16
Vector Calculus in 2D

So far during your journey through calculus you have worked extensively with functions of a single variable, such as:

$$f(x) = x^2 - 3x \cdot \sin(x)$$

that map a single-number input to a single-number output (in this case, $f(\pi) = \pi^2$). More recently, you have worked with functions of two (or more) variables, such as:

$$g(x, y) = \arctan(x + y) - x^2 \cdot y$$

that map a point in two (or higher)–dimensional space to a single-number output (in this case, $g(2, -1) = \frac{\pi}{4} + 4$). And you have studied vector-valued functions of a single variable, such as:

$$\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle$$

that map a single-number input to a two- or three-dimensional vector.

We now turn your attention to functions that map a point in two- (or three-) dimensional space to a vector in the same space. You have already seen examples of this when you computed the gradient of a multivariable function:

