-x^2}$ about the x-axis to get a sphere of radius R. Then:

$$y = \left[R^2-x^2\right]^{\frac{1}{2}} \Rightarrow y' = -x\left[R^2-x^2\right]^{-\frac{1}{2}}$$

$$\Rightarrow 1+\left[y'\right]^2 = 1+\frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

so the slice from $x=a$ to $x=b$ has surface area:

$$\int_a^b 2\pi\sqrt{R^2-x^2}\cdot\sqrt{\frac{1}{R^2-x^2}}\,dx = 2\pi(b-a)$$

which depends only on the width of the slice.

47. Cake: no; frosting: yes.

48. (a) $\int_c^d 2\pi\cdot y\sqrt{1+[g'(y)]^2}\,dy$
    (b) $\int_c^d 2\pi\cdot g(y)\sqrt{1+[g'(y)]^2}\,dy$

49. (a) $\int_0^1 2\pi\cdot y\sqrt{1+[e^y]^2}\,dy \approx 7.055$
    (b) $\int_0^1 2\pi\cdot e^y\sqrt{1+[e^y]^2}\,dy \approx 22.943$

50. (a) $\int_\alpha^\beta 2\pi\cdot y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$
    (b) $\int_\alpha^\beta 2\pi\cdot x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$

51. (a) $\int_0^{\frac{\pi}{2}} 2\pi\sin(t)\sqrt{[-\sin(t)]^2+[\cos(t)]^2}\,dt = 2\pi$
    (b) $\int_0^{\frac{\pi}{2}} 2\pi\cos(t)\sqrt{[-\sin(t)]^2+[\cos(t)]^2}\,dt = 2\pi$

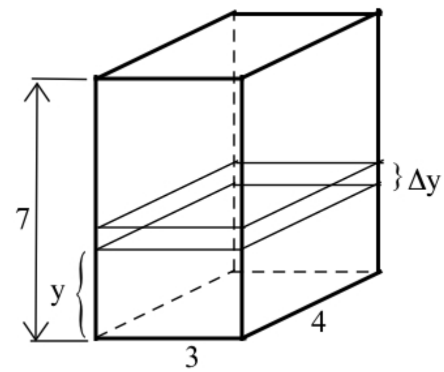
53. $\int_0^{4\pi}\sqrt{[-\sin(t)]^2+[\cos(t)]^2+[1]^2}\,dt = 4\pi\sqrt{2}$

55. $\int_0^1\sqrt{[1]^2+[2t]^2+[3t^2]^2}\,dt \approx 1.863$

57. $\int_0^{2\pi}\sqrt{[-3\sin(t)]^2+[2\cos(t)]^2+[7\cos(7t)]^2}\,dt \approx 33.293$

Section 5.4

1. A "slice" of water y ft high with thickness Δy ft:



has volume $3\cdot4\cdot\Delta y = 12\Delta y$ ft^3, so its weight is $62.5(12)\Delta y$ lb; the work required to lift this slice $(7-y)$ feet is $750(7-y)\Delta y$ ft-lbs. The total work required to pump out all of the water is:

$$\int_0^7 750(7-y)\,dy = 18375 \text{ ft-lbs}$$

3. (a) A slice of oil at initial height y m with thickness Δy m has volume $5\cdot2\cdot\Delta y$ m^3, so its mass is $900\,(10\Delta y) = 9000\Delta y$ kg and the force required to lift it is $(9000\Delta y)(9.81) = 88290\Delta y$ N. Lifting this slice $(4-y)$ m requires $88290(4-y)\Delta y$ N-m of work, so the total work needed to empty the tank is:

$$\int_0^4 88290(4-y)\,dy = 706320\,\text{J}$$

   (b) The top $10\,\text{m}^3$ corresponds to the top 1 m of oil, so the work required is:

$$\int_3^4 88290(4-y)\,dy = 44145\,\text{J}$$

   (c) $\dfrac{706320\,\text{J}}{200\,\frac{\text{J}}{\text{sec}}} = 3531.15\,\text{sec} \approx 59\,\text{min}$

5. (a) A slice of water at initial height y m with thickness Δy m has volume $\pi\,(1)^2\,\Delta y$ m^3, so its mass is $1000\,(\pi\Delta y) = 1000\pi\Delta y$ kg and the force required to lift it is $(1000\pi\Delta y)(9.81) = 9810\pi\Delta y$ N. Lifting this slice $(6-y)$ m requires $9810\pi(6-y)\Delta y$ N-m of work, so the total work needed to empty the tank is:

$$\int_0^6 9810\pi(6-y)\,dy = 176580\pi \approx 554742 \text{ J}$$

(b) The distance each slice must be lifted is now $(6-y)+2 = 8-y$ so the total work done is:

$$\int_5^6 9810\pi(8-y)\,dy = 24525\pi \approx 77048 \text{ J}$$

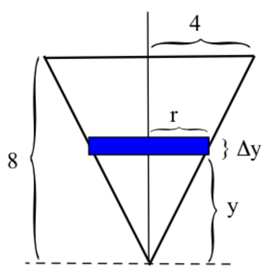
(c) The work needed to remove half the water is:

$$\int_3^6 9810\pi(6-y)\,dy = 44145\pi \approx 138686 \text{ J}$$

and $\frac{1}{2}$ hp $= \frac{1}{2}$ (746 watts) $= 373$ watts so:

$$\frac{138686\,\text{J}}{373\,\frac{\text{J}}{\text{sec}}} = 372\,\text{sec} \approx 6.2\,\text{min}$$

6. (a) Using similar triangles on a cross-section of the cone and a slice of radius r at height y:



reveals that $\frac{r}{y} = \frac{4}{8} \Rightarrow r = \frac{1}{2}y$ so the volume of the slice is $\pi\left[\frac{y}{2}\right]^2\Delta y$ ft^3 and its weight is $25\cdot\pi\frac{y^2}{4}\Delta y$ lb. The work required to lift this slice $8-y$ ft to the top edge of the cone is thus $\frac{25\pi}{4}y^2\,(8-y)\,\Delta y$ ft-lb, so the total work needed to empty the container is:

$$\int_0^8 \frac{25\pi}{4}y^2\,(8-y)\;dy = \frac{6400\pi}{3} \approx 6702 \text{ ft-lbs}$$

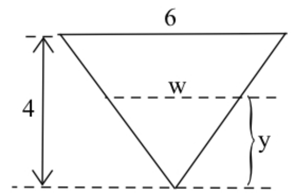
(b) $\displaystyle\int_6^8 \frac{25\pi}{4}y^2\,(8-y)\;dy = \frac{1675\pi}{3} \approx 1754 \text{ ft-lbs}$

7. Half the work is 3351 ft-lbs, so find h so that:

$$3351 = \int_0^h \frac{25\pi}{4}y^2\,(8-y)\;dy = \frac{25\pi}{48}\left[32h^3 - 3h^4\right]$$

Solving this equation using technology yields $h \approx 4.9$ so you should dig from the top down to a depth of 3.1 ft and leave the rest for your friend.

8. Using similar triangles on a cross-section of the trough and a slice of width w at height y:



with thickness Δy reveals that $\frac{w}{y} = \frac{6}{4} \Rightarrow w = \frac{3}{2}y$ so the weight of this slice, which must be lifted $(4-y)$ ft, is $80\left(\frac{3}{2}y\right)(7)\Delta y$ lb and the total work needed is:

$$\int_0^4 840y(4-y)\,dy = 8960 \text{ ft-lbs}$$

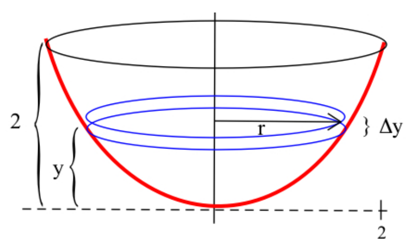
9. If the bottom 70 ft^3 of slop has depth h ft:

$$\frac{1}{2}h\left(\frac{3}{2}h\right)(7) = 70 \Rightarrow h = \sqrt{\frac{40}{3}} \approx 3.65 \text{ ft}$$

so the work needed to remove the top 14 ft^3 is:

$$\int_{3.65}^4 840y(4-y)\,dy \approx 192.21 \text{ ft-lbs}$$

11. (a) A horizontal slice at height $y = \frac{1}{2}x^2$ is (approximately) a disk with radius $x = \sqrt{2y}$ and thickness Δy:



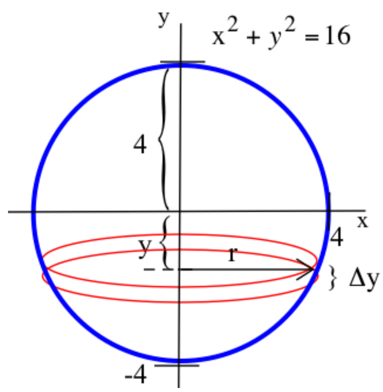
has volume $\pi\left[\sqrt{2y}\right]^2\Delta y = 2\pi y\Delta y$ m^3 and mass $2000\pi y\Delta y$ m^3; that slice must be lifted $(2-y)$ m so the total work required is:

$$\int_0^2 2000\pi y(9.81)(2-y)\,dy \approx 82184 \text{ J}$$

(b) Each slice is lifted $(2-y)+3=(5-y)$ m, so the work needed is now:

$$\int_0^2 2000\pi y(9.81)(5-y)\,dy \approx 452012 \text{ J}$$

13. Rotate the circle $x^2+y^2=16$ about the x-axis to generate the sphere. A slice at height y (for $-4 \le y \le 4$) and thickness Δy is (approximately) a disk with radius $x=\sqrt{16-y^2}$:



so the slice has volume $\pi\left[\sqrt{16-y^2}\right]^2\Delta y$ and mass $1000\pi\left[16-y^2\right]\Delta y$; the slice must be lifted $(4-y)$ m, so the total work is:

$$\int_{-4}^{-2}(9.81)1000\pi\left(16-y^2\right)(4-y)\,dy \approx 2753166 \text{ J}$$

15. (a) The leftmost container: more water is near the bottom, so more of the water must be lifted a greater distance. (b) The rightmost container.

17. (a) 85 ft-lbs (b) 35 ft-lbs

19. (a) $\int_0^3 6x\,dx = 27$ in-oz ≈ 0.14 ft-lbs

(b) $\int_0^6 6x\,dx = 108$ in-oz ≈ 0.56 ft-lbs

21. (a) The average force between $x=23$ and $x=33$ is ≈ 15 dyn so the work done is about:

$$15(33-23) = 150 \text{ dyn-cm} = 0.000015 \text{ J}$$

(b) The area under the graph between $x=28$ and $x=33$ is about 90 dyn-cm $= 0.000009$ J.

23. Converting cm to m and applying Hooke's Law yields $3(9.8)=k(0.15) \Rightarrow k=196$ so the total work is:

$$\int_{0.15}^{0.19} 196x\,dx = \left[98x^2\right]_{0.15}^{0.19} = 1.3328 \text{ J}$$

25. (a) Ceres' radius is 475 km = 475000 m and:

$$GMm = \left(6.673\times 10^{-11}\right)\left(896\times 10^{18}\right)(100)$$

or 5.979×10^{12}; the work to lift the payload from $x=475000$ m to $x=485000$ m is:

$$\int_{475000}^{485000}\frac{5.979\times 10^{12}}{x^2}\,dx \approx 259530 \text{ J}$$

(b) $\displaystyle\int_{475000}^{575000}\frac{5.979\times 10^{12}}{x^2}\,dx \approx 2189110$ J

(c) $\displaystyle\int_{475000}^{975000}\frac{5.979\times 10^{12}}{x^2}\,dx \approx 6455070$ J

27. (a) Answers will depend on your mass. If your mass (including a space suit to ensure your survivial) is 100 kg then:

$$GMm = \left(6.673\times 10^{-11}\right)\left(7.35\times 10^{22}\right)(100)$$

or 4.905×10^{14}; the work done lifting you from $x=1737500$ m to $x=1937500$ m is:

$$\int_{1737500}^{1937500}\frac{4.905\times 10^{14}}{x^2}\,dx \approx 29138818 \text{ J}$$

(b) $\displaystyle\int_{1737500}^{2137500}\frac{4.905\times 10^{14}}{x^2}\,dx \approx 52824758$ J

(c) $\displaystyle\int_{1737500}^{11737500}\frac{4.905\times 10^{14}}{x^2}\,dx \approx 240496104$ J

29. We know that $-0.1 = \frac{k}{10^2} \Rightarrow k=-10$.

(a) $\displaystyle\int_{20}^{10}\frac{-10}{x^2}\,dx = 0.5$ J

(b) $\displaystyle\int_{10}^{1}\frac{-10}{x^2}\,dx = 9$ J

(c) $\displaystyle\int_{1}^{0.1}\frac{-10}{x^2}\,dx = 90$ J

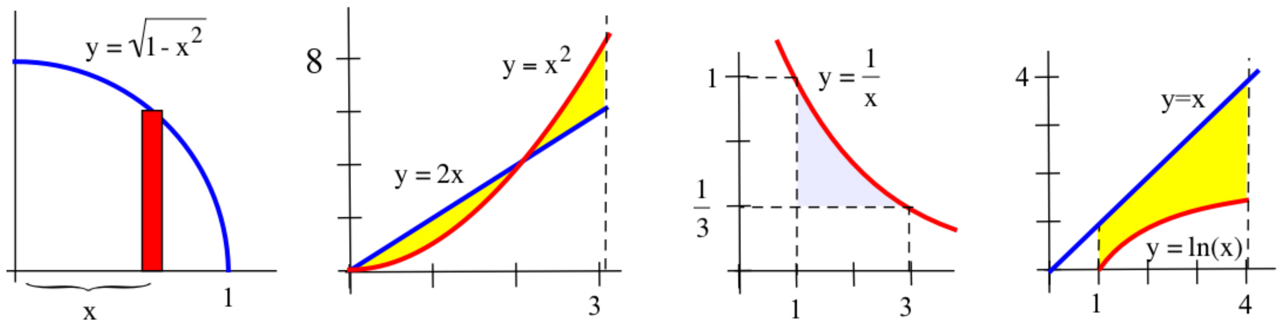
31. $\displaystyle\int_0^{2\pi} t\sqrt{[-\sin(t)]^2+[\cos(t)]^2}\,dt = \int_0^{2\pi} t\,dt = 2\pi^2$

33. $\displaystyle\int_0^1 t\sqrt{[2t]^2+[1]^2}\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt = \frac{5\sqrt{5}-1}{12}$

35. This is the same as Problem 31. To solve without calculus, "unroll" the region to get a triangle with base 2π (the circumference of the circle upon which the region was originally sitting) and height 2π. The area of the triangle is then $\frac{1}{2}(2\pi)(2\pi) = 2\pi^2$.

Section 5.5

Graphs of the regions from Problems 1, 3, 5 and 7 appear below:



1. A vertical slice rotated around the y axis results in a tube: $\int_0^1 2\pi x\sqrt{1-x^2}\,dx = \frac{2\pi}{3} \approx 2.09$

3. We need to split this region into two pieces. The curves intersect where $x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. For $0 \le x \le 2$, $y = 2x$ is the top curve and $y = x^2$ is the bottom curve, so the height of a vertical slice is $2x - x^2$; for $2 \le x \le 3$, the roles are reversed, so the height is $x^2 - 2x$:

$$\int_0^2 2\pi\,(4-x)\left[2x - x^2\right]\,dx + \int_2^3 2\pi\,(4-x)\left[x^2 - 2x\right]\,dx = 8\pi + \frac{7\pi}{2} = \frac{23\pi}{2} \approx 36.13$$

5. Vertical slices yield tubes: $\int_1^3 2\pi\,(5-x)\left[\frac{1}{x} - \frac{1}{3}\right]\,dx = [10\ln(3) - 8]\,\pi \approx 9.38$

7. Vertical slices yield tubes, but the resulting integral requires numerical integration or tables (for now):

$$\int_1^4 2\pi x\,[x - \ln(x)]\,dx \approx 85.83$$

Horizontal slices result in washers and require us to split the region into three pieces, but finding antiderivatives is easier:

$$\int_0^1 \pi\left[(e^y)^2 - 1^2\right]\,dy + \int_1^{\ln(4)} \pi\left[(e^y)^2 - y^2\right]\,dy + \int_{\ln(4)}^4 \pi\left[4^2 - y^2\right]\,dy$$
$$= \frac{\left(e^2-3\right)\pi}{2} + \frac{\left(50 - 3e^2 - 2\left(\ln(4)\right)^3\right)\pi}{6} + \frac{\left(4-\ln(4)\right)^2\left(8+\ln(4)\right)\pi}{3} \approx 85.83$$

9. Washers: $\int_1^2 2\pi x\left[(6-x) - x^2\right]\,dx + \int_2^4 2\pi x\left[x^2 - (6-x)\right]\,dx = \frac{35\pi}{6} + \frac{256\pi}{3} = \frac{541\pi}{6} \approx 283.27$

11. Tubes: $\int_0^{\frac{3}{\sqrt{2}}} 2\pi x\left[\sqrt{9-x^2} - x\right]\,dx = 9\pi\left(2-\sqrt{2}\right) \approx 16.56$

13. $\int_0^1 2\pi x\left[x^2 - x^4\right]\,dx = \frac{\pi}{6} \approx 0.52$

15. $\int_0^{\sqrt{\pi}} 2\pi x \cdot \sin\left(x^2\right)\,dx = 2\pi \approx 6.28$

17. $\int_0^{\frac{1}{2}} 2\pi x \cdot \frac{1}{\sqrt{1-x^2}}\,dx = \pi\left(2-\sqrt{3}\right) \approx 0.84$

19. $\int_0^1 2\pi\,(3-x)\left[x - x^4\right]\,dx = \frac{22\pi}{15} \approx 4.61$

21. $\int_0^1 \pi\left[(x-(-3))^2-\left(x^4-(-3)\right)^2\right]dx=\dfrac{91\pi}{45}$

23. $\int_0^1 2\pi(2-x)\cdot\dfrac{1}{1+x^2}\,dx=\pi[\pi-\ln(2)]\approx 7.69$

25. $\int_0^1 2\pi(x+2)\left[1-\dfrac{1}{1+x^2}\right]dx=\pi[5-\pi-\ln(2)]\approx 3.66$

27. Slicing horizontally results in washers and requires us to split the region into two pieces:

$$\int_{y=0}^{y=1}\pi\left[\left(4-\left(1+y^2\right)\right)^2-\left(4-\left(2+y^2\right)\right)^2\right]dy+\int_{y=1}^{y=\sqrt{2}}\pi\left[\left(4-\left(1+y^2\right)\right)^2-(4-3)^2\right]dy$$

which evaluates to $=\dfrac{13\pi}{3}+\dfrac{\left(24\sqrt{2}-31\right)\pi}{5}\approx 15.46.$

29. Slicing horizontally results in tubes and requires us to split the region into two pieces:

$$\int_{y=0}^{y=1}2\pi(4-y)\left[\left(2+y^2\right)-\left(1+y^2\right)\right]dy+\int_{y=1}^{y=\sqrt{2}}2\pi(4-y)\left[3-\left(1+y^2\right)\right]dy$$

which evaluates to $=7\pi+\dfrac{\left(64\sqrt{2}-83\right)\pi}{6}\approx 25.92.$

Section 5.6

1. (a) The total mass is $m=2+5+5=12$ and $M_0=2\cdot4+5\cdot2+5\cdot6=48$ so $\overline{x}=\dfrac{48}{12}=4.$

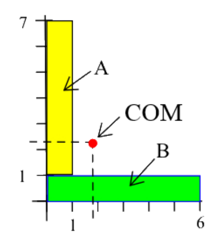
   (b) The total mass is now $m=2+5+5+8=20$ and if the new object is located at $x=b$ then $M_0=2\cdot4+5\cdot2+5\cdot6+8\cdot b=48+8b$ so:
   $$5=\overline{x}=\frac{48+8b}{20}=2.4+0.4b\ \Rightarrow\ b=6.5$$

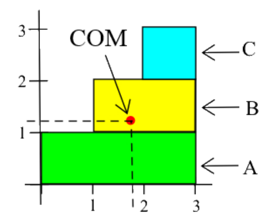
   (c) If the new mass is μ, the total mass becomes $m=12+\mu$ and $M_0=48+\mu\cdot10$ so:
   $$6=\overline{x}=\frac{48+10\mu}{12+\mu}\ \Rightarrow \mu=6$$

3. (a) $m=2+5+5=12$, while $M_y=2\cdot4+5\cdot2+5\cdot6=48$ and $M_x=2\cdot3+5\cdot4+5\cdot2=36$, so $\overline{x}=\dfrac{48}{12}=4$ and $\overline{x}=\dfrac{36}{12}=3$

   (b) The total mass is now $m=2+5+5+10=22$ and if the new object is located at (b,c) then $M_y=48+10b$ and $M_x=36+10c$ so:
   $$5=\overline{x}=\frac{48+10b}{22}=\frac{24}{11}+\frac{5}{11}b\ \Rightarrow\ b=6.2$$
   $$2=\overline{y}=\frac{36+10c}{22}=\frac{18}{11}+\frac{5}{11}c\ \Rightarrow\ c=0.8$$
   You should locate the new object at $(6.2,0.8)$.

5. Split the region into two rectangles, A and B (see figure below left). Then A has mass 6 and center of mass $(0.5,4)$, while B has mass 6 and center of mass $(3,0.5)$. The total mass is $m=12$, $M_y=(6)(0.5)+(6)(3)=21$ and $M_x=(6)(4)+(6)(0.5)=27$, so $\overline{x}=\frac{21}{12}=1.75$ and $\overline{y}=\frac{27}{12}=2.25$. Note the center of mass $(1.75,2.25)$ is not in the region!





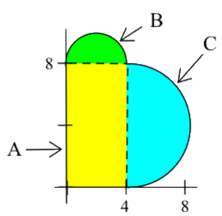
7. Split the region into three rectangles, A, B and C (see figure above right). Then A has mass 3 and center of mass $(1.5,0.5)$, while B has mass 2 and center of mass $(2,1.5)$, and C has mass 1 and center of mass $(2.5,2.5)$. The total mass is $m=6$, $M_y=(3)(1.5)+(2)(2)+(1)(2.5)=11$ and $M_x=(3)(0.5)+(2)(1.5)+(1)(2.5)=7$, so $\overline{x}=\frac{11}{6}$ and $\overline{y}=\frac{7}{6}$.

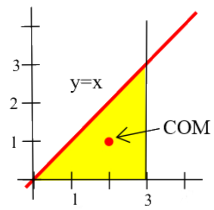
9. Split the region into three pieces, A, B and C (see below left); assume $\rho = 1$. A has mass $(8)(4) = 32$ and center of mass $(2, 4)$, while B has mass $\frac{1}{2}\pi\left(2^2\right) = 2\pi$ and center of mass $\left(2, 8 + \frac{4}{3\pi}(2)\right)$, and C has mass $\frac{1}{2}\pi\left(4^2\right) = 8\pi$ and center of mass $\left(4 + \frac{4}{3\pi}(4), 4\right)$. The total mass is $32 + 10\pi$, while:

$$M_y = (32)(2) + (2\pi)(2) + (8\pi)\left(4 + \frac{16}{3\pi}\right) \approx 219.76$$

$$M_x = (32)(4) + (2\pi)\left(8 + \frac{8}{3\pi}\right) + (8\pi)(4) = 284.13$$

so $\overline{x} \approx \frac{219.76}{63.42} \approx 3.47$ and $\overline{y} \approx \frac{284.13}{63.42} \approx 4.48$



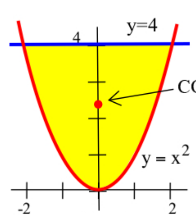


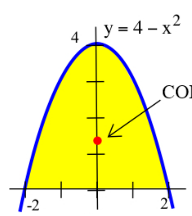
11. See above right. With $\rho = 1$, $m = \int_0^3 x\,dx = \frac{9}{2}$, $M_y = \displaystyle\int_0^3 x \cdot x\,dx = 9$ and $M_x = \displaystyle\int_0^3 \frac{1}{2} \cdot [x]^2\,dx = \dfrac{9}{2}$ so $\overline{x} = \frac{9}{4.5} = 2$ and $\overline{y} = \frac{4.5}{4.5} = 1$.

13. See below left. The region is symmetric about the y-axis, so $\overline{x} = 0$. With $\rho = 1$, $m = \int_{-2}^{2}\left[4 - x^2\right]dx = \frac{32}{3}$ and:

$$M_x = \int_{-2}^{2} \frac{1}{2}\left[(4)^2 - \left(x^2\right)^2\right]dx = \frac{128}{5}$$

so $\overline{y} = \dfrac{\frac{128}{5}}{\frac{32}{3}} = \dfrac{12}{5} = 2.4$.



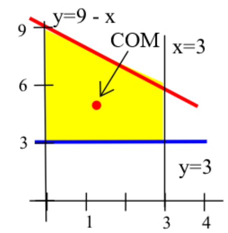


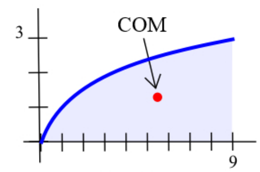
15. See above right. By symmetry, $\overline{x} = 0$. With $\rho = 1$, $m = \int_{-2}^{2}\left(4 - x^2\right)dx = \frac{32}{3}$ and:

$$M_x = \int_{-2}^{2} \frac{1}{2}\left[4 - x^2\right]^2 dx = \frac{256}{15}$$

so $\overline{y} = \dfrac{\frac{256}{15}}{\frac{32}{3}} = \dfrac{8}{5} = 1.6$.

17. If $\rho = 1$, $m = \int_0^3 [(9 - x) - 3]\,dx = \frac{27}{2}$, while $M_y = \int_0^3 x \cdot [(9 - x) - 3]\,dx = 18$ and $M_x = \int_0^3 \frac{1}{2}\left[(9 - x)^2 - 3^2\right]dx = 72$ so $\overline{x} = \dfrac{18}{\frac{27}{2}} = \dfrac{4}{3}$ and $\overline{y} = \dfrac{72}{\frac{27}{2}} = \dfrac{16}{3}$. (See below left.)





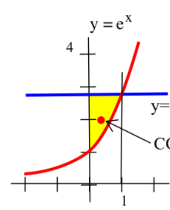
19. See above right. If $\rho = 1$, $m = \int_0^9 \sqrt{x}\,dx = 18$, $M_y = \int_0^9 x \cdot \sqrt{x}\,dx = 97.2$ and $M_x = \int_0^9 \frac{1}{2}\left[\sqrt{x}\right]^2 dx = 20.25$ so $\overline{x} = \dfrac{97.2}{18} = 5.4$ and $\overline{y} = \dfrac{20.25}{18} = 1.125$.

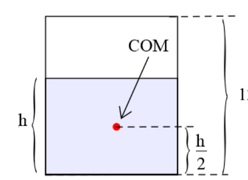
21. If $\rho = 1$, $m = \int_0^1 [e - e^x]\,dx = 1$, while:

$$M_y = \int_0^1 x \cdot [e - e^x]\,dx \approx 0.359$$

$$M_x = \int_0^1 \frac{1}{2}\left[e^2 - (e^x)^2\right]dx = \frac{1}{4}\left(e^2 + 1\right) \approx 2.097$$

so $\overline{x} \approx 0.359$ and $\overline{y} \approx 2.097$. (See below left.)





23. (a) The box has weight 10 and center of mass located 6 inches above the center of its base. If the box is filled with liquid to a height h inches, then the weight of the liquid is $\left(\frac{h}{12}\right)(60) = 5h$ and the liquid's center of mass is located $\frac{h}{2}$ inches above the center of the box's base (see figure above right). The total weight is thus $10 + 5h$ and the moment of the system about the base of the box is $(10)(6) + (5h)\left(\frac{h}{2}\right) = 60 + \frac{5}{2}h^2$, making the height of the system's center of gravity:

$$H = \frac{60 + \frac{5}{2}h^2}{10 + 5h}$$

(b) To minimize H, compute:

$$\frac{dH}{dh} = \frac{(10+5h)\cdot 5h - \left(60+\frac{5}{2}h^2\right)\cdot 5}{(10+5h)^2}$$

Set this equal to 0 to get $h = -2 \pm 2\sqrt{7}$. The $-$ option is unrealistic, so the only critical point occurs where $h = -2 + 2\sqrt{7} \approx 3.3$ inches. Both "endpoints" (empty and full) have center of mass at 6 inches, so $h \approx 3.3$ must yield a minimum.

25. The can has mass 15 g and center of mass located at height 6 cm. If the soda in the can has height h, the mass of the soda is $(400 \text{ g})\left(\frac{h}{12}\right) = \frac{100h}{3}$ g and its center of mass is at height $\frac{h}{2}$. The mass of the can-soda system is $15 + \frac{100h}{3}$ and the moment of the system about the bottom of the can is $(15)(6) + \left(\frac{100h}{3}\right)\left(\frac{h}{2}\right)$ so the height of the system's center of mass is:

$$H = \frac{90 + \frac{50h^2}{3}}{15 + \frac{100h}{3}} = \frac{54 + 10h^2}{9 + 20h}$$

27. On your own.

29. Yes. (What are the only shapes that have exactly two lines of symmetry?)

31. The center of gravity of the water is 2 feet above the bottom of the box, so the total work is $(300 \text{ lb})(8 \text{ ft}) = 2400$ ft-lbs.

33. The center of gravity of the water is at the center of the sphere. The total volume of water is $\frac{4}{3}\pi(2)^3 = \frac{32\pi}{3}$ m^3, so the total mass of water is $\frac{32000\pi}{3}$ kg and the total work is $\left(\frac{32000\pi}{3} \text{ kg}\right)\left(9.81 \ \frac{\text{m}}{\text{sec}^2}\right)(5 \text{ m}) \approx 1643681$ J.

35. The area of the square is 4 cm^2. Using Pappus' Theorem about solids of revolution:
    - (a) $\left(4 \text{ cm}^2\right)(2\pi\cdot 4 \text{ cm}) = 32\pi \text{ cm}^3$
    - (b) $\left(4 \text{ cm}^2\right)(2\pi\cdot 3 \text{ cm}) = 24\pi \text{ cm}^3$
    - (c) $\left(4 \text{ cm}^2\right)(2\pi\cdot 2 \text{ cm}) = 16\pi \text{ cm}^3$
    - (d) $\left(4 \text{ cm}^2\right)(2\pi\cdot 3 \text{ cm}) = 24\pi \text{ cm}^3$
    - (e) $\left(4 \text{ cm}^2\right)\left(2\pi\sqrt{\left(\frac{18}{11}\right)^2 + \left(\frac{32}{11}\right)^2} \text{ cm}\right) \approx 20.97 \text{ cm}^3$

37. The perimeter of the square is 4 cm. Using Pappus' Theorem about surface areas of revolution:
    - (a) $(4 \text{ cm})(2\pi\cdot 4 \text{ cm}) = 32\pi \text{ cm}^2$
    - (b) $(4 \text{ cm})(2\pi\cdot 3 \text{ cm}) = 24\pi \text{ cm}^2$
    - (c) $(4 \text{ cm})(2\pi\cdot 2 \text{ cm}) = 16\pi \text{ cm}^2$
    - (d) $(4 \text{ cm})(2\pi\cdot 3 \text{ cm}) = 24\pi \text{ cm}^2$
    - (e) $(4 \text{ cm})\left(2\pi\sqrt{\left(\frac{18}{11}\right)^2 + \left(\frac{32}{11}\right)^2} \text{ cm}\right) \approx 20.97 \text{ cm}^2$

39. The area of the circle is 4π cm^2 and the perimeter is 4π cm. Using Pappus' Theorems:
    - (a) volume $= \left(4\pi \text{ cm}^2\right)(2\pi\cdot 5 \text{ cm}) = 40\pi^2 \text{ cm}^3$
      surface area $= (4\pi \text{ cm})(2\pi\cdot 5 \text{ cm}) = 40\pi^2 \text{ cm}^2$
    - (b) $\text{V} = \left(4\pi \text{ cm}^2\right)(2\pi\cdot 3 \text{ cm}) = 24\pi^2 \text{ cm}^3$
      $\text{SA} = (4\pi \text{ cm})(2\pi\cdot 3 \text{ cm}) = 24\pi^2 \text{ cm}^2$
    - (c) $\text{V} = \left(4\pi \text{ cm}^2\right)(2\pi\cdot 4 \text{ cm}) = 32\pi^2 \text{ cm}^3$
      $\text{SA} = (4\pi \text{ cm})(2\pi\cdot 4 \text{ cm}) = 32\pi^2 \text{ cm}^2$
    - (d) $\text{V} = \left(4\pi \text{ cm}^2\right)(2\pi\cdot 3 \text{ cm}) = 24\pi^2 \text{ cm}^3$
      $\text{SA} = (4\pi \text{ cm})(2\pi\cdot 3 \text{ cm}) = 24\pi^2 \text{ cm}^2$
    - (e) $\text{V} = \left(4 \text{ cm}^2\right)\left(2\pi\cdot\sqrt{\left(\frac{24}{11}\right)^2 + \left(\frac{36}{11}\right)^2} \text{ cm}\right) \approx 98.86 \text{ cm}^3$
    - (f) $\text{SA} = (4 \text{ cm})\left(2\pi\cdot\sqrt{\left(\frac{24}{11}\right)^2 + \left(\frac{36}{11}\right)^2} \text{ cm}\right) \approx 98.86 \text{ cm}^2$

41. Each rectangle has area 8, perimeter 12 and a centroid 3 ft away from the axis, so each rectangle has volume $8\cdot 2\pi\cdot 3 = 48\pi$ ft^3 and surface area $12\cdot 2\pi\cdot 3 = 72\pi$ ft^2.

43. Colorado? Hawaii?

Section 5.7

1. Replace the upper (infinite) limit of the integral with a massive number M:

$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{M\to\infty}\int_{10}^{M} x^{-3}\,dx = \lim_{M\to\infty}\left[-\frac{1}{2}x^{-2}\right]_{10}^{M}$$
$$= \lim_{M\to\infty}\left[-\frac{1}{2M^2} + \frac{1}{200}\right] = \frac{1}{200}$$

so the improper integral converges to 0.05.

3. Replacing the upper limit of the integral with M:

$$\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \Big[\arctan(x)\Big]_{\sqrt{3}}^{M} = \lim_{M\to\infty} \Big[\arctan(M) - \frac{\pi}{3}\Big] = \frac{\pi}{6}$$

so the improper integral converges to $\frac{\pi}{6}$.

5. Replacing the upper limit of the integral with M:

$$\int_e^{\infty} \frac{5}{x\cdot\ln(x)}\,dx = \lim_{M\to\infty}\int_e^M \frac{5}{x\cdot\ln(x)}\,dx$$

Substitute $u = \ln(x) \Rightarrow du = \frac{1}{x}\,dx$, so $x = e \Rightarrow u = \ln(e) = 1$ and $x = M \Rightarrow u = \ln(M)$:

$$\int_e^M \frac{5}{x\cdot\ln(x)}\,dx = \int_1^{\ln(M)} \frac{5}{u}\,du = \Big[5\ln(u)\Big]_1^{\ln(M)} = 5\ln\left(\ln(M)\right) - 0$$

Returning to the improper integral:

$$\lim_{M\to\infty}\int_e^M \frac{5}{x\cdot\ln(x)}\,dx = \lim_{M\to\infty}\left[5\ln\left(\ln(M)\right)\right] = \infty$$

so the improper integral diverges.

7. Replacing the upper limit of the integral with M:

$$\lim_{M\to\infty}\int_3^M \frac{1}{x-2}\,dx = \lim_{M\to\infty}\Big[\ln\left(|x-2|\right)\Big]_3^M = \lim_{M\to\infty}\left[\ln(M-2) - 0\right] = \infty$$

so the improper integral diverges.

9. Replacing the upper limit of the integral with M:

$$\lim_{M\to\infty}\int_3^M (x-2)^{-3}\,dx = \lim_{M\to\infty}\left[-\frac{1}{2}(x-2)^{-2}\right]_3^M = \lim_{M\to\infty}\left[-\frac{1}{2(M-2)^2} + \frac{1}{2}\right]$$

so the improper integral converges to 0.5.

11. Replacing the upper limit of the integral with M:

$$\lim_{M\to\infty}\int_3^M (x+2)^{-2}\,dx = \lim_{M\to\infty}\left[-(x+2)^{-1}\right]_3^M = \lim_{M\to\infty}\left[-\frac{1}{M+2} + \frac{1}{5}\right]$$

so the improper integral converges to 0.2.

13. Replacing the lower limit of the integral with a:

$$\lim_{a\to0^+}\int_a^4 x^{-\frac{1}{2}}\,dx = \lim_{a\to0^+}\left[2\sqrt{x}\right]_a^4 = \lim_{a\to0^+}\left[4 - 2\sqrt{a}\right] = 4$$

so the improper integral converges to 4.

15. Replacing the lower limit of the integral with a:

$$\lim_{a\to0^+}\int_a^{16} x^{-\frac{1}{4}}\,dx = \lim_{a\to0^+}\left[\frac{4}{3}x^{\frac{3}{4}}\right]_a^{16} = \lim_{a\to0^+}\left[\frac{4}{3}\cdot 8 - \frac{4}{3}a^{\frac{3}{4}}\right] = \frac{32}{3}$$

so the improper integral converges to $\frac{32}{3}$.

17. Replacing the upper limit of the integral with b and using Appendix I:

$$\lim_{b\to2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to2^-}\left[\arcsin\left(\frac{x}{2}\right)\right]_0^b = \lim_{b\to2^-}\arcsin\left(\frac{b}{2}\right) = \frac{\pi}{2}$$

so the improper integral converges to $\frac{\pi}{2}$.

19. Replacing the upper limit of the integral with M:

$$\lim_{M\to\infty}\int_{-2}^M \sin(x)\,dx = \lim_{M\to\infty}\Big[-\cos(x)\Big]_{-2}^M = \lim_{M\to\infty}\left[-\cos(M) + \cos(-2)\right]$$

This limit does not exist, so the integral diverges.

21. Replacing the upper limit of the integral with b:

$$\lim_{b\to\frac{\pi}{2}^-}\int_0^b \tan(x)\,dx = \lim_{b\to\frac{\pi}{2}^-}\Big[-\ln\left(|\cos(x)|\right)\Big]_0^b = \lim_{b\to\frac{\pi}{2}^-}\left[-\ln\left(\cos(b)\right)\right] = \infty$$

so the improper integral diverges.

23. The issue for this integrand occurs not an endpoint but at the point $x = \frac{\pi}{2}$, so we need to split the original integral into two pieces, each of which has only one improper endpoint:

$$\int_0^{\pi}\tan(x)\,dx = \int_0^{\frac{\pi}{2}}\tan(x)\,dx + \int_{\frac{\pi}{2}}^{\pi}\tan(x)\,dx$$

From Problem 21, we know the first of these two new integrals diverges, so the original integral must diverge as well.

25. This integral is improper at both endpoints, so we need to split it into two pieces. We can choose any value we like at which to split the interval of integration, but $x = 0$ works nicely:

$$\int_{\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

The first of these new integrals converges (to $\frac{\pi}{2}$):

$$\lim_{N\to-\infty} \int_{N}^{0} \frac{1}{1+x^2}\,dx = \lim_{N\to-\infty} \left[-\arctan(N)\right] = \frac{\pi}{2}$$

Similarly, the second integral converges (to $\frac{\pi}{2}$):

$$\lim_{M\to\infty} \int_{0}^{M} \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \left[\arctan(M)\right] = \frac{\pi}{2}$$

so the original integral converges to $\frac{\pi}{2} + \frac{\pi}{2} = \pi$.

27. Converges (by the P-Test, with $p = 5 > 1$).

29. Converges (by the P-Test, with $p = \frac{6}{5} > 1$).

31. Diverges (by the P-Test, with $p = \frac{2}{3} < 1$).

33. Converges (by the P-Test, with $p = \frac{7}{4} > 1$).

35. Using the comparison $0 < \frac{1}{x^2+5} < \frac{1}{x^2}$ and the fact that $\int_3^\infty \frac{1}{x^2}\,dx$ converges (P-Test, with $p = 2 > 1$), the smaller integral $\int_3^\infty \frac{1}{x^2+5}\,dx$ converges.

37. When $x > 5$, $x^3 + x > x^3$ so $0 < \frac{1}{x^3+x} < \frac{1}{x^3}$. By the P-Test with $p = 3 > 1$, we know that $\int_3^\infty \frac{1}{x^3}\,dx$ converges, so the smaller integral $\int_3^\infty \frac{1}{x^3+x}\,dx$ also converges.

39. For $x \geq e$, $\ln(x) < x$, so $x + \ln(x) < 2x$ and:

$$\frac{1}{x+\ln(x)} > \frac{1}{2x} \;\Rightarrow\; \frac{7}{x+\ln(x)} > \frac{7}{2}\cdot\frac{1}{x}$$

By the P-Test with $p = 1$, we know that $\int_e^\infty \frac{1}{x}\,dx$ diverges, so $\frac{7}{2}\int_e^\infty \frac{1}{x}\,dx$ diverges and the bigger integral $\int_e^\infty \frac{7}{x+\ln(x)}\,dx$ must diverge as well.

41. For $x \geq \pi$, $-1 \leq \cos(x) \leq 1$ so:

$$0 \leq 1+\cos(x) \leq 2 \;\Rightarrow\; 0 \leq \frac{1+\cos(x)}{x^2} \leq \frac{2}{x^2}$$

We also know that $\int_\pi^\infty \frac{1}{x^2}\,dx$ converges (by the P-Test with $p = 2 > 1$), hence $2\cdot\int_\pi^\infty \frac{1}{x^2}\,dx$ converges, so the smaller integral $\int_\pi^\infty \frac{1+\cos(x)}{x^2}\,dx$ converges as well.

43. For $x \geq 1$, $x^5 + 1 \leq x^5 + x^5 = 2x^5$ so:

$$\frac{1}{x^5+1} \geq \frac{1}{2x^5} \;\Rightarrow\; \frac{x^4}{x^5+1} \geq \frac{x^4}{2x^5} = \frac{1}{2}\cdot\frac{1}{x}$$

Because $\int_1^\infty \frac{1}{x}\,dx$ diverges (by the P-Test with $p = 1$), we know that $\int_1^\infty \frac{1}{2}\cdot\frac{1}{x}\,dx$ diverges, so the bigger integral $\int_1^\infty \frac{x^4}{x^5+1}\,dx$ diverges as well. We can write the original integral as:

$$\int_0^1 \frac{x^4}{x^5+1}\,dx + \int_1^\infty \frac{x^4}{x^5+1}\,dx$$

Because the second of these integrals diverges, the original integral diverges.

45. Using the disk method, the volume is:

$$\int_1^\infty \pi\cdot\left[\frac{1}{x}\right]^2 dx = \lim_{M\to\infty} \pi \int_1^M x^{-2}\,dx$$
$$= \lim_{M\to\infty} \pi \left[-\frac{1}{x}\right]_1^M = \lim_{M\to\infty} \pi \left[-\frac{1}{M} + 1\right]$$

so the volume is finite and is equal to π.

47. Using the disk method, the volume is:

$$\int_0^\infty \pi\cdot\left[\frac{1}{1+x^2}\right]^2 dx = \lim_{M\to\infty} \pi \int_0^M \frac{1}{\left(x^2+1\right)^2}\,dx$$

Using Appendix I to find an antiderivative:

$$\lim_{M\to\infty} \frac{\pi}{2}\left[\frac{x}{x^2+1} + \arctan(x)\right]_0^M$$
$$= \lim_{M\to\infty} \frac{\pi}{2}\left[\frac{M}{M^2+1} + \arctan(M)\right] = \frac{\pi}{2}\cdot\frac{\pi}{2}$$

so the volume is finite and is equal to $\frac{\pi^2}{4}$.

49. Using the tube method, the volume is:

$$\int_0^\infty 2\pi x\left[\frac{1}{1+x^2}\right] dx = \lim_{M\to\infty} \pi \int_0^M \frac{2x}{(x^2+1)}\,dx$$

Using the substitution $u = x^2 + 1$, this becomes:

$$\lim_{M\to\infty} \pi\left[\ln(x^2+1)\right]_0^M = \lim_{M\to\infty} \pi\left[\ln(M^2+1)\right] = \infty$$

so the volume is infinite.

51. The sum (which is a left-endpoint Riemann sum for the integral) is larger.

53. The sum is larger.

55. As long as $s > 0$:

$$F(s) = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{M\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^M$$
$$= \lim_{M\to\infty} -\frac{1}{s} \left[e^{-sM} - 1 \right] = \frac{1}{s}$$

57. As long as $s > 0$:

$$G(s) = \int_2^\infty e^{-st} \cdot 1\, dt = \lim_{M\to\infty} \left[-\frac{1}{s} e^{-st} \right]_2^M$$
$$= \lim_{M\to\infty} -\frac{1}{s} \left[e^{-sM} - e^{-2s} \right] = \frac{e^{-2s}}{s}$$

59. If $b > 0$, the integral $\int_0^b \frac{1}{x^q}\, dx$ converges if $q < 1$ and diverges if $q \geq 1$. The proof mimics the proof of the P-Test. Rewrite the integral as:

$$\lim_{a\to 0^+} \int_a^b \frac{1}{x^q}\, dx$$

then consider the cases $q = 1$, $q < 1$ and $q > 1$.

Section 5.8

1. For window A, the total hydrostatic force is:

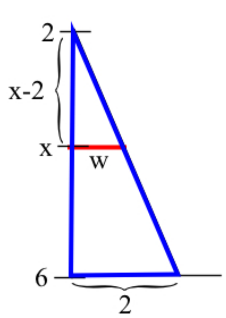
$$\int_2^6 2\rho x\, dx = \rho \left[x^2 \right]_2^6 = 32\rho$$

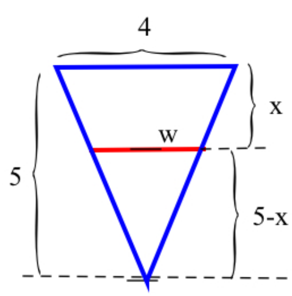
For window B, the width w of a horizontal slice at depth x (see below left) satisfies:

$$\frac{w}{x-2} = \frac{2}{4} \Rightarrow w = \frac{1}{2}(x-2)$$

so the total hydrostatic force is:

$$\int_2^6 2\rho x \cdot \frac{1}{2}(x-2)\, dx = \frac{\rho}{2}\left[\frac{1}{3}x^3 - x^2 \right]_2^6 = \frac{56\rho}{3}$$





3. For the triangular end, the width w of a horizontal slice at depth x (see above right) satisfies:

$$\frac{w}{5-x} = \frac{4}{5} \Rightarrow w = \frac{4}{5}(5-x)$$

so the total hydrostatic force is:

$$\int_0^5 \rho x \cdot \frac{4}{5}(5-x)\, dx = \frac{4\rho}{5}\left[\frac{5}{2}x^2 - \frac{1}{3}x^3 \right]_0^5 = \frac{50\rho}{3}$$

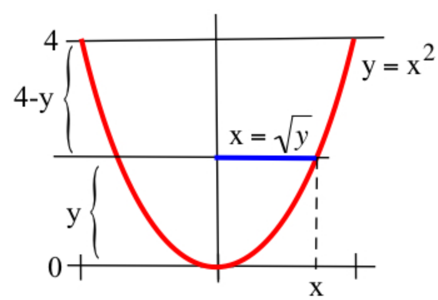
For the rectangular end, the total force is:

$$\int_0^5 \rho x \cdot 4\, dx = 2\rho \left[x^2 \right]_0^5 = 50\rho$$

The length plays no role in the above computations, so doubling the length does not change the value of either hydrostatic force.

5. A slice at height y above the bottom of the tank has width $2\sqrt{y}$ (see below) so the total force is:

$$\int_0^4 \rho\,(4-y) \cdot 2\sqrt{y}\, dy = 2\rho \int_0^4 \left[4y^{\frac{1}{2}} - y^{\frac{3}{2}} \right] dy$$
$$= 2\rho \left[\frac{8}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} \right]_0^4 = \frac{256\rho}{15}$$



7. The one with the largest perimeter (not the left one, probably the middle one).

9. (a) A "slice" of the cylinder at depth x of thickness Δx has area $2\pi \cdot 20 \cdot \Delta x$ so the total force against the bottom 2 feet of the cylinder is:

$$\int_{28}^{30} \rho x \cdot 40\pi\, dx = 20\pi\rho \left[x^2 \right]_{28}^{30} = 2320\pi\rho$$

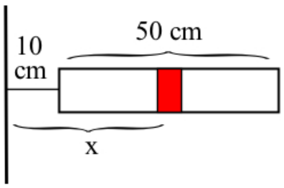
(b) Only the limits of integration change:

$$\int_{33}^{35} \rho x \cdot 40\pi\, dx = 20\pi\rho \left[x^2 \right]_{33}^{35} = 2720\pi\rho$$

11. (a) $m = 20$ g and $v = 3 \cdot 2\pi \cdot 15 = 90\pi$ cm/sec, so KE $= \frac{1}{2} \cdot 20\,(90\pi)^2 = 81000\pi^2 \approx 799438$ ergs
(b) $m = 20$ g, $v = 3 \cdot 2\pi \cdot 20 = 120\pi$ cm/sec, so KE $= \frac{1}{2} \cdot 20\,(120\pi)^2 = 144000\pi^2 \approx 1421223$ ergs

13. The kinetic energy of a vertical slice of the bar (see below) of width Δx cm located x cm from the axis has kinetic energy $\frac{1}{2}\,(3\,\Delta x)\,(2\cdot 2\pi x)^2 = 24\pi^2 x^2\,\Delta x$ so the object's total kinetic energy is:

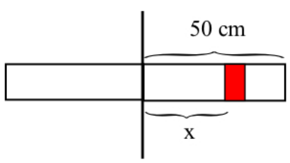
$$\int_{10}^{60} 24\pi^2 x^2\,dx = 8\pi^2 x^3\Big|_{10}^{60} \approx 1.7\text{ J}$$



15. The density of the bar is 0.2 g/cm so the kinetic energy of a vertical slice of the bar (see below) of width Δx cm located x cm from the axis has kinetic energy $\frac{1}{2}\,(0.2\,\Delta x)\,(2\pi x)^2 = 0.4\pi^2 x^2\,\Delta x$ so the kinetic energy of the right half of the object is:

$$\int_{0}^{50} 0.4\pi^2 x^2\,dx = \frac{0.4}{3}\pi^2 x^3\Big|_{0}^{50} \approx 164493.4\text{ ergs}$$

By symmetry, the total kinetic energy of the object is $2\cdot 164493.4 = 328,986.8$ ergs.



17. (a) Proceeding as in Example 5, an annular "slice" of the first washer of thickness Δx and radius x has (approximate) mass $1\cdot 2\pi x\Delta x\cdot 1$ g and velocity $6\pi x$ cm/sec so the kinetic energy of the slice is (approximately) $\frac{1}{2}\,(2\pi x\delta x)\,(6\pi x)^2 = 36\pi^3 x^3\Delta x$ and the total kinetic energy of the washer is thus:

$$\int_{1}^{3} 36\pi^3 x^3\,dx = 9\pi^3 x^4\Big|_{1}^{3} = 720\pi^3\text{ ergs}$$

(b) Only the height (2 cm instead of 1 cm) and the radii of the second washer differ from the first washer, so the total kinetic energy of the second washwer is:

$$\int_{3}^{4} 72\pi^3 x^3\,dx = 18\pi^3 x^4\Big|_{3}^{4} = 3150\pi^3\text{ ergs}$$

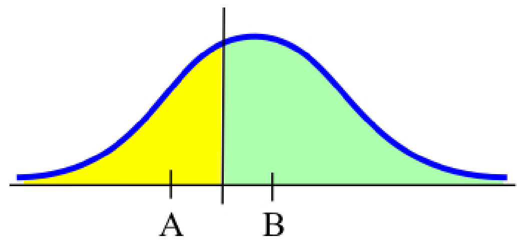
19. The mass of a vertical slice of the place of thickness Δx cm located x cm right of the axis is $3\cdot 10\cdot\Delta x$ g and its velocity is $2\cdot 2\pi x = 4\pi x$ cm/sec so its kinetic energy is $\frac{1}{2}\cdot 30\Delta x\,(4\pi x)^2 = 240\pi^2 x^2\delta x$ and the kinetic energy of the right half of the plate is:

$$\int_{0}^{3} 240\pi^2 x^2\,dx = 80\pi^2 x^3\Big|_{0}^{3} = 2160\pi^2\text{ ergs}$$

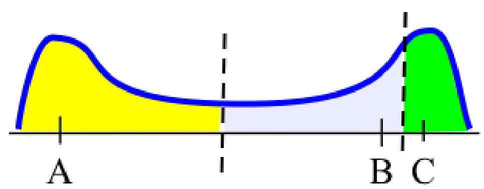
and the kinetic energy of the entire plate is $2\cdot 2160\pi^2 = 4320\pi^2$ ergs.

21. a votes for candidate A, b for A and c for C.

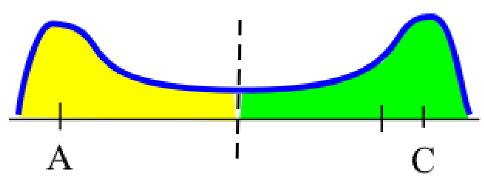
23. B wins:



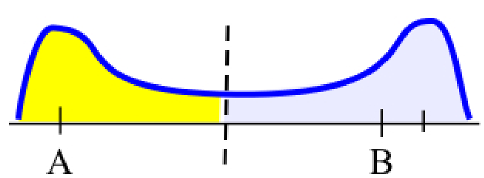
25. (a) A wins:



(b) C wins:



(c) B wins:



27. (a) A (b) A (c) looks like C

29. On your own.

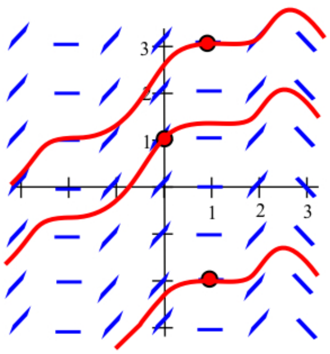
Section 6.1

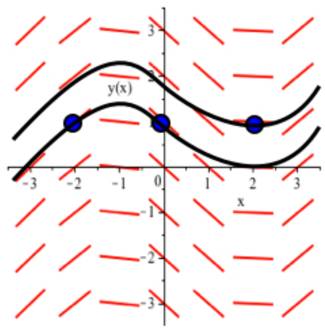
1. $y = e^{-3x} + 2 \Rightarrow y' = -3e^{-3x}$ so $y' + 3y = \left(-3e^{-3x}\right) + 3\left(e^{-3x} + 2\right) = -3e^{-3x} + 3e^{-3x} + 6 = 6$
3. $y = x^2 + 2x \Rightarrow y' = 2x + 2 \Rightarrow y' = 2$ so $y'' - y' + y = 2 - (2x+2) + \left(x^2 + 2x\right) = 2 - 2x - 2 + x^2 + 2x = x^2$
5. $y = 7x^3 - x^2 \Rightarrow y' = 21x^2 - 2x$ so $xy' - 3y = x\left(21x^2 - 2x\right) - 3\left(7x^3 - x^2\right) = 21x^3 - 2x^2 - 21x^3 + 3x^2 = x^2$
7. $y = \frac{1}{2}e^x + 2e^{-x} \Rightarrow y' = \frac{1}{2}e^x - 2e^{-x}$ so $y' + y = \left(\frac{1}{2}e^x - 2e^{-x}\right) + \left(\frac{1}{2}e^x + 2e^{-x}\right) = e^x$
9. $y = \left(7 - x^2\right)^{\frac{1}{2}} \Rightarrow y' = \dfrac{1}{2}\left(7 - x^2\right)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{7 - x^2}} = -\dfrac{x}{y}$
11. $y = 2x^3 - 3x + 3 \Rightarrow y(1) = 2 \cdot 1^3 - 3 \cdot 1 + 3 = 2$ (OK) and $y' = 6x^2 - 3$ (OK)
13. $y = \sin(2x) + 1 \Rightarrow y(0) = \sin(0) + 1 = 1$ (OK) and $y' = 2\cos(2x)$ (OK)
15. $y = 7e^{5x} \Rightarrow y(0) = 7e^{5 \cdot 0} = 7$ (OK) and $y' = 7e^{5x} \cdot 5 - 5y$ (OK)
17. $y = -\dfrac{4}{x} \Rightarrow y(1) = -\dfrac{4}{1} = -4$ (OK) and $y' = \dfrac{4}{x^2}$ so $x \cdot y' = x\left(\dfrac{4}{x^2}\right) = \dfrac{4}{x} = -y$
19. $y = 5\ln(x) - 2 \Rightarrow y(e) = 5\ln(e) - 2 = 5 - 2 = 3$ (OK) and $y' = 5 \cdot \dfrac{1}{x}$ (OK)
21. $7 = y(3) = 3^2 + C = 9 + C \Rightarrow C = -2$
23. $5 = y(0) = Ce^{3 \cdot 0} = C \cdot 1 \Rightarrow C = 5$
25. $4 = y(0) = 2\sin(3 \cdot 0) + C = 0 + C \Rightarrow C = 4$
27. $2 = y(e) = \ln(e) + C = 1 + C \Rightarrow C = 1$
29. $10 = y(2) = -\dfrac{C}{2} \Rightarrow C = -20$
31. $y = \displaystyle\int \left[4x^2 - x\right] dx = \dfrac{4}{3}x^3 - \dfrac{1}{2}x^2 + C$ so $7 = y(1) = \frac{4}{3} - \frac{1}{2} + C \Rightarrow C = \frac{37}{6}$ and $y = \frac{4}{3}x^3 - \frac{1}{2}x^2 + \frac{37}{6}$
33. $y = \displaystyle\int \dfrac{3}{x}\, dx = 3\ln(|x|) + C$ so $2 = y(1) = 3\ln(1) + C = 0 + C$ so $C + 2$ and $y = 3\ln(|x|) + 2$
35. $y = \displaystyle\int 6e^{2x}\, dx = 3e^{2x} + C$ so $1 = y(0) = 3e^{2 \cdot 0} + C = 3 + C \Rightarrow C = -2$ and $y = 3e^{2x} - 2$
37. $y = \displaystyle\int x \cdot \sin\left(x^2\right) dx = -\dfrac{1}{2}\cos\left(x^2\right) + C$ so $3 = y(0) = -\frac{1}{2}\cos(0) + C = -\frac{1}{2} + C \Rightarrow C = \frac{7}{2}$ and
    $y = -\dfrac{1}{2}\cos\left(x^2\right) + \dfrac{7}{2}$
39. $y' = \dfrac{1}{x}\left(6x^3 - 10x^2\right) = 6x^2 - 10x \Rightarrow y = \displaystyle\int \left[6x^2 - 10x\right] dx = 2x^3 - 5x^2 + C$ so
    $5 = y(2) = 2 \cdot 2^3 - 5 \cdot 2^2 + C = 16 - 20 + C = -4 + C \Rightarrow C = 9$ and $y = 2x^3 - 5x^2 + 9$
41. We know that $f'(x) + 5 \cdot f(x) = 0$ and $g'(x) + 5 \cdot g(x) = 0$, so:
    $y = 3f(x) \Rightarrow y' = 3f'(x) \Rightarrow y' + 5y = 3f'(x) + 5 \cdot 3f(x) = 3\left[f'(x) + 5 \cdot f(x)\right] = 3 \cdot 0 = 0$
    $y = 7g(x) \Rightarrow y' = 7g'(x) \Rightarrow y' + 5y = 7g'(x) + 5 \cdot 7g(x) = 7\left[g'(x) + 5 \cdot g(x)\right] = 3 \cdot 0$
    $y = f(x) + g(x) \Rightarrow y' = f'(x) + g'(x) \Rightarrow y' + 5 \cdot y = [f'(x) + g'(x)] + 5 \cdot [f(x) + g(x)]$
    $\Rightarrow y = f'(x) + g'(x) + 5f(x) + 5g(x) = [f'(x) + 5f(x)] + [g'(x) + 5g(x)] = 0 + 0 = 0$
    $y = A \cdot f(x) + B \cdot g(x) \Rightarrow y' = A \cdot f'(x) + B \cdot g'(x) \Rightarrow y' + 5 \cdot y = [A \cdot f'(x) + B \cdot g'(x)] + 5 \cdot [A \cdot f(x) + B \cdot g(x)]$
    $\Rightarrow y = Af'(x) + Bg'(x) + 5Af(x) + 5Bg(x) = A\left[f'(x) + 5f(x)\right] + B\left[g'(x) + 5g(x)\right] = A \cdot 0 + B \cdot 0 = 0$
43. $y = \sin(x) + x \Rightarrow y' = \cos(x) + 1 \Rightarrow y'' = -\sin(x) \Rightarrow y'' + y = [-\sin(x)] + [\sin(x) + x] = x$ (OK)
    $y = \cos(x) + x \Rightarrow y' = -\sin(x) + 1 \Rightarrow y'' = -\cos(x) \Rightarrow y'' + y = [-\cos(x)] + [\cos(x) + x] = x$ (OK)
    $y = 3[\sin(x) + x] \Rightarrow y' = 3\cos(x) + 3 \Rightarrow y'' = -3\sin(x) \Rightarrow y'' + y = -3\sin(x) + 3[\sin(x) + x] = 3x$ (NO)
    Similarly, $y = [\sin(x) + x] + [\cos(x) + x] \Rightarrow y'' + y = 2x$ (NO)

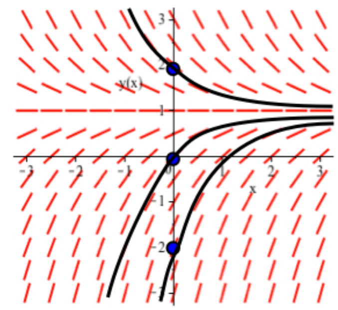
45. $y=\frac{A}{B}-C\cdot e^{-Bt}\Rightarrow \frac{dy}{dt}=0-C\left[-Be^{-Bt}\right]=BCe^{-Bt}$
$\Rightarrow A-By=A-B\left[\frac{A}{B}-C\cdot e^{-Bt}\right]=A-A+BCe^{-Bt}=BCe^{-Bt}=\frac{dy}{dt}$ (OK)

47. $I=\frac{E}{R}\left[1-\cdot e^{-\frac{Rt}{L}}\right]\Rightarrow \frac{dI}{dt}=\frac{E}{R}\left[0-\left(-\frac{R}{L}\right)e^{-\frac{Rt}{L}}\right]=\frac{E}{R}e^{-\frac{Rt}{L}}$
$\Rightarrow L\cdot\frac{dI}{dt}+R\cdot I=l\left[\frac{E}{R}e^{-\frac{Rt}{L}}\right]+r\cdot\frac{E}{R}\left[1-\cdot e^{-\frac{Rt}{L}}\right]=E\cdot e^{-\frac{Rt}{L}}+E\left[1-e^{-\frac{Rt}{L}}\right]=E$ (OK)

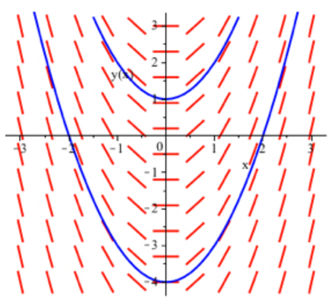
Section 6.2

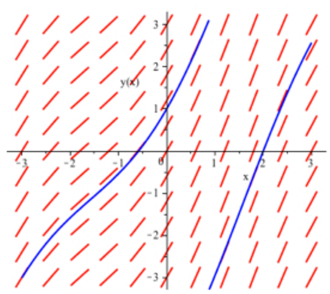
1. 

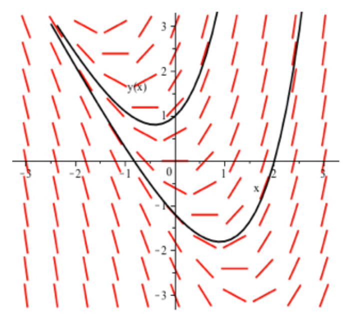
3. 

5. 

7. All solutions appear to approach the horizontal line $y=1$: for any solution $y(x)$, $\lim_{x\to\infty} y(x)=1$.

9. 

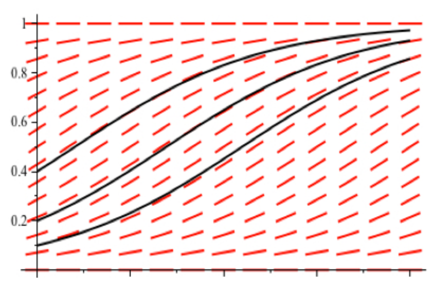
11. 

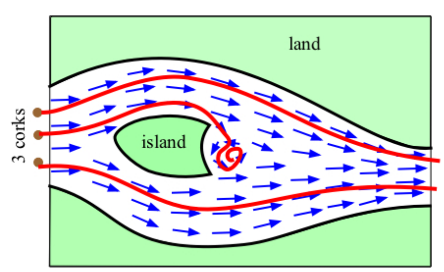
13. 

15. (a) $y=x^2-3x+C$ (b) $4=y(1)=1^2-3\cdot1+C\Rightarrow 4=C-2\Rightarrow C=6$ so $y=x^2-3x+6$ (c) $(-\infty,\infty)$

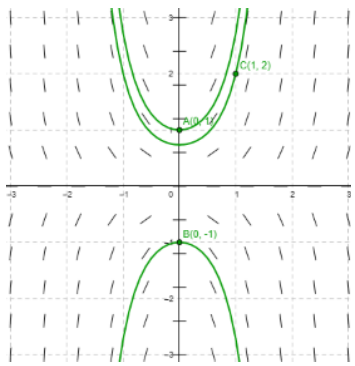
17. (a) $\Rightarrow y=e^x+\sin(x)+C$ (b) $7=y(0)=1+0+C\Rightarrow C=6$ so $y=e^x+\sin(x)+6$ (c) $(-\infty,\infty)$

19. (a) $y'=\dfrac{6}{2x+1}+\sqrt{x}\Rightarrow y=3\ln(|2x+1|)+\dfrac{2}{3}x^{\frac{3}{2}}+C$ (b) $4=y(1)=3\ln(3)+\frac{2}{3}+C\Rightarrow C=\frac{10}{3}-3\ln(3)$ so $3\ln(|2x+1|)+\frac{2}{3}x^{\frac{3}{2}}+\frac{10}{3}-3\ln(3)$ (c) $(0,\infty)$

21. 

23. 

Section 6.3

1. 

3. If $y \neq 0$, divide both sides by y to separate:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x \Rightarrow \frac{1}{y}\,dy = 2x\,dx$$

and then integrate both sides:

$$\int \frac{1}{y}\,dy = \int 2x\,dx \Rightarrow \ln(|y|) = x^2 + C$$

then exponentiate both sides to solve for y:

$$e^{\ln(|y|)} = e^{x^2+C} \Rightarrow |y| = e^C \cdot e^{x^2} \Rightarrow y = \pm e^C \cdot e^{x^2}$$

so $y = Ae^{x^2}$ is a solution for any $A \neq 0$. But the constant function $y(x) = 0$ is also a solution to $y' = 2xy$ so the general solution is $y = Ae^{x^2}$ for any constant A.

5. Divide both sides of the ODE by $1 + x^2$ to separate, then integrate:

$$\frac{dy}{dx} = \frac{3}{1+x^2} \Rightarrow y = 3\arctan(x) + C$$

7. Divide both sides by $e^y \cdot \cos(x)$ to separate:

$$\frac{1}{e^y} \cdot \frac{dy}{dx} = \frac{1}{\cos(x)} \Rightarrow e^{-y}\,dy = \sec(x)\,dx$$

and use Appendix I to assist with the integration:

$$\int e^{-y}\,dy = \int \sec(x)\,dx$$
$$\Rightarrow -e^{-y} = \ln(|\sec(x) + \tan(x)|) + C$$

then solve for y:

$$e^{-y} = K - \ln(|\sec(x) + \tan(x)|) \Rightarrow$$
$$-y = \ln(K - \ln(|\sec(x) + \tan(x)|))$$

so the general solution is:

$$y = -\ln(K - \ln(|\sec(x) + \tan(x)|))$$

9. Note that $y = 0$ is a solution to $y' = 4y$. If $y \neq 0$, divide both sides of the ODE by y to separate:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 4 \Rightarrow \frac{1}{y}\,dy = 4\,dx$$

then integrate both sides:

$$\int \frac{1}{y}\,dy = \int 4\,dx \Rightarrow \ln(|y|) = 4x + C$$

and solve for y:

$$e^{\ln(|y|)} = e^{4x+C} \Rightarrow |y| = e^C \cdot e^{4x} \Rightarrow y = \pm e^C \cdot e^{4x}$$

so $y = Ae^{4x}$ is a solution for any constant $A \neq 0$ but $y(x) = 0$ is also a solution, so the general solution is $y(x) = Ae^{4x}$ for any constant A.

11. From Problem 3, we know that $y = Ae^{x^2}$, so using each initial condition:

$$3 = y(0) = A \cdot e^{0^2} = A \Rightarrow A = 3 \Rightarrow y = 3e^{x^2}$$
$$5 = y(0) = A \cdot e^{0^2} = A \Rightarrow A = 5 \Rightarrow y = 5e^{x^2}$$
$$2 = y(1) = A \cdot e \Rightarrow A = 2e^{-1} \Rightarrow y = 2e^{x^2-1}$$

13. Mimic the solution to Problem 9 to arrive at the general solution: $y = Ae^{3x}$. Then use each initial condition:

$$4 = y(0) = A \cdot e^{3 \cdot 0} = A \Rightarrow A = 4 \Rightarrow y = 4e^{3x}$$
$$7 = y(0) = A \cdot e^{3 \cdot 0} = A \Rightarrow A = 7 \Rightarrow y = 7e^{3x}$$
$$3 = y(1) = A \cdot e^{3 \cdot 1} = Ae^3 \Rightarrow A = 3e^{-3} \Rightarrow y = 3e^{3x-3}$$

15. From Problem 10, we know $y = 2 + Ae^{-5x}$, so using each initial condition:

$$5 = y(0) = 2 + A \cdot 1 \Rightarrow A = 3 \Rightarrow y = 2 + 3e^{-5x}$$
$$-3 = y(0) = 2 + A \cdot 1 \Rightarrow A = -5 \Rightarrow y = 2 - 5e^{-5x}$$

17. From Problem 5, we know $y = 3\arctan(x) + C$, so using each initial condition:

$$4 = y(1) = 3 \cdot \frac{\pi}{4} + C \Rightarrow y = 3\arctan(x) + 4 - \frac{3\pi}{4}$$
$$2 = y(0) = 3 \cdot 0 + C \Rightarrow y = 3\arctan(x) + 2$$

19. No. Putting $x = 0$ and $y = 2$ into the ODE:

$$0 \cdot y'(0) = 2 + 3 \Rightarrow 0 = 5$$

which is a contradiction.

Section 6.4

1. The population is 10,000 around 1966 and is 20,000 around 1978, so $1978 - 1966 = 12$ years. The population grows from 15,000 around 1974 to 30,000 around 1985, so $1985 - 1974 = 11$ years. The doubling time is approximately 12 years.

3. (a) If $P(t)$ represents the population (in thousands) t years after 1990, then $P(t) = 48e^{kt}$ for some constant k. We also know that:

$$64 = 48e^{20k} \;\Rightarrow\; \frac{4}{3} = e^{20k} \;\Rightarrow\; k = \frac{1}{20}\ln\left(\frac{4}{3}\right)$$

so that $P(t) = 48\left(\frac{4}{3}\right)^{\frac{t}{20}} \approx 48e^{0.01438t}$.

(b) $P(30) = 48\left(\frac{4}{3}\right)^{\frac{30}{20}} \approx 73.901$, so if the model holds, in 2020 the community's population will be approximately 74,000.

(c) Solving $P(t) = 100$ for t:

$$100 = 48\left(\frac{4}{3}\right)^{\frac{t}{20}} \;\Rightarrow\; \frac{100}{48} = \left(\frac{4}{3}\right)^{\frac{t}{20}}$$

$$\Rightarrow\; \ln\left(\frac{25}{12}\right) = \frac{t}{20}\ln\left(\frac{4}{3}\right) \;\Rightarrow t = \frac{20\ln\left(\frac{25}{12}\right)}{\ln\left(\frac{4}{3}\right)}$$

so about 51 years later (in 2041).

(d) $\dfrac{\ln(2)}{k} = \dfrac{20\ln(2)}{\ln\left(\frac{4}{3}\right)} \approx 48.19$ years

5. If $A(t)$ is the value of the investment t years later:

$$A(t) = 5000(1.15)^t = 5000e^{t\cdot\ln(1.15)}$$

so the doubling time is $\dfrac{\ln(2)}{\ln(1.15)} \approx 4.96$ years and the tripling time is $\dfrac{\ln(3)}{\ln(1.15)} \approx 7.86$ years.

7. In 1950 the population is approximately 5,000, so if $P(t)$ represents the size of the population t years after 1950, then $P(t) = 5000e^{kt}$ for some constant k. If the doubling time is 12 years:

$$\frac{\ln(2)}{k} = 12 \Rightarrow k = \frac{\ln(2)}{12} \Rightarrow P(t) = 5000e^{\frac{t}{12}\ln(2)}$$

so $P(t) = 5000\,(2)^{\frac{t}{12}}$.

9. $k = \dfrac{\ln(2)}{50} \Rightarrow P(t) = P(0)e^{\frac{t}{50}\cdot\ln(2)} = P(0)\cdot 2^{\frac{t}{50}}$ so $\dfrac{P(1)-P(0)}{P(0)} = 2^{\frac{1}{50}-1} \approx 0.014 = 1.4\%$

11. $6(1.03)^t = 4(1.06)^t \Rightarrow t = \dfrac{\ln\left(\frac{3}{2}\right)}{\ln\left(\frac{1.06}{1.03}\right)} \approx 14.12$ years

13. After t months, you have $S(t) = 8000e^{0.14\left(\frac{t}{12}\right)}$ snails before harvest.

(a) $S(2) = 8000e^{0.14\left(\frac{2}{12}\right)} \approx 8,188$, so after harvest $S = 6188$; $S(4) = 6188e^{0.14\left(\frac{2}{12}\right)} \approx 6334$, so after harvest $S = 4,334$; after third harvest, 2436 remain; after fourth harvest, 494 remain; after fifth harvest, no snails remain.

(b) No snails remain after the third harvest.

(c) The population growth is 188 snails after two months, so you can harvest 188 snails every two months and maintain a stable population (between 8,000 and 8,188).

15. $A(t) = A(0)e^{kt} = 10e^{kt}$ and $A(14) = 2$

(a) $2 = 10e^{14k} \Rightarrow k \approx -0.115 \Rightarrow A(t) = 10e^{\breve{}0.115t}$

(b) $\dfrac{-\ln(2)}{-0.115} \approx 6$ days

(c) $0.7 = 10e^{-0.115t} \Rightarrow t = \dfrac{\ln(0.7)}{-0.115} \approx 23$ days

17. (a) $143 = 187e^{2k} \Rightarrow k = \dfrac{\ln\left(\frac{143}{187}\right)}{2} \approx -0.134$, hence $\dfrac{-\ln(2)}{k} \approx 5.17$ days

(b) $20 = 187e^{-0.134t} \Rightarrow t = \dfrac{\ln\left(\frac{20}{187}\right)}{-0.134} \approx 16.7$ days

19. $3.5 = 8e^{6k} \Rightarrow k = \dfrac{\ln\left(\frac{3.5}{8}\right)}{6} \approx -0.138$, so $A(t) = 8000e^{-0.138t}$ counts per minute after t days.

21. Carbon-14 has a half-life of 5,700 years, so $k = \dfrac{-\ln(2)}{5700} \approx -0.00012$, hence $0.975 = e^{-0.00012t} \Rightarrow t = \dfrac{\ln(0.975)}{-0.00012} \approx 211$ years, but Newton died in 1727, over 288 years ago, so the letter is a fake.

23. $k = \dfrac{-\ln(2)}{6} \approx -0.116$, so $A(t) = 30e^{-0.116t}$ and $A(t) \geq 10 \Rightarrow -0.116t \geq \ln\left(\dfrac{10}{30}\right) \Rightarrow t \leq \dfrac{-1.009}{-0.116} \approx 9.47$ hours. After about 9.5 hours, the concentration of medicine is no longer effective.

25. $k = \dfrac{-\ln(2)}{15} \approx -0.046$, so $A(t) = 9e^{-0.046t}$, hence $A(8) = 9e^{-0.046(8)} \approx 6.23$ mg, resulting in a "decay" of $9 - 6.23 = 2.77$ mg during these 8 hours. Taking a 2.77 mg dose every 8 hours keeps level of the medicine in the safe and effective range over a long period of time.

27. The half-life of iodine-131 is 8.07 days, hence $k = \dfrac{-\ln(2)}{8.07} \approx -0.086$, thus $A(t) = 5Se^{-0.086t}$. If S is highest safe level, $S = 5Se^{-0.086t} \Rightarrow 0.2 = e^{-0.086t}$ so that $t = \dfrac{\ln(0.2)}{-0.086} \approx 18.7$ days.

29. For the population $P(t)$, $P(0) = 4$ and $P(1) = (1.05)(4) = 4e^{k(1)} \Rightarrow k = \ln(1.05) \approx 0.049$, so $P(t) = 4e^{0.049t}$ (in millions). The size of the forest is $F(t) = 10000000 - 300000t$ acres after t years (the entire forest will be gone in 33.3 years).
   (a) $\dfrac{100-3t}{40e^{0.049t}}$ acres per person
   (b) $\mathbf{D}\left(\dfrac{100-3t}{40e^{0.049t}}\right) = \dfrac{-7.9+0.147t}{40e^{0.049t}}$
   (c) Solve $\dfrac{100-3t}{40e^{0.049t}} = 1$ using technology to determine that $t \approx 10.75$ years.

Section 6.5

1. The temperature of the cheesecake t minutes later is given by:

$$f(t) = 35 + [165-35]\,e^{kt} = 35 + 130e^{kt}$$

Solving $150 = 35 + 130e^{10k}$ for k yields:

$$k = \frac{1}{10}\ln\left(\frac{115}{130}\right) \approx -0.01226$$

so that $f(t) = 35 + 130e^{-0.01226t}$. Solving $f(T) = 70$ for T then yields:

$$70 = 35 + 130e^{-0.01226t} \Rightarrow T = -\frac{1}{0.01226}\ln\left(\frac{35}{130}\right)$$

so you will need to wait about 107 minutes.

3. (a) The temperature of the water t minutes later is given by:

$$f(t) = 40 + [200-40]\,e^{kt} = 40 + 160e^{kt}$$

Solving $150 = 40 + 160e^{4k}$ for k yields:

$$k = \frac{1}{4}\ln\left(\frac{110}{160}\right) \approx -0.09367$$

so that $f(t) = 40 + 160e^{-0.09367t}$.

   (b) Solving $100 = 40 + 160e^{-0.09367T}$ for T yields:

$$T = -\frac{1}{0.09367}\ln\left(\frac{60}{160}\right) \approx 10.5 \text{ minutes}$$

   (c) $-\dfrac{1}{0.09367}\ln\left(\dfrac{40}{160}\right) \approx 14.8$ minutes
   (d) ≈ 22.2 minutes

5. If $A(t)$ represents the amount of salt in the tank t minutes later, then $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = -\frac{A}{100}\cdot 3, \quad A(0) = 50$$

Solving this separable ODE yields:

$$\int \frac{1}{A}\,dA = \int -0.03\,dt \Rightarrow \ln(A) = -0.03t + C$$

The initial condition $A(0) = 50$ tells us that $\ln(50) = C$, so:

$$\ln(A) = -0.03t + \ln(50) \Rightarrow A(t) = 50e^{-0.03t}$$

Hence $A(60) = 50e^{-0.03(60)} \approx 8.26$ pounds.

7. If $A(t)$ represents the amount of salt in the tank t minutes later, then $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = -\frac{A}{100+t}\cdot 2, \quad A(0) = 50$$

Solving this separable ODE yields:

$$\int \frac{1}{A}\,dA = \int -\frac{2}{100+t}\,dt$$

so that $\ln(A) = \ln(100+t)^{-2} + C$. The initial condition $A(0) = 50$ then tells us that:

$$\ln(50) = -2\ln(100) + C \Rightarrow C = \ln(500000)$$

hence $A(t) = \dfrac{500000}{(100+t)^2}$, so $A(60) = \dfrac{500000}{160^2} \approx 19.53$ pounds.

Section 7.1

1. $f(x)$ and $y = 3 - x$ are one-to-one because their graphs pass the horizontal line test (HLT); $g(x)$ is not one-to-one because $g(1) = g(4)$; h is not because its graph fails the HLT.

3. The graph of $f(x)$ fails the HLT; $y = e^x - 2$ is one-to-one because $y' = e^x > 0$ (see Problem 13); $g(x)$ is one-to-one because the output values are all distinct; the graph of h passes the HLT.

5. The relation is a function if the domain is all people with a Social Security number; ideally, it is a one-to-one function (but due to identity theft and other issues may not be so in practice).

7. No two students received the same score.

9. At most one.

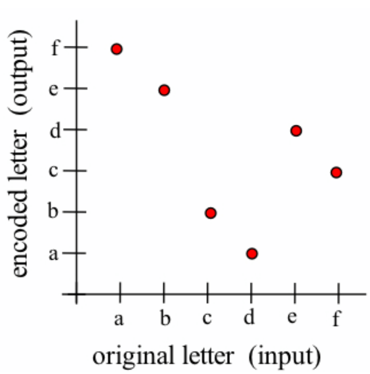
11. (a) Yes (graph passes HLT).
    (b) No, $f(0) > f(1)$.
    (c) No, $f(0) < f(0.5)$.

13. $f'(x) = \frac{1}{x} > 0$ for all $x > 0$. If $0 < a < b$ with $f(a) = f(b)$, then Rolle's Theorem guarantees a c with $a < c < b$ and $f'(c) = 0$, which is impossible. Hence $f(a) \neq f(b)$for any $a \neq b$ on $(0, \infty)$, which proves that $f(x)$ is one-to-one.

15. (a) Yes (each input has exactly one output).
    (b) Yes (each input appears only once).
    (c) cde
    (d) Interchanging inputs and outputs:

a	b	c	d	e	f
f	e	b	a	d	c

    (e) Interchanging inputs and outputs:

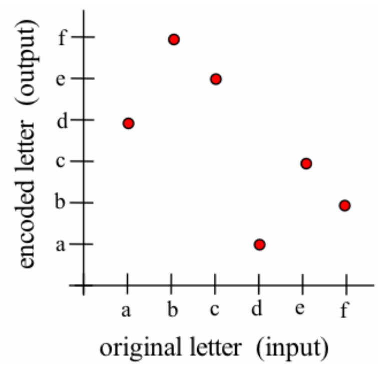


    (f) They are reflections of each other.

17. (a) Yes (each output appears only once).
    (b) Yes (each input appears only once).
    (c) fda
    (d) Interchanging inputs and outputs:

a	b	c	d	e	f
d	c	f	e	b	a

    (e) Interchanging inputs and outputs:



    (f) It's the same! (Function is its own inverse.)

Section 7.2

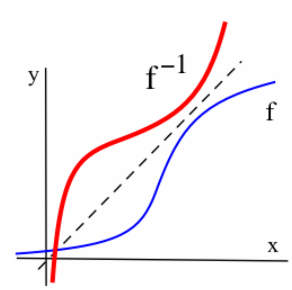
1.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
1	3	-3	2	$\frac{1}{2}$
2	1	2	3	$\frac{1}{3}$
3	2	3	1	$-\frac{1}{3}$

3.

x	$h(x)$	$h'(x)$	$h^{-1}(x)$	$\left(h^{-1}\right)'(x)$
1	2	2	3	undefined
2	3	-2	1	$\frac{1}{2}$
3	1	0	2	$-\frac{1}{2}$

5. The graphs of f and f^{-1} appear below:



7. $a = b$

9. (1) Multiply by 4, (2) add 5 and (3) divide by 7: $f^{-1}(x) = \dfrac{4x+5}{7}$

11. $g^{-1}(x) = \frac{x-1}{2}$, hence $g^{-1}(g(1)) = \frac{3-1}{2} = 1$ and $g^{-1}(g(7)) = \frac{15-1}{2} = 7$.

13. $w^{-1}(x) = e^{x-5}$, so $w^{-1}(w(1)) = e^{5-5} = 1$.

15. The graph of f^{-1} goes through $(3, 1)$ and $\left(f^{-1}\right)'(3) = \dfrac{1}{f'(1)} > 0$.

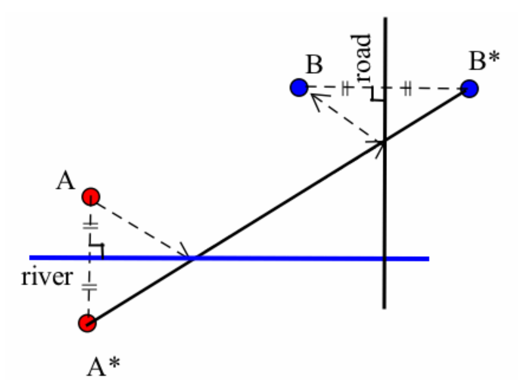
17. $\left(f^{-1}\right)'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)} > 0$, so both f and f^{-1} are increasing functions.

19. $f(x) = x + 2 \Rightarrow f^{-1}(x) = x - 2$ both have slope 1 (hence are parallel) but different y-intercepts.

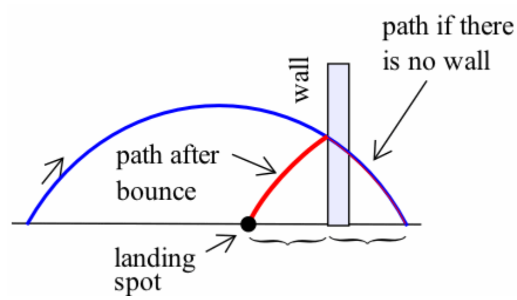
21. $f'(x) = 3 + \cos(x) \geq 2 > 0$, so $f(x)$ is increasing, hence one-to-one.

23. $x = \dfrac{ay+b}{cy-a} \Rightarrow cxy - ax = ay + b \Rightarrow (cx-a)y = ax + b \Rightarrow y = \dfrac{ax+b}{cx-a}$ so the function in (d) is its own inverse, and the functions in (a), (b) and (c) are all special cases of (d).

25. Fold across the river to generate point A^* (see figure below), then unfold; fold across the road to generate the point B^*, then unfold; connect A^* and B^* with a line; fold the line across the river and the road to get the shortest path:



27. Fold across the wall:



The distance from the wall to the landing spot is the same whether the ball bounces off the wall or passes through it.

Section 7.3

1. $\dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$

3. (a) $\arcsin(0.3) < 1$ but $\pi - \arcsin(0.3) \approx 2.837$ and $\arcsin(0.3) + 2\pi \approx 6.588$

   (b) $\arcsin(-0.4) < 1$ but $\pi - \arcsin(-0.4) \approx 3.553$ and $\arcsin(-0.4) + 2\pi \approx 5.872$

   (c) $\dfrac{5\pi}{6}, \dfrac{13\pi}{6}$

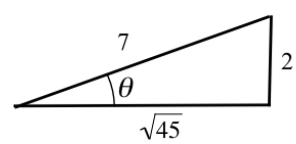
5. (a) $\arctan(3.2) \approx 1.268$, $\pi + 1.268 \approx 4.410$

   (b) $\arctan(-0.2) + \pi \approx 2.944$, $\pi + 2.944 \approx 6.086$

7. (a) $\dfrac{4}{5}$ (b) $\dfrac{4}{3}$ (c) $\dfrac{5}{3}$ (d) $\dfrac{3}{5}$

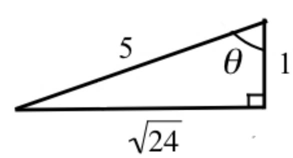
9. (a) $\dfrac{5}{13}$ (b) $\dfrac{5}{12}$ (c) $\dfrac{13}{12}$ (d) $\dfrac{12}{13}$

11. The triangle below should help:



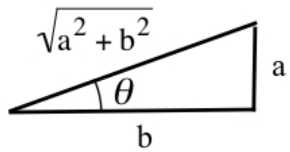
(a) $\dfrac{2}{\sqrt{45}}$ (b) $\dfrac{\sqrt{45}}{7}$ (c) $\dfrac{7}{2}$ (d) $\dfrac{\sqrt{45}}{2}$

13. The triangle below should help:



(a) $\sqrt{24}$ (b) $\dfrac{\sqrt{24}}{5}$ (c) $\dfrac{5}{\sqrt{24}}$ (d) $\dfrac{1}{\sqrt{24}}$

15. The triangle below should help:



(a) $\frac{a}{b}$ (b) $\frac{a}{\sqrt{a^2+b^2}}$ (c) $\frac{b}{\sqrt{a^2+b^2}}$ (d) $\frac{b}{a}$

17. For $0 < \theta < \frac{\pi}{2}$, $\cos^2(\theta) + \sin^2(\theta) = 1 \Rightarrow \sin(\theta) = \sqrt{1-\cos^2(\theta)}$ and $\cos(\theta) = \sqrt{1-\sin^2(\theta)}$ so:

(a) $\sin(\arccos(x)) = \sqrt{1-x^2}$

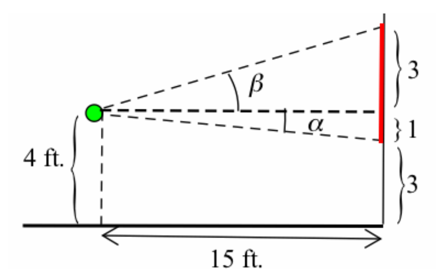
(b) $\cos(\arcsin(x)) = \sqrt{1-x^2}$

(c) $\sec(\arccos(x)) = \frac{1}{x}$

19. (a) No: $2\arcsin(1) = 2 \cdot \frac{\pi}{2} = \pi$, but $\arcsin(2)$ is not defined.

(b) No: $2\arccos(1) = 2 \cdot 0 = 0$, but $\arccos(2)$ is not defined.

21. Let α represent the angle of declination from the viewer to the bottom of the whiteboard, and β the angle of elevation to the top:

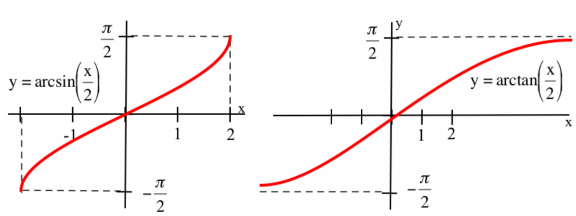


(a) $\alpha + \beta = \arctan\left(\frac{1}{15}\right) + \arctan\left(\frac{3}{15}\right) \approx 0.264$, or about $15.12°$.

(b) Replacing 15 with x:

$$\alpha + \beta = \arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{3}{x}\right)$$

23. The graphs appear below:



25. Differentiating implicitly, $\frac{dh}{dt} = 20\cos(\theta) \cdot \frac{d\theta}{dt}$, so $\left.\frac{dh}{dt}\right|_{\theta=1.3} = 20\cos(1.3) \cdot 12 \approx 64.20$.

27. Differentiating implicitly, $-\sin(\theta) \cdot \frac{d\theta}{dt} = 3 \cdot \frac{dh}{dt}$, so $\left.\frac{dh}{dt}\right|_{\theta=1.3} = -\frac{1}{3}\sin(1.3) \cdot 12 \approx -3.85$.

29. Differentiating implicitly, $\cos(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{38} \cdot \frac{dh}{dt}$, so $\left.\frac{d\theta}{dt}\right|_{\theta=1.3} = \frac{4}{38}\sec(1.3) \approx 0.394$.

31. Differentiating implicitly, $-\sin(\theta) \cdot \frac{d\theta}{dt} = 7 \cdot \frac{dh}{dt}$, so $\left.\frac{d\theta}{dt}\right|_{\theta=1.3} = -28\csc(1.3) \cdot 12 \approx -29.059$.

33. If h is the height of the rocket, then $\tan(\theta) = \frac{h}{4000} \Rightarrow h = 4000\tan(\theta) \Rightarrow \frac{dh}{dt} = 4000\sec^2(\theta) \cdot \frac{d\theta}{dt}$, so $\left.\frac{dh}{dt}\right|_{\theta=\frac{\pi}{3}} = 4000 \cdot 4 \cdot \frac{\pi}{12} \approx 4189$ ft/sec.

35. (a) $\alpha = \arcsin\left(\frac{A}{C}\right)$ (b) $\beta = \arccos\left(\frac{A}{C}\right)$ (c) $\arcsin\left(\frac{A}{C}\right) + \arccos\left(\frac{A}{C}\right) = \alpha + \beta = \frac{\pi}{2}$

37. (a) $\alpha = \operatorname{arcsec}\left(\frac{C}{B}\right)$ (b) $\beta = \operatorname{arccsc}\left(\frac{C}{B}\right)$ (c) $\operatorname{arcsec}\left(\frac{C}{B}\right) + \operatorname{arccsc}\left(\frac{C}{B}\right) = \alpha + \beta = \frac{\pi}{2}$

39. (a) $\frac{5}{12}$ (b) $\frac{12}{5}$

41. (a) $\frac{12}{13}$ (b) $\frac{13}{12}$

43. 1.231

45. π

47. $\frac{2\pi}{3}$

49. $\frac{\pi}{4}$

51. 0.322

53. (a) $\frac{7}{25}$ (b) $\frac{24}{25}$ (c) $\arcsin\left(\frac{7}{25}\right) = \arccos\left(\frac{24}{25}\right)$

55. If $x \neq 0$ then:

$$\sin(\operatorname{arccsc}(x)) = \frac{1}{\csc(\operatorname{arccsc}(x))} = \frac{1}{x}$$

Applying arcsin to each side yields the result.

57. We know that $\tan(\operatorname{arcsec}(x)) = \sqrt{x^2-1}$ from Example 5, so $\operatorname{arcsec}(x) = \arctan\left(\sqrt{x^2-1}\right)$.

Section 7.4

1. $\dfrac{1}{\sqrt{1-(3x)^2}} \cdot 3 = \dfrac{3}{\sqrt{1-9x^2}}$

3. $\dfrac{1}{1+(x+5)^2} = \dfrac{1}{x^2+10x+26}$

5. $\dfrac{1}{1+\left(\sqrt{x}\right)^2} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}\,(1+x)}$

7. $\dfrac{1}{\arctan(x)} \cdot \dfrac{1}{1+x^2}$

9. $3\left(\text{arcsec}(x)\right)^2 \cdot \dfrac{1}{|x|\sqrt{x^2-1}}$

11. $\dfrac{1}{1+(\ln(x))^2} \cdot \dfrac{1}{x}$

13. $e^x \cdot \dfrac{2}{1+4x^2} + \arctan(2x) \cdot e^x$

15. $\dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$

17. $\dfrac{1}{2\sqrt{\arcsin(x)}} \cdot \dfrac{1}{\sqrt{1-x^2}}$

19. $\cos\left(3+\arctan(x)\right) \cdot \dfrac{1}{1+x^2}$

21. $\arctan\left(\dfrac{1}{x}\right) - \dfrac{x}{x^2+1}$

23. (a) Use the result of Section 7.3 Problem 21:

$$\theta = \arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{3}{x}\right)$$

(b) Differentiating yields:

$$\frac{d\theta}{dx} = \frac{-1}{x^2+1} - \frac{3}{x^2+9} = \frac{-4\left(x^2+3\right)}{(x^2+1)(x^2+9)} < 0$$

so there are no critical points. As $x \to \infty$, $\theta \to 0$ (a minimum), and as $x \to 0$, $\theta \to \pi$: the viewer's nose touches the whiteboard!

25. With $y = \arccos(x)$, $\cos(y) = x$ so that:

$$-\sin(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin(y)}$$

The range of $\arccos(y)$ is $0 \le y \le \pi$, and on this interval $\sin(y) \ge 0$, so:

$$\cos^2(y) + \sin^2(y) = 1 \Rightarrow \sin(y) = \sqrt{1-\cos^2(y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}}$$

27. For $0 < x < \dfrac{\pi}{2}$: $\arccos(x) + \arcsin(x) = \dfrac{\pi}{2}$

Section 7.5

1. $7 \cdot \displaystyle\int \frac{1}{\sqrt{3^2-x^2}}\,dx = 7\arcsin\left(\frac{x}{3}\right) + C$

3. $$3 \cdot \int_0^1 \frac{1}{5^2+x^2}\,dx = \left[\frac{3}{5}\arctan\left(\frac{x}{5}\right)\right]_0^1 = \frac{3}{5}\arctan\left(\frac{1}{5}\right) \approx 0.1184$$

5. $9 \cdot \displaystyle\int \frac{1}{\sqrt{7^2-x^2}}\,dx = 9\arcsin\left(\frac{x}{7}\right) + C$

7. $$3 \cdot \int_6^{10} \frac{1}{x\sqrt{x^2-5^2}}\,dx = \left[\frac{3}{5}\text{arcsec}\left(\frac{x}{5}\right)\right]_6^{10} = \frac{3}{5}\left[\text{arcsec}\,(2) - \text{arcsec}\left(\frac{6}{5}\right)\right] \approx 0.2769$$

9. With $u = x-1 \Rightarrow du = dx$ the integral becomes:

$$\int \frac{1}{1+u^2}\,du = \arctan(u) + C = \arctan(x-1) + C$$

11. With $u = e^x \Rightarrow du = e^x\,dx$ the integral becomes:

$$\int_{e^{-1}}^{e} \frac{1}{1+u^2}\,du = \arctan(e) - \arctan\left(e^{-1}\right) \approx 0.8658$$

13. With $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ this becomes:

$$\int \frac{1}{\sqrt{3^2-u^2}}\,du = \arcsin\left(\frac{u}{3}\right) + C = \arcsin\left(\frac{\sin(\theta)}{3}\right) + C$$

15. With $u = 9+x^2 \Rightarrow du = 2x\,dx$ this becomes:

$$\frac{3}{2}\int u^{-\frac{1}{2}}\,du = 3\sqrt{u} + C = 3\sqrt{9+x^2} + C$$

17. With $u = x^2 \Rightarrow du = 2x\,dx$ the integral becomes:

$$3\int \frac{1}{3^2+u^2}\,du = \arctan\left(\frac{u}{3}\right) + C = \arctan\left(\frac{x^2}{3}\right) + C$$

19. With $u = 2x \Rightarrow du = 2\,dx$ the integral becomes:

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(2x) + C$$

21. This is an improper integral:

$$\lim_{M\to\infty}\int_0^M \frac{1}{\left(\sqrt{3}\right)^2+x^2}\,dx = \lim_{M\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^M = \lim_{M\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{M}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}$$

23. This is an improper integral:

$$\lim_{b\to\sqrt{7}^-}\int_0^b \frac{1}{\sqrt{\left(\sqrt{7}\right)^2-x^2}}\,dx = \lim_{b\to\sqrt{7}^-}\left[\arcsin\left(\frac{x}{\sqrt{7}}\right)\right]_0^b = \lim_{b\to\sqrt{7}^-}\arcsin\left(\frac{b}{\sqrt{7}}\right) = \arcsin(1) = \frac{\pi}{2}$$

25. Separating variables:

$$\int\frac{1}{y}\,dy = \int\frac{1}{\sqrt{1-x^2}}\,dx \Rightarrow \ln\left(|y|\right) = \arcsin(x)+C$$

Using $y(0)=e$:

$$\ln(e) = \arcsin(0)+C \Rightarrow 1 = 0+C \Rightarrow C=1$$

Solving for y:

$$\ln\left(|y|\right) = \arcsin(x)+1 \Rightarrow |y| = e^{\arcsin(x)+1}$$

so $y = e^{\arcsin(x)+1}$ (because $y(0)=e>0$).

27. Separating variables:

$$\int\frac{1}{y^2}\,dy = \int\frac{1}{3^2+x^2}\,dx \Rightarrow -\frac{1}{y} = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C$$

Using $y(1)=2$:

$$-\frac{1}{2} = \frac{1}{3}\arctan\left(\frac{1}{3}\right)+C \Rightarrow C = -\frac{1}{2}-\frac{1}{3}\arctan\left(\frac{1}{3}\right)$$

Solving for y yields:

$$y = \frac{-1}{\frac{1}{3}\arctan\left(\frac{x}{3}\right)-\frac{1}{2}-\frac{1}{3}\arctan\left(\frac{1}{3}\right)}$$

29. Rewrite the integral as:

$$\int\frac{4\cdot 2x}{3^2+x^2}\,dx + \int\frac{5}{3^2+x^2}\,dx$$

and integrate each term separately to get:

$$4\ln\left(x^2+9\right)+\frac{5}{3}\arctan\left(\frac{x}{3}\right)+C$$

31. Rewrite the integral as:

$$\frac{7}{2}\int\frac{2x}{\left(\sqrt{10}\right)^2+x^2}\,dx + 3\int\frac{1}{\left(\sqrt{10}\right)^2+x^2}\,dx$$

and integrate each term separately to get:

$$\frac{7}{2}\ln\left(x^2+10\right)+\frac{3}{\sqrt{10}}\arctan\left(\frac{x}{\sqrt{10}}\right)+C$$

33. Completing the square in the denominator, the integral becomes:

$$8\int\frac{1}{1+(x+3)^2}\,dx = 8\arctan(x+3)+C$$

35. Rewrite the integral as:

$$\int\frac{2(2x+6)}{x^2+6x+10}\,dx + \int\frac{8}{1+(x+3)^2}\,dx$$

and integrate each term separately to get:

$$2\ln(x^2+6x+10)+8\arctan(x+3)+C$$

37. Use $u = x^2+4x+5 \Rightarrow du = (2x+4)\,dx$:

$$\int\frac{6(2x+4)}{x^2+4x+5}\,dx = 6\int\frac{1}{u}\,du = 6\ln\left(|u|\right)+C$$

and resubstitute to get $6\ln(x^2+4x+5)+C$

39. Rewrite the integral as:

$$\int\frac{3(2x+4)}{x^2+4x+20}\,dx + \int\frac{3}{4^2+(x+2)^2}\,dx$$

and integrate each term separately to get:

$$3\ln(x^2+4x+20)+\frac{3}{4}\arctan\left(\frac{x+2}{4}\right)+C$$

Section 8.1

1. Set $u = x^2 + 7 \Rightarrow du = 2x\,dx$ so that:

$$\int 6x\left(x^2+7\right)^2 dx = \int 3u^2\,du = u^3 + C = \left(x^2+7\right)^3 + C$$

3. Set $u = t^2 - 3 \Rightarrow du = 2t\,dt$ so $t = 2 \Rightarrow u = 1$, $t = 4 \Rightarrow u = 13$ and:

$$\int_2^4 \frac{6t}{\sqrt{t^2-3}}\,dt = \int_1^{13} 3u^{-\frac{1}{2}}\,du = \left[6u^{\frac{1}{2}}\right]_1^{13} = 6\sqrt{13} - 6 \approx 15.63$$

5. Set $u = x^2 + 3 \Rightarrow du = 2x\,dx$ so that:

$$\int \frac{12x}{x^2+3}\,dx = \int \frac{6}{u}\,du = \ln(|u|) + C = 6\ln\left(x^2+3\right) + C$$

7. Set $u = 3y + 2 \Rightarrow du = 3\,dy \Rightarrow \frac{1}{3}\,du = dy$ so that:

$$\int \sin(3y+2)\,dy = \int \frac{1}{3}\sin(u)\,du = -\frac{1}{3}\cos(u) + C = -\frac{1}{3}\cos(3y+2) + C$$

9. Set $u = e^x + 3 \Rightarrow du = e^x\,dx$ so that $x = -1 \Rightarrow u = e^{-1} + 3$, $x = 0 \Rightarrow 4$ and:

$$\int_{-1}^{0} e^x \sec^2(e^x+3)\,dx = \int_{e^{-1}+3}^{4} \sec^2(u)\,du = \Big[\tan(u)\Big]_{e^{-1}+3}^{4} = \tan(4) - \tan\left(e^{-1}+3\right)$$

or approximately 0.9276.

11. Set $u = \ln(x) \Rightarrow du = \dfrac{1}{x}\,dx$ so that:

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left[\ln(x)\right]^2 + C$$

13. Set $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ so that:

$$\int \cos(\theta)e^{\sin(\theta)}\,d\theta = \int e^u\,du = e^u + C = e^{\sin(\theta)} + C$$

15. Set $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ so $x = 1 \Rightarrow u = 3$, $x = 3 \Rightarrow 9$ and:

$$\int_1^3 \frac{5}{1+9x^2}\,dx = \frac{5}{3}\int_1^9 \frac{1}{1+u^2}\,du = \frac{5}{3}\Big[\arctan(u)\Big]_3^9 = \frac{5}{3}\left[\arctan(9) - \arctan(3)\right] \approx 0.3518$$

17. Set $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx$ so that $x = 1 \Rightarrow u = 1$, $x = 2 \Rightarrow \frac{1}{2}$ and:

$$\int_1^2 \frac{1}{x^2}\cdot\cos\left(\frac{1}{x}\right)dx = -\int_1^{\frac{1}{2}} \cos(u)\,du = \int_{\frac{1}{2}}^{1} \cos(u)\,du = \Big[\sin(u)\Big]_{\frac{1}{2}}^{1} = \sin(1) - \sin\left(\frac{1}{2}\right) \approx 0.3620$$

19. Set $u = 5 + \sin^2(\theta) \Rightarrow du = 2\sin(\theta)\cos(\theta)\,d\theta$ so:

$$\int \frac{6\sin(\theta)\cos(\theta)}{5+\sin^2(\theta)}\,d\theta = \int \frac{3}{u}\,du = 3\ln(|u|) + C = 3\ln\left(5+\sin^2(\theta)\right) + C$$

21. Set $u = 2x + 5 \Rightarrow du = 2\,dx$ so that:

$$\int \frac{10}{2x+5}\,dx = \int \frac{5}{u}\,du = 5\ln(|u|) + C = 5\ln(|2x+5|) + C$$

23. Set $u = 5x^2 + 3 \Rightarrow du = 10x\,dx$ so $x = 1 \Rightarrow u = 8$, $x = 3 \Rightarrow u = 48$ and:

$$\int_1^3 \frac{20x}{5x^2+3}\,dx = \int_8^{48} \frac{2}{u}\,du = \Big[2\ln(|u|)\Big]_8^{48} = 2\ln(48) - 2\ln(8) = \ln(36) \approx 3.5835$$

25. Set $u = x + 3 \Rightarrow du = dx$ so $x = 0 \Rightarrow u = 3$, $x = 1 \Rightarrow u = 4$ and:

$$\int_0^1 \frac{7}{(x+3)^2+4}\,dx = \int_3^4 \frac{7}{u^2+2^2}\,du = \left[\frac{7}{2}\arctan(u)\right]_3^4 = \frac{7}{2}\left[\arctan(2) - \arctan\left(\frac{3}{2}\right)\right] \approx 0.4352$$

27. Set $u = e^t \Rightarrow du = e^t\,dt$ so:

$$\int \frac{e^t}{1+e^{2t}}\,dt = \int \frac{1}{1+u^2}\,du = \arctan(u) = \arctan\left(e^t\right) + C$$

29. Set $u = 1 + \ln(x) \Rightarrow du = \dfrac{1}{x}\,dx$ so $x = 1 \Rightarrow u = 1$, $x = 3 \Rightarrow u = 1 + \ln(3)$ and:

$$\int_1^3 \frac{3}{x\left[1+\ln(x)\right]}\,dx = \int_1^{1+\ln(3)} \frac{3}{u}\,du = 3\Big[\ln(u)\Big]_1^{1+\ln(3)} = 3\ln(1+\ln(3)) \approx 2.2238$$

31. Set $u = 1 - x^2 \Rightarrow du = -2x\,dx$ so $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$ and:

$$\int_0^1 2x\sqrt{1-x^2}\,dx = -\int_1^0 \sqrt{u}\,du = \int_0^1 u^{\frac{1}{2}}\,du = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

33. Set $u = 1 + \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ so:

$$\int \cos(\theta)\left[1+\sin(\theta)\right]^3\,d\theta = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\left[1+\sin(\theta)\right]^4 + C$$

35. Set $u = \ln(x) \Rightarrow du = \frac{1}{x}\,dx$ so $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$ and:

$$\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx = \int_0^1 \sqrt{u}\,du = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

37. Set $u = 5 + \tan(\theta) \Rightarrow du = \sec^2(\theta)\,d\theta$ so:

$$\int \frac{\sec^2(\theta)}{5+\tan(\theta)}\,d\theta = \int \frac{1}{u}\,du = \ln\left(|5+\tan(\theta)|\right) C$$

39. $\ln\left(|\sec\left(y-5\right)|\right) + C$

41. $\frac{1}{5}\left[e^{5u}\right]_0^1 = \frac{1}{5}\left[e^5 - 1\right] \approx 29.483$

43. $\frac{1}{6}\ln\left(\left|\sec\left(2+3t^2\right) + \tan\left(2+3t^2\right)\right|\right) + C$

45. 0

47. Set $u = 3x^2 \Rightarrow du = 6x\,dx \Rightarrow \frac{1}{6}\,du = x\,dx$ so $x = 1 \Rightarrow u = 3$, $x = \infty \Rightarrow u = \infty$ and:

$$\begin{aligned}\int_1^\infty \frac{x}{1+9x^4}\,dx &= \frac{1}{6}\int_3^\infty \frac{1}{1+u^2}\,du \\ &= \lim_{M\to\infty} \frac{1}{6}\int_1^M \frac{1}{1+u^2}\,du \\ &= \lim_{M\to\infty} \frac{1}{6}\Big[\arctan(u)\Big]_3^M \\ &= \lim_{M\to\infty} \frac{1}{6}\Big[\arctan(M) - \arctan(3)\Big] \\ &= \frac{1}{6}\left[\frac{\pi}{2} - \arctan(3)\right] \approx 0.536\end{aligned}$$

49. $\displaystyle\int \frac{7}{(x+2)^2+1^2}\,dx = 7\arctan(x+2) + C$

51. $\displaystyle\int \frac{2}{(x-3)^2+7^2}\,dx = \frac{2}{7}\arctan\left(\frac{x-3}{7}\right) + C$

53. $\displaystyle\int \frac{3}{(x+5)^2+2^2}\,dx = \frac{3}{2}\arctan\left(\frac{x+5}{2}\right) + C$

55. $\displaystyle\int \frac{2x+4}{x^2+4x+5}\,dx + \int \frac{7}{(x+2)^2+1^2}\,dx = \ln\left(x^2+4x+5\right) + 7\arctan\left(x+2\right) + C$

57. $\displaystyle\int \frac{2(2x-6)}{x^2-6x+10}\,dx + \int \frac{19}{(x-3)^2+1^2}\,dx = 2\ln\left(x^2-6x+10\right) + 19\arctan\left(x-3\right) + C$

59. $\displaystyle\int \frac{3(2x-4)}{x^2-4x+13}\,dx + \int \frac{17}{(x-2)^2+3^2}\,dx = 3\ln\left(x^2-4x+13\right) + \frac{17}{3}\arctan\left(\frac{x-2}{3}\right) + C$

61. $\displaystyle\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2} + C$

63. $\displaystyle\int \frac{x+3}{(x-3)^2}\,dx = \int \frac{x-3}{(x-3)^2}\,dx + \int \frac{6}{(x-3)^2}\,dx = \ln\left(|x-3|\right) - \frac{6}{x-3} + C$

65. Rewrite $\displaystyle\int_3^\infty \frac{1}{(x-3)^2}\,dx$ as:

$$\lim_{a\to 3^+}\int_b^4 (x-3)^{-2}\,dx + \lim_{M\to\infty}\int_4^M (x-3)^{-2}\,dx$$

The first limit diverges, so the integral diverges.

Section 8.2

1. Setting $u = \ln(x)$ leaves $dv = 12x\,dx$ so we have $du = \frac{1}{x}\,dx$, $v = 6x^2$ and therefore:

$$\int 12x\ln(x)\,dx = 6x^2\ln(x) - \int 6x^2\cdot\frac{1}{x}\,dx = 6x^2\ln(x) - \int 6x\,dx = 6x^2\ln(x) - 3x^2 + C$$

3. Setting $dv = x^4\,dx$ leaves $u = \ln(x)$ so $du = \frac{1}{x}\,dx$, $v = \frac{1}{5}x^5$ and:

$$\int x^4\ln(x)\,dx = \frac{1}{5}x^5\ln(x) - \int \frac{1}{5}x^5\cdot\frac{1}{x}\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{5}\int x^4\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$

5. Setting $dv = x\,dx$ leaves $u = \arctan(x)$ so $du = \frac{1}{1+x^2}\,dx$, $v = \frac{1}{2}x^2$ and:

$$\begin{aligned}\int x\arctan(x)\,dx &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int\left[1 - \frac{1}{1+x^2}\right]dx \\ &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C\end{aligned}$$

7. Setting $u = x$ leaves $dv = e^{-3x}\,dx$ so $du = dx$, $v = -\frac{1}{3}e^{-3x}$ and:

$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \left(-\frac{1}{3}\right)\int e^{-3x}\,dx$$
$$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

9. Setting $u = x$ leaves $dv = \sec(x)\tan(x)\,dx$ so $du = dx$, $v = \sec(x)$ and:

$$\int x\sec(x)\tan(x)\,dx = x\sec(x) - \int \sec(x)\,dx$$
$$= x\sec(x) - \ln\left(|\sec(x)+\tan(x)|\right) + C$$

or approximately -2.887.

11. Setting $u = 7x$ leaves $dv = \cos(3x)\,dx$ so that $du = 7\,dx$, $v = \frac{1}{3}\sin(3x)$ and:

$$\int 7x\cos(3x)\,dx = \frac{7}{3}x\sin(3x) - \frac{7}{3}\int \sin(3x)\,dx$$
$$= \frac{7}{3}x\sin(3x) + \frac{7}{9}\cos(3x) + C$$

and the definite integral is then:

$$\left[\frac{7}{3}x\sin(x) + \frac{7}{9}\cos(x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{7}{9} - \frac{7\pi}{6}$$

13. Don't use integration by parts on this one! Use the substitution $w = 3x^2 \Rightarrow dw = 6x\,dx$:

$$\int 12x\cos(3x^2)\,dx = \int 2\cos(w)\,dw = 2\sin(w) + C$$
$$= 2\sin(3x^2) + C$$

15. Setting $u = \ln(2x+5)$ leaves $dv = dx$ so that $du = \frac{2}{2x+5}\,dx$, $v = x$ and:

$$\int \ln(2x+5)\,dx = x\ln(2x+5) - \int \frac{2x}{2x+5}\,dx$$
$$= x\ln(2x+5) - \int \left[1 - \frac{5}{2x+5}\right]\,dx$$
$$= x\ln(2x+5) - x + \frac{5}{2}\ln(2x+5) + C$$

and the definite integral is then:

$$\left[x\ln(2x+5) - x + \frac{5}{2}\ln(2x+5)\right]_1^3$$
$$= \left[3\ln(11) - 3 + \frac{5}{2}\ln(11)\right] - \left[\ln(7) - 1 + \frac{5}{2}\ln(7)\right]$$
$$= \frac{11}{2}\ln(11) - \frac{7}{2}\ln(7) - 2 \approx 4.38$$

17. Setting $u = (\ln(x))^2$ leaves $dv = dx$ so that $du = 2\ln(x)\cdot\frac{1}{x}\,dx$, $v = x$ and:

$$\int (\ln(x))^2\,dx = x(\ln(x))^2 - 2\int \ln(x)\,dx$$
$$= x(\ln(x))^2 - 2\left[x\ln(x) - x\right] + C$$
$$= x(\ln(x))^2 - 2x\ln(x) + 2x + C$$

and the definite integral is then:

$$\left[x(\ln(x))^2 - 2x\ln(x) + 2x\right]_1^e$$
$$= [e - 2e + 2e] - [0 - 0 + 2] = e - 2 \approx 0.728$$

19. Setting $u = \arcsin(x)$ leaves $dv = dx$ so that $du = \frac{1}{\sqrt{1-x^2}}\,dx$, $v = x$ and:

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

For this new integral use the substitution $w = 1 - x^2 \Rightarrow dw = -2x\,dx \Rightarrow -\frac{1}{2}\,dw = x\,dx$ so that:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int w^{-\frac{1}{2}}\,dw = -w^{\frac{1}{2}} + K$$
$$= -\sqrt{1-x^2} + K$$

and the original integral is then:

$$\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + C$$

21. Setting $u = \arctan(3x)$ leaves $dv = x\,dx$ so that $du = \frac{3}{1+9x^2}\,dx$, $v = \frac{1}{2}x^2$ and:

$$\int x\arctan(3x)\,dx = \frac{1}{2}x^2\arctan(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\,dx$$
$$= \frac{1}{2}x^2\arctan(3x) - \frac{3}{2}\cdot\frac{1}{9}\int\left[1 - \frac{1}{1+9x^2}\right]\,dx$$
$$= \frac{1}{2}x^2\arctan(3x) - \frac{1}{6}\left[x - \frac{1}{3}\arctan(3x)\right] + C$$
$$= \frac{1}{2}x^2\arctan(3x) - \frac{1}{6}x + \frac{1}{18}\arctan(3x) + C$$

23. Don't use integration by parts on this one! Use the substitution $w = \ln(x) \Rightarrow dw = \frac{1}{x}\,dx$ so that:

$$\int \frac{\ln(x)}{x}\,dx = \int w\,dw = \frac{1}{2}w^2 + C = \frac{1}{2}(\ln(x))^2 + C$$

and the definite integral is: $\frac{1}{2}(\ln(2))^2 \approx 0.240$.

25. On your own.

27. Write $\sec^n(x) = \sec^{n-2}(x) \cdot \sec^2(x)$ and set $u = \sec^{n-2}(x)$, leaving $dv = \sec^2(x)\,dx$, so that:

$$du = (n-2)\sec^{n-3}(x) \cdot \sec(x)\tan(x)\,dx = (n-2)\sec^{n-2}(x)\tan(x)\,dx \quad \text{and} \quad v = \tan(x)$$

Integration by parts then says:

$$\int \sec^{n-2}(x)\,dx = \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,dx$$

Now use the identity $\tan^2(x) = \sec^2(x) - 1$ to write this new integral as:

$$\int \sec^{n-2}(x)\left[\sec^2(x) - 1\right]\,dx = \int \sec^n(x)\,dx - \int \sec^{n-2}(x)\,dx$$

and combining these results yields:

$$\int \sec^{n-2}(x)\,dx = \sec^{n-2}(x)\tan(x) - (n-2)\left[\int \sec^n(x)\,dx - \int \sec^{n-2}(x)\,dx\right]$$

Moving the first of the two integrals on the right side to the left side:

$$(n-1)\int \sec^{n-2}(x)\,dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx$$

and solving for the original integral yields:

$$\int \sec^{n-2}(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

29. $\displaystyle\int \sin^3(x)\,dx = \frac{1}{3}\left[-\sin^2(x)\cos(x) + 2\int \sin(x)\,dx\right] = -\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x) + C$

31. $\displaystyle\int \sin^5(x)\,dx = \frac{1}{5}\left[-\sin^4(x)\cos(x) + 4\int \sin^3(x)\,dx\right]$ and using the result of Problem 29 yields:

$$\int \sin^5(x)\,dx = -\frac{1}{5}\sin^4(x)\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) - \frac{8}{15}\cos(x) + C$$

33. $\displaystyle\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left[\frac{1}{2}\cos(x)\sin(x) + x\right] + C$

35. $\displaystyle\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left(|\sec(x) + \tan(x)|\right) + C$

37. $\displaystyle\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx$ and using the result of Problem 33 yields:

$$\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\ln\left(|\sec(x) + \tan(x)|\right) + C$$

39. $\displaystyle\int \cos^3(u)\,du = \frac{1}{3}\cos^2(u)\sin(u) + \frac{2}{3}\int \cos(u)\,du == \frac{1}{3}\cos^2(u)\sin(u) + \frac{2}{3}\sin(u) + C$ so that:

$$\begin{aligned}\int \cos^3(2x+3)\,dx &= \frac{1}{2}\left[\frac{1}{3}\cos^2(2x+3)\sin(2x+3) + \frac{2}{3}\sin(2x+3)\right] + C \\ &= \frac{1}{6}\cos^2(2x+3)\sin(2x+3) + \frac{1}{3}\sin(2x+3) + C\end{aligned}$$

41. Set $u = x^n$ and $dv = e^{ax}\,dx$ so $du = n \cdot x^{n-1}\,dx$, $v = \frac{1}{a}e^{ax}$ and $\displaystyle\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

43. Set $u = (\ln(x))^n$, leaving $dv = x\,dx$, so that $du = n\,(\ln(x))^{n-1} \cdot \frac{1}{x}\,dx$, $v = \frac{1}{2}x^2$ and:

$$\int x\,(\ln(x))^n\ dx = \frac{1}{2}x^2\,(\ln(x))^n - \frac{n}{2}\int x\,(\ln(x))^{n-1}\ dx$$

45. (a) Set $u = x \Rightarrow du = dx$ and $dv = (2x+5)^{19}\ dx$ $\Rightarrow v = \frac{1}{40}\,(2x+5)^{20}$:

$$\begin{aligned}\int x\,(2x+5)^{19}\ dx \\ &= \frac{x}{40}\,(2x+5)^{20} - \frac{1}{40}\int (2x+5)^{20}\ dx \\ &= \frac{x}{40}\,(2x+5)^{20} - \frac{1}{40}\cdot\frac{1}{42}\,(2x+5)^{21} + C\end{aligned}$$

(b) Set $w = 2x+5 \Rightarrow dw = 2\,dx \Rightarrow \frac{1}{2}\,dw = dx$ and note that $x = \frac{1}{2}(w-5)$:

$$\begin{aligned}\int x\,(2x+5)^{19}\ dx &= \int \frac{1}{2}(w-5)\cdot w^{19}\cdot\frac{1}{2}\,dw \\ &= \frac{1}{2}\int\left[w^{20} - 5w^{19}\right]\ dw \\ &= \frac{1}{4}\left[\frac{1}{21}w^{21} - \frac{5}{20}w^{20}\right] + K \\ &= \frac{1}{84}\,(2x+5)^{21} - \frac{1}{16}\,(2x+5)^{20} + K\end{aligned}$$

These answers look different, but you can verify that the derivative of each answer is $x(2x+5)^{19}$.

47. Use the result of Problem 43 (twice) to get:

$$\frac{1}{2}x^2\left[(\ln(x))^2 - \ln(x) + \frac{1}{2}\right] + C$$

49. Apply integration by parts twice to get a reappearing integral and, eventually:

$$\frac{1}{2}\left[-e^{-1}\cos(1) - e^{-1}\sin(1) + 1\right] \approx 0.24584$$

51. Substitute $y = \ln(x) \Rightarrow x = e^y \Rightarrow dx = e^y\,dy$ to turn the integral into $\int \sin(y)\cdot e^y\,dy$ and then apply integration by parts twice to get a reappearing integral and, eventually:

$$\frac{1}{2}x\left[\sin\left(\ln(x)\right) - \cos\left(\ln(x)\right)\right] + C$$

53. Substitute $y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow dx = 2y\,dy$, then apply integration by parts to get:

$$2\left[\sqrt{x}\sin\left(\sqrt{x}\right) + \cos\left(\sqrt{x}\right)\right] + C$$

55. Integration by parts (twice) results in a reappearing integral and, eventually:

$$\frac{1}{10}e^{3x}\left[3\sin(x) - \cos(x)\right] + C$$

57. Integration by parts yields $-(x+1)e^{-x} + C$ so:

$$\int_0^M xe^{-x}\,dx = \Big[-(x+1)e^{-x}\Big]_0^M = -\frac{M+1}{e^M} + 1$$

and $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{M\to\infty}\left[1 - \frac{M+1}{e^M}\right] = 0.$

59. After integrating by parts (twice):

$$\lim_{M\to 0}\frac{1}{2}\left[-e^{-x}\cos(x) - e^{-x}\sin(x)\right]_0^M = \frac{1}{2}$$

61. The substitution $w = x+1$ results in:

$$\int (w-1)\sqrt{w}\,dw = \frac{2}{5}w^{\frac{5}{2}} - \frac{2}{3}w^{\frac{3}{2}} + C$$

and then resubstitute $w = x+1$.

63. Use the substitution $w = x^2$ to get $\frac{1}{2}\sin(x^2) + C$.

65. Use integration by parts twice, starting with $u = x^2$ and $dv = \cos(x)\,dx$ to get:

$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

67. Use the substitution $y = x^2+1$ so so that:

$$\int \frac{1}{2}(y-1)y^{\frac{1}{3}}\,dy = \frac{3}{14}\left(x^2+1\right)^{\frac{7}{3}} - \frac{3}{8}(x^2+1)^{\frac{4}{3}} + C$$

69. Integration by parts on the right side yields $y = -x\cos(x) + \sin(x) + C$ and using the initial condition tells us $0 = y(0) = 0 + C \Rightarrow C = 0$.

71. Separate variables to get $\int e^y\,dy = \int xe^{-x}\,dx$ so, using integration by parts on the right side:

$$e^y = -(x+1)e^{-x} + C \Rightarrow e^1 = -1 + C$$

resulting in $y = \ln\left(e + 1 - (x+1)e^{-x}\right)$.

73. (a) When $0 \le x \le 1$, $x\sin(x) \le \sin(x)$, so you should expect the second integral to be larger. (b) Using integration by parts on the first integral yields $\sin(1) - \cos(1) \approx 0.3012$, while the value of the second integral is $1 - \cos(1) \approx 0.4597$.

75. (a) Make an informed prediction. (b) From Problem 17, we know that the first volume is $\pi(e-2) \approx 2.257$, while using integration by parts on the second integral yields $2\pi \approx 6.283$.

77. Using the tube method, the volume is given by $\int_0^{\pi} 2\pi x \cdot \sin(x)\, dx$ and, using integration by parts with $u = 2\pi x \Rightarrow du = 2\pi\, dx$ so that $dv = \sin(x)\, dx \Rightarrow v = -\cos(x)$, this yields:

$$\Big[-2\pi x\cos(x)\Big]_0^{\pi} - \int_0^{\pi} [-\cos(x)] \cdot 2\pi\, dx$$
$$= 2\pi^2 + 2\pi \int_0^{\pi} \cos(x)\, dx$$
$$= 2\pi^2 + 2\pi\Big[\sin(x)\Big]_0^{\pi} = 2\pi^2$$

79. Using the tube method, the volume is given by:

$$\int_0^{\pi} 2\pi x \cdot x\sin(x)\, dx = 2\pi\int_0^{\pi} x^2\sin(x)\, dx$$

Using integration by parts twice yields:

$$\Big[-x^2\cos(x) + 2\pi x\sin(x) + 2\cos(x)\Big]_0^{\pi}$$

which evaluates to $\pi^2 - 4 \approx 5.8670$.

81. The area of the region is given by:

$$\int_0^{\infty} xe^{-x}\, dx = \lim_{M\to\infty}\int_0^{M} xe^{-x}\, dx$$

Using integration by parts yields:

$$\lim_{M\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^{M} = \lim_{M\to\infty}\left[-\frac{M}{e^M} - \frac{1}{e^M} + 1\right]$$

which equals 1, so the area is finite.

83. Using the disk method, the volume is given by:

$$\int_0^{\infty} \pi\left[xe^{-x}\right]^2\, dx = \lim_{M\to\infty}\int_0^{M} x^2e^{-2x}\, dx$$

Using integration by parts (twice) yields:

$$\lim_{M\to\infty}\left[-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^{M}$$
$$= \lim_{M\to\infty}\left[-\frac{M^2}{2e^{2M}} - \frac{M}{2e^{2M}} - \frac{1}{4e^M} + \frac{1}{4}\right]$$

which equals $\frac{1}{4}$, so the volume is finite.

85. Using the tube method, the volume is given by:

$$\int_0^{\infty} 2\pi x \cdot xe^{-x}\, dx = \lim_{M\to\infty} 2\pi\int_0^{M} x^2e^{-x}\, dx$$

Using integration by parts (twice) yields:

$$\lim_{M\to\infty}\Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_0^{M}$$

which becomes:

$$\lim_{M\to\infty}\left[-\frac{M^2}{e^M} - \frac{2M}{e^M} - \frac{2}{e^M} + 2\right] = 2$$

so the volume is finite.

Section 8.3

1. Decompose the integrand as:

$$\frac{7x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Multiplying by $x(x+1)$ yields:

$$7x + 2 = A(x+1) + Bx$$

If $x = 0$ this tells us that $2 = A$; if $x = -1$, $-5 = -B \Rightarrow B = 5$, so the integral becomes:

$$\int\left[\frac{2}{x} + \frac{5}{x+1}\right]\, dx = 2\ln(|x|) + 5\ln(|x+1|) + C$$

3. Factor the denominator and decompose as:

$$\frac{11x+25}{(x+1)(x+8)} = \frac{A}{x+1} + \frac{B}{x+8}$$

Multiplying by $(x+1)(x+8)$ yields:

$$11x + 25 = A(x+8) + B(x+1)$$

If $x = -1$, $14 = 7A \Rightarrow A = 2$; if $x = -8$, $-63 = -7B \Rightarrow B = 9$, so:

$$\int\left[\frac{2}{x+1} + \frac{9}{x+8}\right]\, dx = \ln\left((x+1)^2 \cdot |x+8|^9\right) + C$$

5. First use polynomial division, then factor the denominator to rewrite the integral as:

$$\int\left[2 + \frac{5}{x}\right]\, dx = 2x + 5\ln(|x|) + C$$

7. Decompose the integrand as:

$$\frac{6x^2+9x-15}{x(x+5)(x-1)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-1}$$

Multiplying by $x(x+5)(x-1)$:

$$\begin{aligned} &6x^2 + 9x - 15 \\ &\quad = A(x+5)(x-1) + Bx(x-1) + Cx(x+5) \end{aligned}$$

If $x = 0$, $-15 = -5A \Rightarrow A = 3$; if $x = -5$, then $90 = 30B \Rightarrow B = 3$; and if $x = 1$, $0 = 6C \Rightarrow C = 0$. So the integral becomes:

$$\int\left[\frac{3}{x} + \frac{3}{x+5}\right]\, dx = 3\ln(|x|) + 3\ln(|x+5|) + C$$

9. Factor the denominator and decompose:

$$\frac{8x^2 - x + 3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiplying by the denominator $x(x^2+1)$ yields:

$$8x^2 - x + 3 = [A+B]x^2 + Cx + A$$

so $A = 3$, $C = -1$, $A + B = 8 \Rightarrow B = 5$ and:

$$\int \left[\frac{3}{x} + \frac{5x-1}{x^2+1}\right] dx$$
$$= \int \left[\frac{3}{x} + \frac{5}{2} \cdot \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right] dx$$
$$= 3\ln(|x|) + \frac{5}{2}\ln\left(x^2+1\right) - \arctan(x) + C$$

11. Decompose the integrand as:

$$\frac{11x^2 + 23x + 6}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

and multiply both sides by $x^2(x+2)$ to get:

$$\begin{aligned} 11x^2 + 23x + 6 &= Ax(x+2) + B(x+2) + Cx^2 \\ &= [A+C]x^2 + [2A+B]x + 2B \end{aligned}$$

so $2B = 6 \Rightarrow B = 3$, $2A + B = 23 \Rightarrow 2A = 20 \Rightarrow$ $A = 10$, $A + C = 11 \Rightarrow C = 1$ and:

$$\int \left[\frac{10}{x} + \frac{3}{x^2} + \frac{1}{x+2}\right] dx$$
$$= 10\ln(|x|) - \frac{3}{x} + \ln(|x+2|) + C$$

13. Decompose the integrand as:

$$\frac{3x+13}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}$$

and multiply by the denominator $(x+2)(x-5)$:

$$3x + 13 = A(x-5) + B(x+2)$$

With $x = -2$, $7 = -7A \Rightarrow A = -1$; with $x = 5$, $28 = 7B \Rightarrow B = 4$ so the integral becomes:

$$\int \left[\frac{-1}{x+2} + \frac{4}{x-5}\right] dx = \ln\left(\frac{(x-5)^4}{|x+2|}\right) + C$$

15. The integrand decomposes as:

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

so the integral becomes:

$$\int_2^5 \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx = \left[\ln\left(\left|\frac{x-1}{x+1}\right|\right)\right]_2^5$$

which evaluates to $\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\right) = \ln(2)$.

17. Use polynomial division to rewrite the integrand:

$$\int \left[2 + \frac{5x-1}{(x-1)(x+1)}\right] dx$$
$$= \int \left[2 + \frac{2}{x-1} + \frac{3}{x+1}\right] dx$$
$$= 2x + 2\ln(|x-1|) + 3\ln(|x+1|) + C$$

19. Use polynomial division to rewrite the integrand:

$$\int \left[3 + \frac{x+9}{(x+1)(x+5)}\right] dx$$
$$= \int \left[3 + \frac{2}{x+1} - \frac{1}{x+5}\right] dx$$
$$= 3x + 2\ln(|x+1|) - \ln(|x+5|) + C$$

21. Factor the denominator and decompose:

$$\frac{3x^2 - 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

and then multiply by the denominator $x(x^2+1)$:

$$\begin{aligned} 3x^2 - 1 &= A(x^2+1) + (Bx+C)x \\ &= [A+B]x^2 + Cx + A \end{aligned}$$

so $A = -1$, $C = 0$ and $A + B = 3 \Rightarrow B = 4$ and:

$$\int \left[\frac{-1}{x} + \frac{4x}{x^2+1}\right] dx = -\ln(|x|) + 2\ln\left(x^2+1\right) + C$$

23. Use polynomial division to rewrite the integrand:

$$\int \left[x + \frac{6(x+5)}{(x-2)(x+5)}\right] dx = \frac{1}{2}x^2 + 6\ln(|x-2|) + C$$

25. Factor the denominator and decompose:

$$\frac{12x^2 + 19x - 6}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

then multiply by the denominator $x^2(x+3)$:

$$12x^2 + 19x - 6 = Ax(x+3) + B(x+3) + Cx^2$$

When $x = 0$, $-6 = 3B \Rightarrow B = -2$; when $x = -3$, $45 = 9C \Rightarrow C = 5$; and when $x = 1$, $25 = 4A - 2(4) + 5(1)^2 \Rightarrow 4A = 28 \Rightarrow A = 7$ so:

$$\int \left[\frac{7}{x} - \frac{2}{x^2} + \frac{5}{x+3}\right] dx$$
$$= 7\ln(|x|) + \frac{2}{x} + 5\ln(|x+3|) + C$$
$$= \frac{2}{x} + \ln\left(|x|^7 \cdot |x+3|^5\right) + C$$

27. Factor the denominator and decompose:

$$\frac{7x^2+3x+7}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

then multiply by the denominator $x(x^2+1)$:

$$7x^2+3x+7 = [A+B]x^2 + Cx + A$$

so $A = 7$, $C = 3$, $A + B = 7 \Rightarrow B = 0$ and:

$$\int \left[\frac{7}{x} + \frac{3}{x^2+1}\right] dx = 7\ln(|x|) + 3\arctan(x) + C$$

29. Using the result of Problem 15:

$$\int_2^\infty \frac{2}{x^2-1}\,dx = \lim_{M\to\infty} \left[\ln\left(\left|\frac{x-1}{x+1}\right|\right)\right]_2^M$$
$$= \lim_{M\to\infty} \left[\ln\left(\left|\frac{M-1}{M+1}\right|\right) - \ln\left(\frac{1}{3}\right)\right] = \ln(3)$$

31. Decompose the integrand as:

$$\frac{6x^2+5x+61}{(x-1)(x^2+4x+13)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+13}$$

and multiply by the common denominator to get:

$$6x^2+5x+61 = A(x^2+4x+13) + (Bx+C)(x-1)$$
$$= [A+B]x^2 + [4A-B+C]x + [13A-C]$$

so $A + B = 6$ and $4A - B + C = 5 \Rightarrow 5A + C = 11$ while $13A - C = 61$, so $18A = 72 \Rightarrow A = 4 \Rightarrow B = 2 \Rightarrow C = -9$ and the integral becomes:

$$\int \left[\frac{4}{x-1} + \frac{2x-9}{x^2+4x+13}\right] dx$$
$$= \int \left[\frac{4}{x-1} + \frac{2x+4}{x^2+4x+13} - \frac{13}{(x+2)^2+3^2}\right] dx$$
$$= 4\ln(|x-1|) + \ln(x^2+4x+13)$$
$$- \frac{13}{3}\arctan\left(\frac{x+2}{3}\right) + C$$

33. (a) $\displaystyle\int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C$

(b) $\displaystyle\int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + C$

(c) $\displaystyle\int \frac{1}{x(x+2)}\,dx = \frac{1}{2}\int\left[\frac{1}{x} - \frac{1}{x+2}\right] dx$

$$= \frac{1}{2}\ln\left(\left|\frac{x}{x+2}\right|\right) + C$$

35. Using the decomposition from Problem 1:

$$f(x) = 2x^{-1} + 5(x+1)^{-1}$$
$$f'(x) = -2x^{-2} - 5(x+1)^{-2}$$
$$f''(x) = 4x^{-3} + 10(x+1)^{-3}$$

37. Using the decomposition from Problem 3:

$$g(x) = 2(x+1)^{-1} + 9(x+8)^{-1}$$
$$g'(x) = -2(x+1)^{-2} - 9(x+8)^{-2}$$
$$g''(x) = 4(x+1)^{-3} + 18(x+8)^{-3}$$

39. Using results from Problem 5:

$$h(x) = 2 + 3x^{-1}$$
$$h'(x) = -3x^{-2}$$
$$h''(x) = 6x^{-3}$$

41. Using $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$:

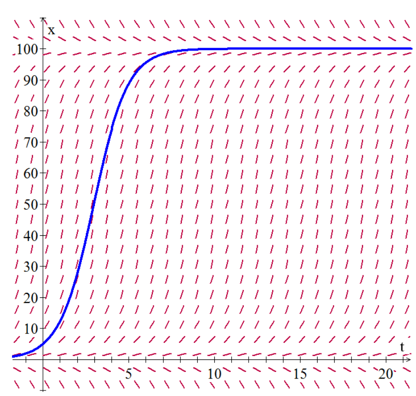
$$\int \frac{1}{1-u^2}\,du = \int \frac{1}{(1-u)(1+u)}\,du$$
$$= \frac{1}{2}\int\left[\frac{1}{1-u} + \frac{1}{1+u}\right] du$$
$$= -\frac{1}{2}\ln(|1-u|) + \frac{1}{2}\ln(|1+u|) + C$$
$$= \ln\left(\sqrt{\left|\frac{1+u}{1-u}\right|}\right) + C$$

Replacing u with $\sin(\theta)$ yields:

$$\frac{1+u}{1-u} = \frac{1+\sin(\theta)}{1-\sin(\theta)} \cdot \frac{1+\sin(\theta)}{1+\sin(\theta)} = \frac{(1+\sin(\theta))^2}{1-\sin^2(\theta)}$$
$$= \left[\frac{1+\sin(\theta)}{\cos(\theta)}\right]^2 = [\sec(\theta)+\tan(\theta)]^2$$

Combining results: $\ln(|\sec(\theta)+\tan(\theta)|) + C$.

43. (a) Here is the direction field for the ODE, along with a graph of the solution to the IVP:



(b) To solve $\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)$, first separate the variables:

$$\int \frac{1}{x\left(1-\frac{x}{100}\right)}\,dx = \int t\,dt$$

Then decompose the integrand on the left:

$$\frac{1}{x\left(1-\frac{x}{100}\right)} = \frac{1}{x} + \frac{\frac{1}{100}}{1-\frac{x}{100}} = \frac{1}{x} + \frac{1}{100-x}$$

Integrating both sides of the integral equation:

$$\ln\left(\left|\frac{x}{100-x}\right|\right) = t + C$$

Using the initial condition:

$$\ln\left(\left|\frac{5}{95}\right|\right) = t + C \;\Rightarrow\; C = -ln(19)$$

Solving for $x(t)$:

$$\frac{x}{100-x} = \frac{1}{19}e^t \;\Rightarrow\; 19x = (100-x)e^t$$

finally results in:

$$x(t) = \frac{100e^t}{e^t+19} = \frac{100}{1+19e^{-t}}$$

(c) See graph for (a).

(d) Using an intermediate equation from (b):

$$\ln\left(\left|\frac{20}{100-20}\right|\right) = t - \ln(19) \Rightarrow t = \ln\left(\frac{19}{4}\right)$$

or about 1.56. Similarly, $x = 50 \Rightarrow t = \ln(19) \approx 2.94$ and $x = 90 \Rightarrow t = \ln(171) \approx 5.14$; $x = 100$ is impossible.

(e) $x(t) \to 100$ (the carrying capacity)

(f) The bacteria begin to grow exponentially, but soon the growth rate slows and the number of bacteria approaches the carrying capacity.

(g) $\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)$ is biggest when $x = 50$, which by part (d) occurs when $t = \ln(19)$.

(h) $x = 50$

(i) The number of bacteria would decrease, approaching the carrying capacity.

45. (a) Following the same steps as in Problem 43 yields:

$$x(t) = \frac{M}{1+e^{-t}\left(\frac{M}{x_0}-1\right]}$$

(b) $x(t) \to M$

(c) When $x = \dfrac{M}{2} \Rightarrow t = \ln\left(\left|\dfrac{M-x_0}{x_0}\right|\right)$.

(d) $x = \frac{M}{2}$

47. (a) Separate variables and use partial fractions to get the integral equation:

$$\int \left[\frac{\frac{1}{b-a}}{a-x} + \frac{\frac{1}{a-b}}{b-x}\right] dx = \int 1\,t$$

Integrate both sides to get:

$$\ln\left(\left|\frac{b-x}{a-x}\right|\right) = (b-a)t + K$$

Use $x(0) = 0$ to get $K = \ln\left(\frac{b}{a}\right)$ so that:

$$\frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)t}$$

and solve for x to get:

$$x(t) = \frac{b\left[e^{(b-a)t}-1\right]}{\frac{b}{a}e^{(b-a)t}-1}$$

(b) Separate variables to get:

$$\int \frac{1}{(c-x)^2}\,dx = \int 1\,dt$$

and integrate to get:

$$\frac{1}{c-x} = t + C$$

The initial condition tells us $\frac{1}{c} = C$ so:

$$\frac{1}{x-c} = t + \frac{1}{c} \;\Rightarrow\; x(t) = c + \frac{1}{t+\frac{1}{c}}$$

Section 8.4

1. $x = 7\sin(\theta)$

3. $x = 9\tan(\theta)$

5. $x = \sqrt{7}\sec(\theta)$

7. $x = 10\sin(\theta)$

9. $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$ and:
$$\frac{1}{\sqrt{9-(3\sin(\theta))^2}} = \frac{1}{\sqrt{9\left(1-\sin^2(\theta)\right)}} = \frac{1}{3\cos(\theta)}$$

11. $x = 3\sec(\theta) \Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$ and:
$$\frac{1}{\sqrt{(3\sec(\theta))^2-9}} = \frac{1}{\sqrt{9\left(\sec^2(\theta)-1\right)}} = \frac{1}{3\tan(\theta)}$$

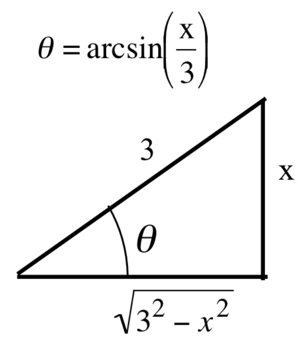
13. $x = \sqrt{2}\tan(\theta) \Rightarrow dx = \sqrt{2}\sec^2(\theta)\,d\theta$ and:
$$\frac{1}{\sqrt{2+\left(\sqrt{2}\tan(\theta)\right)^2}} = \frac{1}{\sqrt{2}\sec(\theta)} = \frac{1}{\sqrt{2}}\cos(\theta)$$

15. (a) $\theta = \arcsin\left(\frac{x}{3}\right)$
(b) $f(\theta) = \cos(\theta)\tan(\theta)$ becomes:
$$\cos\left(\arcsin\left(\frac{x}{3}\right)\right)\tan\left(\arcsin\left(\frac{x}{3}\right)\right)$$
(c) $\dfrac{\sqrt{9-x^2}}{3}\cdot\dfrac{x}{\sqrt{9-x^2}} = \dfrac{x}{3}$

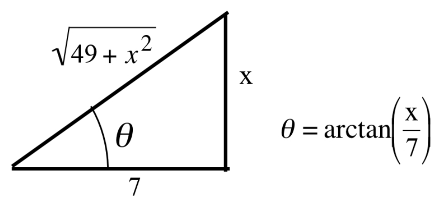
17. (a) $\theta = \operatorname{arcsec}\left(\frac{x}{3}\right)$
(b) $f(\theta) = \sqrt{1+\sin^2(\theta)}$ becomes:
$$\sqrt{1+\sin^2\left(\operatorname{arcsec}\left(\frac{x}{3}\right)\right)} = \sqrt{1+\left[\frac{\sqrt{x^2-9}}{x}\right]^2}$$
(c) $\sqrt{1+\dfrac{x^2-9}{x^2}} = \sqrt{2-\dfrac{9}{x^2}}$

19. (a) $\theta = \arctan\left(\frac{x}{5}\right)$
(b) $f(\theta) = \dfrac{\cos^2(\theta)}{1+\cot(\theta)}$ becomes:
$$\frac{\left[\arctan\left(\frac{x}{5}\right)\cos\left(\right)\right]^2}{1+\cot\left(\arctan\left(\frac{x}{5}\right)\right)} = \frac{\frac{25}{25+x^2}}{1+\frac{5}{x}}$$
(c) $\dfrac{25x}{(25+x^2)(x+5)}$

21. Using $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$:
$$\int\frac{1}{x\sqrt{9-x^2}}\,dx = \int\frac{3\cos(\theta)}{3\sin(\theta)\cdot 3\cos(\theta)}\,d\theta$$
$$= \frac{1}{3}\int\csc(\theta)\,d\theta = -\frac{1}{3}\ln\left(|\csc(\theta)+\cot(\theta)|\right)+C$$
$$= -\frac{1}{3}\ln\left(\left|\frac{3}{x}+\frac{\sqrt{9-x^2}}{x}\right|\right)+C$$
Remember to draw a triangle!



23. Using $x = 7\tan(\theta) \Rightarrow dx = 7\sec^2(\theta)\,d\theta$:
$$\int\frac{1}{\sqrt{49+x^2}}\,dx = \int\frac{7\sec^2(\theta)}{7\sec(\theta)}\,d\theta = \int\sec(\theta)\,d\theta$$
$$= \ln\left(|\sec(\theta)+\tan(\theta)|\right)+C$$
$$= \ln\left(\frac{\sqrt{49+x^2}}{7}+\frac{x}{7}\right)+C$$
$$= \ln\left(x+\sqrt{49+x^2}\right)+K$$
Remember to draw a triangle!



25. Using $x = 6\sin(\theta) \Rightarrow dx = 6\cos(\theta)\,d\theta$:
$$\int\sqrt{36-x^2}\,dx = \int\sqrt{36-36\sin^2(\theta)}\cdot 6\cos(\theta)\,d\theta$$
$$= 36\int\cos^2(\theta)\,d\theta = 18\int\left[1+\cos(2\theta)\right]\,d\theta$$
$$= 18\theta+9\sin(2\theta)+C = 18\theta+18\sin(\theta)\cos(\theta)+C$$
$$= 18\arcsin\left(\frac{x}{6}\right)+18\cdot\frac{x}{6}\cdot\frac{\sqrt{36-x^2}}{6}$$
$$= 18\arcsin\left(\frac{x}{6}\right)+\frac{x}{2}\sqrt{36-x^2}+C$$
Remember to draw a triangle!

27. This is very similar to Problem 23:

$$\int \frac{1}{36+x^2}\,dx = \ln\left(x+\sqrt{36+x^2}\right)+K$$

29. Using $x = 7\sin(\theta) \Rightarrow dx = 7\cos(\theta)\,d\theta$:

$$\int \frac{x^2}{\sqrt{49-x^2}}\,dx = \int \frac{49\sin^2(\theta)}{\sqrt{49-49\sin^2(\theta)}}\cdot 7\cos(\theta)\,d\theta$$
$$= 49\int \sin^2(\theta)\,d\theta = \frac{49}{2}\int [1-\cos(2\theta)]\,d\theta$$
$$= \frac{49}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]+C$$
$$= 49\left[\theta + \sin(\theta)\cos(\theta)\right]+C$$
$$= \frac{49}{2}\arcsin\left(\frac{x}{7}\right) - \frac{49}{2}\cdot\frac{x}{7}\cdot\frac{\sqrt{49-x^2}}{7}$$
$$= \frac{49}{2}\arcsin\left(\frac{x}{7}\right) + \frac{x}{2}\sqrt{49-x^2}+C$$

Remember to draw a triangle!

31. This does not require a trig substitution! Use $u = 25-x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = dx$:

$$\int \frac{x}{\sqrt{25-x^2}}\,dx = -\sqrt{25-x^2}+C$$

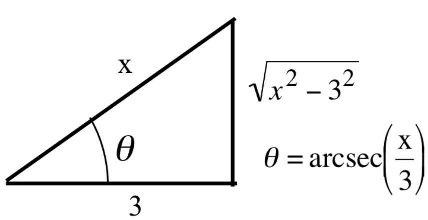
33. This does not require a trig substitution! Use $u = x^2+49 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = dx$ to get:

$$\int \frac{x}{x^2+49}\,dx = \frac{1}{2}\ln\left(x^2+49\right)+C$$

35. Using $x = 3\sec(\theta) \Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$:

$$\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\,dx = \int \frac{3\sec(\theta)\tan(\theta)}{\left[\sqrt{9\sec^2(\theta)-9}\right]^3}\,d\theta$$
$$= \int \frac{3\sec(\theta)\tan(\theta)}{27\tan^3(\theta)}\,d\theta = \frac{1}{9}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$
$$= \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = -\frac{1}{9}\cdot\frac{1}{\sin(\theta)}+C$$
$$= -\frac{1}{9}\cdot\frac{x}{\sqrt{x^2-9}}+C = \frac{-x}{9\sqrt{x^2-9}}+C$$

Remember to draw a triangle!



37. This does not require a trig substitution! If $x > 5$:

$$\int \frac{5}{2x\sqrt{x^2-25}}\,dx = \frac{1}{2}\,\text{arcsec}\left(\frac{x}{5}\right)+C$$

39. This does not require a trig substitution! Decompose the integrand using partial fractions:

$$\frac{1}{(5-x)(5+x)} = \frac{\frac{1}{10}}{5-x}+\frac{\frac{1}{10}}{5+x}$$

and then integrate:

$$\int \frac{1}{25-x^2}\,dx = \frac{1}{10}\left[-\ln\left(|5-x|\right)+\ln\left(|5+x|\right)+C\right.$$
$$= \frac{1}{10}\ln\left(\left|\frac{5+x}{5-x}\right|\right)+C$$

41. This resembles Problem 23 with a replacing 7:

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \ln\left(x+\sqrt{a^2+x^2}\right)+C$$

43. Using $x = a\tan(\theta) \Rightarrow dx = a\sec^2(\theta)\,d\theta$:

$$\int \frac{1}{x^2\sqrt{a^2+x^2}}\,dx = \int \frac{a\sec^2(\theta)}{a^2\tan^2(\theta)\cdot a\sec(\theta)}\,d\theta$$
$$= \frac{1}{a^2}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$
$$= \frac{1}{a^2}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = \frac{1}{a}\cdot\frac{-1}{\sin(\theta)}+C$$
$$= -\frac{1}{a^2}\cdot\frac{\sqrt{a^2+x^2}}{x}+C = \frac{-\sqrt{a^2+x^2}}{a^2x}+C$$

45. Set $u = x+1 \Rightarrow du = dx$ so the integral becomes:

$$\int \frac{1}{\sqrt{u^2+3^2}}\,du = \ln\left(\left|u+\sqrt{u^2+9}\right|\right)+C$$

(using the same pattern as Problem 23) and then replace u with $x+1$ to get:

$$\ln\left(\left|x+1+\sqrt{(x+1)^2+9}\right|\right)+C$$

47. Complete the square in the denominator:

$$x^2+10x+29 = x^2+10x+25+4 = (x+5)^2+2^2$$

to get an integrand that does not require trig substitution:

$$\int \frac{1}{(x+5)^2+2^2}\,dx = \frac{1}{2}\arctan\left(\frac{x+5}{2}\right)+C$$

49. Complete the square in the denominator: $x^2+4x+3 = x^2+4x+4-1 = (x+2)^2-1^2$. Then substitute $x+2 = \sec(\theta) \Rightarrow dx = \sec(\theta)\tan(\theta)\,d\theta$ (or substitute $u = x+2$ and then do the trig substitution) to get:

$$\int \frac{1}{\sqrt{(x+2)^2-1}}\,dx = \int \frac{\sec(\theta)\tan(\theta)}{\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(|\sec(\theta)+\tan(\theta)|\right)+C$$
$$= \ln\left(\left|x+2+\sqrt{x^2+4x+3}\right|\right)+C$$

51. (a) Using $x = \tan(\theta) \Rightarrow dx = \sec^2(\theta)\,d\theta$:

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{\sec^2(\theta)}{\left[\sec^2(\theta)\right]^2}\,d\theta = \int \cos^2(\theta)\,d\theta = \int \left[\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right]d\theta = \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C$$
$$= \frac{1}{2}\theta+\frac{1}{2}\sin(\theta)\cos(\theta)+C = \frac{1}{2}\arctan(x)+\frac{1}{2}\cdot\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}+C$$
$$= \frac{1}{2}\arctan(x)+\frac{x}{2(x^2+1)}+C$$

(b) First write the denominator as $1 = 1+x^2-x^2$ so that:

$$\int \frac{(1+x^2)-x^2}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{1+x^2}\,dx = \arctan(x) - \int x\cdot\frac{x}{(1+x^2)^2}\,dx$$

For the second integral, use $u = x \Rightarrow du = dx$ and $dv = \dfrac{x}{(1+x^2)^2}\,dx \Rightarrow v = -\dfrac{1}{2}\cdot\dfrac{1}{1+x^2}$ so that:

$$\int x\cdot\frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}\cdot\frac{x}{1+x^2}+\frac{1}{2}\int\frac{1}{1+x^2}\,dx = \frac{-x}{2(1+x^2)}+\frac{1}{2}\arctan(x)+K$$

Putting this all together yields the same result as part (a).

(c) Trig substitution requires less cleverness.

53. Substitute $u = x^2+25 \Rightarrow du = 2x\,dx$ so that the integral does not require trig substitution:

$$\int \frac{4\cdot 2x}{(x^2+25)^2}\,dx = 4\int\frac{1}{u^2}\,du = -\frac{4}{u}+C = -\frac{4}{x^2+25}+C$$

Section 8.5

1. $\displaystyle\int \sin^2(3x)\,dx = \frac{x}{2}-\frac{\sin(6x)}{12}+C = \frac{x}{2}-\frac{\sin(3x)\cos(3x)}{6}+C$

3. With $u = e^x \Rightarrow du = e^x\,dx$ the integral becomes $\sin(u)\cos(u)\,du = \dfrac{1}{2}\sin^2(u)+C = \dfrac{1}{2}\sin^2\left(e^x\right)+C$. With $w = \cos\left(e^x\right) \Rightarrow dw = -e^x\sin\left(e^x\right)\,dx$, the integral becomes $-\displaystyle\int w\,dw = -\frac{1}{2}w^2+K = -\frac{1}{2}\cos^2\left(e^x\right)+K$.

5. Substituting $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ and using the result of Example 1 yields:

$$\frac{1}{3}\left[\frac{3}{8}(3x)-\frac{1}{4}\sin(6x)+\frac{1}{32}\sin(12x)\right]_0^{\pi} = \frac{3\pi}{8}$$

7. First split off one factor of $\cos(5x)$ to write the integrand as $\cos^2(5x)\cdot\cdot(5x) = \left[1-\sin^2(5x)\right]\cos(5x)$ and then substitute $u = \sin(5x) \Rightarrow du = 5\cos(5x)\,dx \Rightarrow \frac{1}{5}\,du = \cos(5x)\,dx$ and note that $x = 0 \Rightarrow u = 0$ and $x = \pi \Rightarrow u = 0$ so the integral becomes:

$$\int_{x=0}^{x=\pi}\cos^2(5x)\cdot\cos(5x)\,dx = \int_{x=0}^{x=\pi}\left[1-\sin^2(5x)\right]\cos(5x)\,dx = \frac{1}{5}\int_{u=0}^{u=0}\left[1-u^2\right]du = 0$$

9. Substitute $u = \sin(7x) \Rightarrow du = 7\cos(7x)\,dx$:

$$\int \sin(7x)\cos(7x)\,dx = \frac{1}{7}\int u\,du = \frac{1}{14}u^2 + C = \frac{1}{14}\sin^2(7x) + C$$

Substituting $w = \cos(7x)$ yields an equivalent (but different-looking) result.

11. Substitute $u = \cos(7x) \Rightarrow du = -7\sin(7x)\,dx$:

$$\int \sin(7x)\cos^3(7x)\,dx = -\frac{1}{7}\int u^3\,du = -\frac{1}{28}u^4 + C = -\frac{1}{28}\cos^4(7x) + C$$

13. One option involves writing $\sin^2(3x)\cos^2(3x)$ as:

$$\sin^2(3x)\left[1 - \sin^2(3x)\right] = \sin^2(3x) - \sin^4(3x)$$

and then using the formula for $\int \sin^2(au)\,du$ and the result of Example 1. Or you can write:

$$\sin^2(3x)\cos^2(3x) = [\sin(3x)\cos(3x)]^2 = \left[\frac{1}{2}\sin(2\cdot 3x)\right]^2 = \frac{1}{4}\sin^2(6x)$$

and then use the formula for $\int \sin^2(au)\,du$:

$$\frac{1}{4}\int \sin^2(6x)\,dx = \frac{1}{4}\left[\frac{x}{2} - \frac{\sin(2\cdot 6x)}{4\cdot 6}\right] + C = \frac{x}{8} - \frac{1}{96}\sin(12x) + C$$

15. Split off one factor of $\sin(x)$, writing:

$$\sin^5(x)\cos^2(x) = \sin(x)\left[\sin^2(x)\right]^2\cos^2(x) = \sin(x)\left[1 - \cos^2(x)\right]^2\cos^2(x)$$

then substitute $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,dx &= -\int \left[1 - u^2\right]^2 u^2\,du \\ &= -\int \left[1 - 2u^2 + u^4\right] u^2\,du \\ &= \int \left[-u^2 + 2u^4 - u^6\right]\,du \\ &= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \\ &= -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C\end{aligned}$$

17. First split $\sec^4(4x) = \sec^2(4x)\cdot\sec^2(4x)$ and write this as $\left[1 + \tan^2(4x)\right]\sec^2(4x)$, then substitute $u = \tan(4x) \Rightarrow du = 4\sec^2(4x)]\,dx$ so that:

$$\int \left[1 + \tan^2(4x)\right]\sec^2(4x)\,dx = \frac{1}{4}\int \left[1 + u^2\right]\,du = \frac{1}{4}u + \frac{1}{12}u^3 + C = \frac{1}{4}\tan(4x) + \frac{1}{12}\tan^3(4x) + C$$

19. Substitute $u = 4x \Rightarrow du = 4\,dx \Rightarrow \frac{1}{4}\,du = dx$ and apply the reduction formula repeatedly:

$$\begin{aligned}\int \tan^5(4x)\,dx &= \frac{1}{4}\int \tan^5(u)\,du \\ &= \frac{1}{4}\left[\frac{1}{4}\tan^4(u) - \int \tan^3(x)\,dx\right] \\ &= \frac{1}{4}\left[\frac{1}{4}\tan^4(u) - \left(\frac{1}{2}\tan^2(x) - \int \tan(x)\,dx\right)\right] \\ &= \frac{1}{16}\tan^4(4x) - \frac{1}{8}\tan^2(4x) + \frac{1}{4}\ln\left(|\sec(4x)|\right) + C\end{aligned}$$

21. Substitute $u = \tan(5x) \Rightarrow du = 5\sec^2(5x)\,dx \Rightarrow \frac{1}{5}\,du = \sec^2(5x)\,dx$:

$$\frac{1}{5}\int u\,du = \frac{1}{10}u^2 + C = \frac{1}{10}\tan^2(5x) + C$$

23. Split $\sec^3(5x)\tan(5x) = \sec^2(5x)\cdot\sec(5x)\tan(5x)$ so $u = \sec(5x) \Rightarrow du = 5\sec(5x)\tan(5x)\,dx$ yields the integral:

$$\frac{1}{5}\int u^2\,du = \frac{1}{15}u^3 + C = \frac{1}{15}\sec^3(5x) + C$$

25. This is quite similar to Problem 23:

$$\int \sec^4(\theta)\tan(\theta)\,d\theta = \frac{1}{4}\sec^4(\theta) + C$$

27. Write the integrand as $\sec^2(\theta)\left[1 + \tan^2(\theta)\right]\tan^4(\theta)$ and use $u = \tan(\theta) \Rightarrow du = \sec^2(\theta)\,d\theta$ so that:

$$\int \left[1 + u^2\right]u^4\,du = \frac{1}{5}u^5 + \frac{1}{7}u^7 + C = \frac{1}{5}\tan^5(\theta) + \frac{1}{7}\tan^7(\theta) + C$$

29. The integrand is just $\tan^2(\theta) = \sec^2(\theta) - 1$ so the integral evaluates to $\tan(\theta) - \theta + C$.

31. The integrand simplifies to $\sin^4(\theta)$, so use the result of Example 1.

33. Use a product-to-sum identity to write:

$$\sin(x)\cos(3x) = \frac{1}{2}\left[\sin(4x) + \sin(2x)\right]$$

and integrate to get: $-\frac{1}{8}\cos(8x) - \frac{1}{4}\cos(4x) + C$

35. Use a product-to-sum identity to write:

$$\sin(x)\sin(3x) = \frac{1}{2}\left[\cos(2x) - \cos(4x)\right]$$

and integrate to get: $\frac{1}{4}\sin(2x) - \frac{1}{8}\sin(4x) + C$

37. If n is odd, we can write $n = 2k+1$ where k is some other integer. Then write:

$$\begin{aligned}\sin^{2k+1}(x) &= \sin^{2k}(x)\cdot\sin(x)\\ &= \left[\sin^2(x)\right]^k \sin(x)\\ &= \left[1-\cos^2(x)\right]^k \sin(x)\end{aligned}$$

and use $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$. This substitution changes the limits of integration from $x = 0$ to $u = \cos(0) = 1$ and from $x = 2\pi$ to $u = \cos(2\pi) = 1$. The integral thus becomes:

$$\int_1^1 \left[1-u^2\right]\,du = 0$$

39. Use a product-to-sum formula to rewrite the integrand $\sin(mx)\cdot\sin(nx)$ as:

$$\frac{1}{2}\left[\sin\left((m+n)x\right) + \sin\left((m-n)x\right)\right]$$

If $k = m+n$ or $k = m-n$ (neither of which is 0:

$$\int_0^{2\pi} \sin(kx)\,dx = \left[-\frac{1}{k}\cos(kx)\right]_0^{2\pi} = 0$$

so the original integral must equal 0 as well.

41. The integrand is just $\sin^2(mx)$, so:

$$\int_0^{2\pi} \sin^2(mx)\,dx = \left[\frac{x}{2} - \frac{\sin(2mx)}{4m}\right]_0^{2\pi} = \pi$$

43. Integrating the product of $\sin(2x)$ and any term of $P(x)$ other than $-4\sin(2x)$ yields 0, so a_2 is:

$$\frac{1}{\pi}\int_0^{2\pi} \sin(2x)\cdot\left[-4\sin(2x)\right]\,dx = \frac{-4}{\pi}\cdot\pi = -4$$

45. $a_4 = 0$ because $P(x)$ has no $\sin(4x)$ term.

47. On your own.

Section 8.6

In place of full solutions, the "answers" for this section suggest an integration method, or a first step.

1. Substitute $u = 1 - x$.

3. Substitute $u = a^2 - x^2$.

5. Substitute $u = a + bx$.

7. Substitute $u = x^2$, or factor the denominator:

$$1 - x^4 = (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2)$$

and then use partial fractions.

9. Divide first.

11. Substitute $u = a^{\frac{2}{3}} - x^{\frac{2}{3}}$.

13. Expand by squaring.

15. Substitute $u = 1 + x^3$.

17. Substitute $u = e^x + x^{-x}$. (Or note that the integrand is equivalent to $\tanh(x)$.)

19. Start with long division.

21. Start with long division.

23. Substitute $u = a + b\cos(2\theta)$.

25. Substitute $u = \cos(\theta)$.

27. Complete the square.

29. Start with long division.

31. Substitute $u = 1 - e^{2t}$.

33. Substitute $x = u^2$.

35. Substitute $u = 1 + x^2$.

37. Start with integration by parts.

39. Substitute $u = a^2 - x^2$.

41. Substitute $u = \sin(\theta)$.

43. Use integration by parts with $u = x$.

45. Substitute $x = u^2$.

47. Substitute $u = 1 + 2x$.

49. Substitute $y = e^x$.

51. Use integration by parts with $u = x^2$.

53. Substitute $u = 1 - x$.
55. Start with long division.
57. Substitute $u = 1 + \cos\left(\frac{\theta}{2}\right)$.
59. Substitute $u = 1 + e^{-x}$.
61. Substitute $u = 1 - x$.
63. Substitute $u = \sin(\theta)$.
65. Substitute $u = x^3$.
67. Substitute $u = \frac{x}{2}$.
69. Start with long division.
71. Substitute $u = x^2$.
73. Use integration by parts with $u = \theta$.
75. Substitute $x = t^2$.
77. Start with long division.
79. Expand by squaring.
81. Complete the square.
83. Substitute $u = 1 - 3e^x$.
85. Use integration by parts with $u = [\ln(x)]^2$.
87. Substitute $t = -x^3$.
89. Factor the denominator:
$$\begin{aligned} x^2(x+1) - 4(x+1) &= (x^2-4)(x+1) \\ &= (x-2)(x+2)(x+1) \end{aligned}$$
and use partial fractions.
91. Start with long division.
93. Substitute $x = e^t$.
95. Use partial fractions.
97. Use partial fractions.
99. Use partial fractions (graph the polynomial in the denominator to assist with factoring).
101. Start with long division.
103. Use partial fractions.
105. Complete the square.
107. Substitute $u = 1 + x^2$.
109. Write $\cot(\theta) = \dfrac{1}{\tan(\theta)}$ and simplify.
111. Expand the denominator and use long division.
113. Substitute $x = e^t - 1$.
115. Substitute $u = x^2$.
117. Complete the square.
119. Complete the square.
121. Use a trigonometric identity.
123. Use a trigonometric identity.
125. Integration by parts.
127. Integration by parts (twice).
129. Write $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$.
131. Integration by parts.
133. Integration by parts.

Section 8.7

1. $P(x) = 5 + 3x$
3. $P(x) = 4 - x$
5. $P(x) = 4$
7. $P(0) = A$, $P'(0) = B$
9. $P(x) = -2 + 7x + 3x^2$
11. $P(x) = 8 + 5x + 5x^2$
13. $P(x) = -3 - 2x + 2x^2$
15. $P(x) = 5 + 3x + 2x^2 + x^3$
17. $P(x) = 4 - x - x^2 - 2x^3$
19. $P(x) = 4 - 2x^2 + 6x^3$
21. $P(0) = A$, $P'(0) = B$, $P''(0) = 2C$, $P'''(0) = 6D$
23. To five decimal places:

x	$f(x))$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0.0	0.00000	0.00000	0.00000
0.1	0.09983	0.09983	0.00000
0.2	0.19867	0.19867	0.00000
0.3	0.29552	0.29552	0.00000
1.0	0.84147	0.84167	0.00020
2.0	0.90930	0.93333	0.02404

25. To five decimal places:

x	$f(x))$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0.0	1.00000	1.00000	0.00000
0.1	0.99500	0.99500	0.00000
0.2	0.98007	0.98007	0.00000
0.3	0.95534	0.95534	0.00000
1.0	0.54030	0.54167	0.00136
2.0	-0.41615	-0.33333	0.08281

27. To five decimal places:

x	$f(x))$	$P(x)$	$\lvert f(x)-P(x)\rvert$
0.0	1.00000	1.00000	0.00000
0.1	1.10517	1.10517	0.00000
0.2	1.22140	1.22133	0.00007
0.3	1.34986	1.34950	0.00036
1.0	2.71828	2.66667	0.05162
2.0	7.38906	6.33333	1.05572

29. Starting with:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3$$

and substituting $u = 2x$ yields:

$$e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

31. $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$ so $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$ and $f'''(0) = 6$:

$$P(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 = 1 + x + x^2 + x^3$$

33. $f(x) = \ln(1+x) \Rightarrow f'(x) = (1+x)^{-1} \Rightarrow f''(x) = -(1+x)^{-2} \Rightarrow f'''(x) = 2(1+x)^{-3} \Rightarrow f^{(4)}(x) = -6(1+x)^{-4}$ so $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$ and $f^{(4)}(0) = -6$:

$$\begin{aligned} P(x) &= 0 + x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \end{aligned}$$

35. Starting with:

$$\cos(u) \approx 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \frac{1}{6!}u^6$$

and substituting $u = x^2$ yields:

$$\cos(x^2) \approx 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{720}x^{12}$$

37. Starting with:

$$\sin(u) \approx u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7$$

and substituting $u = x^2$ yields:

$$\sin(x^2) \approx x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} - \frac{1}{5040}x^{14}$$

so multiplying both sides by x^3 results in:

$$x^3 \cdot \sin(x^2) \approx x^5 - \frac{1}{6}x^9 + \frac{1}{120}x^{13} - \frac{1}{5040}x^{17}$$

39. Starting with:

$$\sin(u) \approx u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7$$

and substituting $u = x^3$ yields:

$$\sin(x^3) \approx x^3 - \frac{1}{6}x^9 + \frac{1}{120}x^{15} - \frac{1}{5040}x^{21}$$

so integrating both sides from 0 to 1 results in:

$$\begin{aligned} \int_0^1 \sin(x^3)\,dx &\approx \left[\frac{1}{4}x^4 - \frac{1}{60}x^{10} + \frac{1}{1920}x^{16} - \frac{1}{110880}x^{22}\right]_0^1 \\ &= \frac{1}{4} - \frac{1}{60} + \frac{1}{1920} - \frac{1}{110880} \approx 0.2338 \end{aligned}$$

41. Starting with:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3$$

and substituting $u = -x^3$ yields:

$$e^{-x^3} \approx 1 - x^3 + \frac{1}{2}x^6 - \frac{1}{6}x^9$$

so integrating both sides from 0 to $\frac{1}{2}$ results in:

$$\begin{aligned} \int_0^1 e^{-x^3}\,dx &\approx \left[x - \frac{1}{4}x^4 + \frac{1}{14}x^7 - \frac{1}{60}x^{10}\right]_0^{\frac{1}{2}} \\ &= \frac{1}{2} - \frac{1}{64} + \frac{1}{1792} - \frac{1}{61440} \approx 0.4849 \end{aligned}$$

D
Derivative Facts

Basic Patterns

$\mathbf{D}(k) = 0$ $\qquad$ $\mathbf{D}(k \cdot f) = k \cdot \mathbf{D}(f)$ $\qquad$ k represents a constant

$\mathbf{D}(f+g) = \mathbf{D}(f) + \mathbf{D}(g)$ $\qquad$ $\mathbf{D}(f-g) = \mathbf{D}(f) - \mathbf{D}(g)$

$\mathbf{D}(f \cdot g) = f \cdot \mathbf{D}(g) + g \cdot \mathbf{D}(f)$ $\qquad$ $\mathbf{D}\left(\frac{f}{g}\right) = \frac{g \cdot \mathbf{D}(f) - f \cdot \mathbf{D}(g)}{g^2}$ $\qquad$ Product Rule and Quotient Rule

Power Rules

$\mathbf{D}(x^p) = p \cdot x^{p-1}$ $\qquad$ $\mathbf{D}\left(f^n\right) = n \cdot f^{n-1} \cdot \mathbf{D}(f)$

Chain Rule

$\mathbf{D}\left(f(g(x))\right) = f'\left(g(x)\right) \cdot g'(x)$ $\qquad$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Exponential and Logarithmic Functions

$\mathbf{D}\left(e^u\right) = e^u$ $\qquad$ $\mathbf{D}\left(a^u\right) = a^u \cdot \ln(a)$

$\mathbf{D}\left(\ln(|u|)\right) = \frac{1}{u}$ $\qquad$ $\mathbf{D}\left(\log_a(|u|)\right) = \frac{1}{u \cdot \ln(a)}$ $\qquad$ $\mathbf{D}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$

Trigonometric Functions

$\mathbf{D}\left(\sin(u))\right) = \cos(u))$ $\qquad$ $\mathbf{D}(\tan(u)) = \sec^2(u)$ $\qquad$ $\mathbf{D}(\sec(u)) = \sec(u) \cdot \tan(u)$

$\mathbf{D}(\cos(u)) = -\sin(u)$ $\qquad$ $\mathbf{D}(\cot(u)) = -\csc^2(u)$ $\qquad$ $\mathbf{D}(\csc(u)) = -\csc(u) \cdot \cot(u)$

Inverse Trigonometric Functions

$$\mathbf{D}(\arcsin(u)) = \frac{1}{\sqrt{1-u^2}} \qquad \mathbf{D}(\arctan(u)) = \frac{1}{1+u^2} \qquad \mathbf{D}(\text{arcsec}(u)) = \frac{1}{|u|\sqrt{u^2-1}}$$

$$\mathbf{D}(\arccos(u)) = \frac{-1}{\sqrt{1-u^2}} \qquad \mathbf{D}(\text{arccot}(u)) = \frac{-1}{1+u^2} \qquad \mathbf{D}(\text{arccsc}(u)) = \frac{-1}{|u|\sqrt{u^2-1}}$$

Hyperbolic Functions

$$\mathbf{D}\left(\sinh(u)\right)) = \cosh(u)) \qquad \mathbf{D}(\cosh(u)) = \sinh(u)$$

$$\mathbf{D}(\tanh(u)) = \text{sech}^2(u) \qquad \mathbf{D}(\coth(u)) = -\text{csch}^2(u)$$

$$\mathbf{D}(\text{sech}(u)) = -\text{sech}(u)\cdot\tanh(u) \qquad \mathbf{D}(\text{csch}(u)) = -\text{csch}(u)\cdot\coth(u)$$

Inverse Hyperbolic Functions

$$\mathbf{D}(\text{argsinh}(u)) = \frac{1}{\sqrt{1+u^2}} \qquad \mathbf{D}(\text{argcosh}(u)) = \frac{1}{\sqrt{u^2-1}} \quad (\text{for } u > 1)$$

$$\mathbf{D}(\text{argtanh}(u)) = \frac{1}{1-u^2} \quad (\text{for } |u| < 1) \qquad \mathbf{D}(\text{argcoth}(u)) = \frac{1}{1-u^2} \quad (\text{for } |u| > 1)$$

$$\mathbf{D}(\text{argsech}(u)) = \frac{-1}{|u|\sqrt{1-u^2}} \quad (\text{for } 0 < u < 1) \qquad \mathbf{D}(\text{argcsch}(u)) = \frac{-1}{|u|\sqrt{u^2+1}} \quad (\text{for } u \neq 0)$$

H
How to Succeed in Calculus

The following comments are based on over 30 years of watching students succeed and fail in calculus courses at universities, colleges and community colleges and of listening to their comments as they went through their study of calculus. This is the best advice we can give to help you succeed.

Calculus takes time. Almost no one fails calculus because they lack sufficient "mental horsepower." Most people who do not succeed are unwilling (or unable) to devote the necessary time to the course. The "necessary time" depends on how smart you are, what grade you want to earn and on how competitive the calculus course is. Most calculus teachers and successful calculus students agree that two (or three) hours every weeknight and six or seven hours each weekend is a good way to begin if you seriously expect to earn an A or B grade. If you are only willing to devote five or 10 hours a week to calculus outside of class, you should consider postponing your study of calculus.

Do NOT fall behind. The brisk pace of the calculus course is based on the idea that "if you are in calculus, then you are relatively smart, you have succeeded in previous mathematics courses, and you are willing to work hard to do well." It is terribly difficult to **catch up** and **keep up** at the same time. A much safer approach is to work very hard for the first month and then evaluate your situation. If you do fall behind, spend a part of your study time catching up, but spend most of it trying to follow and understand what is going on in class.

Go to class, every single class. We hope your calculus teacher makes every idea crystal clear, makes every technique seem obvious and easy, is enthusiastic about calculus, cares about you as a person, and even makes you laugh sometimes. If not, you still need to attend class. You need to hear the vocabulary of calculus spoken and to see how mathematical ideas are strung together to reach conclusions. You need to see how an expert problem-solver approaches problems. You need to hear the announcements about homework and tests. And you need to get to know some of the other students in the class. Unfortunately, when students get a bit behind or confused, they are most likely to miss a class or two (or five). That is absolutely the worst time to miss classes. Attend class anyway. Ask where on campus you can get some free tutoring or counseling. Ask a classmate to help you for an hour after class. If you must miss a class, ask a classmate what material was covered and skim those sections before the next class. Even if you did not read the material, return to class as soon as possible.

Work together. Study with a friend. Work in small groups. It is much more fun and is very effective for doing well in calculus. Recent studies — and our personal observations — indicate that students who regularly work together in small groups are less likely to drop the course and are more likely to get A's or B's. You need lots of time to work on the material alone, but study groups of 3–5 students, working together two or three times a week for a couple hours, seem to help everyone in the group. Study groups offer you a way to get and give help on the material and they can provide an occasional psychological boost ("misery loves company?"); they are a place to use the mathematical language of the course, to trade mathematical tips, and to "cram" for the next day's test. Students in study groups are less likely to miss important points in the course, and they get to know some very nice people — their classmates.

Use the textbook effectively. There are a number of ways to use a mathematics textbook:

- to gain an overview of the concepts and techniques,
- to gain an understanding of the material,
- to master the techniques, and
- to review the material and see how it connects with the rest of the course.

The first time you read a section, just try to see what problems are being discussed. Skip around, look at the pictures, and read some of the problems and the definitions. If something looks complicated, skip it. If an example looks interesting, read it and try to follow the explanation. This is an exploratory phase. Don't highlight or underline at this stage — you don't know what is important yet and what is just a minor detail.

The next time through the section, proceed in a more organized fashion, reading each introduction, example, explanation, theorem and proof. This is the beginning of the "mastery" stage. If you don't understand the explanation of an example, put a question mark (in pencil) in the margin and go on. Read and try to understand each step in the proofs and ask yourself why that step is valid. If you don't see what justified moving from one step to another in the proof, pencil in question marks in the margin. This second phase will go more slowly than the first, but if you don't understand some details, just keep going. Don't get bogged down yet.

Now worry about the details. Go quickly over the parts you already understand, but slow down and try to figure out the parts marked with question marks. Try to solve the example problems before you refer to the explanations. If you now understand parts that were giving you trouble, cross out the question marks. If you still don't understand something, put in another question mark and **write down** your question to ask a teacher, tutor or classmate.

Finally, it is time to try the problems at the end of the section. Many of them are similar to examples in the section, but now *you* need to solve them. Some of the problems are more complicated than the examples, but they still require the same basic techniques. Some of the problems will require that you use concepts and facts from earlier in the course, a combination of old and new concepts and techniques. Working lots of problems is the "secret" of success in calculus.

Work the Problems. Many students read a problem, work it out and check the answer in the back of the book. If their answer is correct, they go on to the next problem. If their answer is wrong, they manipulate (finagle, fudge, massage) their work until their new answer is correct, and then they go on to the next problem. **Do not try the next problem yet!** Before going on, spend a short time, just half a minute, thinking about what you have just done in solving the problem. Ask yourself: "What was the point of this problem?" "What big steps did I need to take to solve this problem?" "What was the **process**?" Do not simply review every single step of the solution process. Instead, look at the outline of the solution, the process, the "big picture." If your first answer was wrong, ask yourself, "What about this problem should have suggested the right process the *first* time?" As much learning and retention can take place in the 30 seconds you spend reviewing the process as took place in the 10 minutes you took to solve the problem. A correct answer is important, but **a correct process — carefully used — will get you many correct answers**.

There is one more step that too many students omit. **Go back and quickly look over the section one more time.** Don't worry about the details, just try to understand the overall logic and layout of the section. Ask yourself, "What was I expected to learn in this section?" Typically, this last step — a review and overview — goes quickly, but it is very valuable. It can help you see and retain the important ideas and connections.

I

Integral Table

To save space (and ink), only one member of each antiderivative family appears for most integrals below; for example, you should interpret $\int \cos(x)\,dx = \sin(x)$ as $\int \cos(x)\,dx = \sin(x) + C$, where C is an arbitrary constant.

Basic Patterns

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx \qquad \int F'(ax+b)\,dx) = \frac{1}{a} \cdot F(x) \qquad \int F'(g(x)) \cdot g'(x)\,dx) = F(g(x))$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \qquad \int [f(x) \cdot g'(x)]\,dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx$$

Powers

$$\int x^p\,dx = \frac{1}{p+1} x^{p+1} \qquad (p \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x|$$

Exponential Functions

$$\int e^x\,dx = e^x \qquad \int a^x\,dx = \frac{a^x}{\ln(a)} \qquad (a > 0) \qquad \int x \cdot e^x\,dx = (x-1)e^x$$

$$\int x^2 \cdot e^x\,dx = (x^2 - 2x + 2)e^x \qquad \int x^n \cdot e^x\,dx = x^n \cdot e^x - n \int x^{n-1} e^x\,dx$$

Logarithmic Functions

$$\int \ln(x)\,dx = x\ln(x) - x \qquad \int \ln(a^2 + x^2)\,dx = x \cdot \ln(a^2 + x^2) + 2a \arctan\left(\frac{x}{a}\right) - 2x$$

$$\int \frac{\ln(x)}{x}\,dx = \frac{1}{2}[\ln(x)]^2 \qquad \int x^p \cdot \ln(x)\,dx = x^{p+1}\left[\frac{\ln(x)}{p+1} - \frac{1}{(p+1)^2}\right] \qquad (p \neq -1)$$

Trigonometric Functions

$\int \sin(x)\,dx = -\cos(x)$

$\int \tan(x)\,dx = \ln(|\sec(x)|)$

$\int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|)$

$\int \cos(x)\,dx = \sin(x)$

$\int \cot(x)\,dx = \ln(|\sin(x)|)$

$\int \csc(x)\,dx = \ln(|\csc(x) - \cot(x)|)$

$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x)$

$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$

$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x)$

$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$

$\int \sec^2(x)\,dx = \tan(x)$

$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln(|\sec(x) + \tan(x)|)$

$\int \sec(x)\tan(x)\,dx = \sec(x)$

$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$

$\int \csc^2(x)\,dx = -\cot(x)$

$\int \csc^3(x)\,dx = -\frac{1}{2}\csc(x)\cot(x) + \frac{1}{2}\ln(|\csc(x) - \cot(x)|)$

$\int \csc(x)\cot(x)\,dx = -\csc(x)$

$\int \csc^n(x)\,dx = -\frac{1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b)} \quad (a \neq \pm b)$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad (a \neq \pm b)$$

$\int x\sin(x)\,dx = -x\cos(x) + \sin(x)$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad (a \neq \pm b)$$

$\int x\cos(x)\,dx = x\sin(x) + \cos(x)$

$\int x^n\sin(x)\,dx = -x^n\cos(x) + n\int x^{n-1}\cos(x)\,dx$

$\int x^n\cos(x)\,dx = x^n\sin(x) - n\int x^{n-1}\sin(x)\,dx$

Hyperbolic Functions

$\int \sinh(x)\,dx = \cosh(x)$

$\int \tanh(x)\,dx = \ln(\cosh(x))$

$\int \mathrm{sech}(x)\,dx = \arctan(\sinh(x))$

$\int \cosh(x)\,dx = \sinh(x)$

$\int \coth(x)\,dx = \ln(|\sinh(x)|)$

$\int \mathrm{csch}(x)\,dx = \ln(|\coth(x) - \mathrm{csch}(x)|)$

$\int \mathrm{sech}^2(x)\,dx = \tanh(x)$

$\int \mathrm{sech}(x)\tanh(x)\,dx = -\mathrm{sech}(x)$

$\int \mathrm{csch}^2(x)\,dx = -\coth(x)$

$\int \mathrm{csch}(x)\coth(x)\,dx = -\mathrm{csch}(x)$

Inverse Trigonometric Functions

$$\int \arcsin(x)\,dx = x \cdot \arcsin(x) + \sqrt{1-x^2}$$

$$\int \arctan(x)\,dx = x \cdot \arctan(x) - \frac{1}{2}\ln(1+x^2)$$

Rational Functions

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\left|\frac{x+a}{x-a}\right|\right) = \frac{1}{a}\operatorname{argtanh}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{(a^2+x^2)^2}\,dx = \frac{1}{2a^3}\left[\frac{ax}{a^2+x^2} + \arctan\left(\frac{x}{a}\right)\right]$$

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}\ln\left(\left|\frac{x-a}{x-b}\right|\right)$$

Radical Functions

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{argsinh}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) \quad (x > a)$$

Products of Exponentials and Trigonometric or Hyperbolic Functions

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right]$$

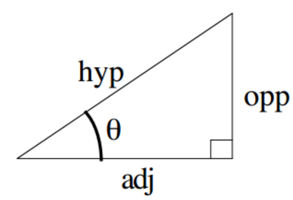
$$\int e^{ax}\sinh(bx)\,dx = \frac{e^{ax}}{a^2-b^2}\left[a\sinh(bx) - b\cosh(bx)\right]$$

$$\int e^{ax}\cosh(bx)\,dx = \frac{e^{ax}}{a^2-b^2}\left[a\cosh(bx) - b\sinh(bx)\right]$$

T
Trigonometry Facts

Right Angle Trigonometry

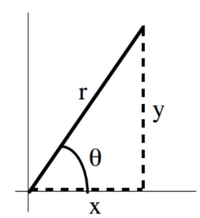
$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \tan(\theta) = \frac{\text{opp}}{\text{adj}}$$



$$\csc(\theta) = \frac{\text{hyp}}{\text{opp}} \qquad \sec(\theta) = \frac{\text{hyp}}{\text{adj}} \qquad \cot(\theta) = \frac{\text{adj}}{\text{opp}}$$

Trigonometric Functions

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad \tan(\theta) = \frac{y}{x}$$



$$\cot(\theta) = \frac{x}{y} \qquad \sec(\theta) = \frac{r}{x} \qquad \csc(\theta) = \frac{r}{y}$$

Fundamental Identities

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)}$$

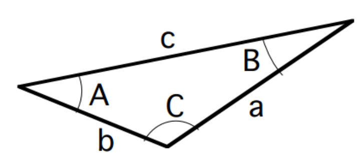
$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \Rightarrow \quad 1 + \tan^2(\theta) = \sec^2(\theta) \quad \Rightarrow \quad \cot^2(\theta) + 1 = \csc^2(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta) \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$$

Law of Sines: $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$



Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$

Angle Addition and Subtraction Formulas

$$\sin(x+y) = \sin(x)\cdot\cos(y) + \cos(x)\cdot\sin(y)$$

$$\sin(x-y) = \sin(x)\cdot\cos(y) - \cos(x)\cdot\sin(y)$$

$$\cos(x+y) = \cos(x)\cdot\cos(y) - \sin(x)\cdot\sin(y)$$

$$\cos(x-y) = \cos(x)\cdot\cos(y) + \sin(x)\cdot\sin(y)$$

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\cdot\tan(y)} \qquad \tan(x-y) = \frac{\tan(x)+\tan(y)}{1+\tan(x)\cdot\tan(y)}$$

Product-to-Sum Formulas

$$\sin(x)\cdot\sin(y) = \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y)$$

$$\cos(x)\cdot\cos(y) = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)$$

$$\sin(x)\cdot\cos(y) = \tfrac{1}{2}\sin(x+y) + \tfrac{1}{2}\sin(x-y)$$

Sum-to-Product Formulas

$$\sin(x)+\sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x)+\cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)$$

$$\tan(x)+\tan(y) = \frac{\sin(x+y)}{\cos(x)\cdot\cos(y)}$$

Double-Angle Formulas

$$\sin(2x) = 2\sin(x)\cdot\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

Half-Angle Formulas

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1-\cos(x)\right)}$$

The quadrant of $\frac{x}{2}$ determines the $\pm$.

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1+\cos(x)\right)}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{\sin(x)}$$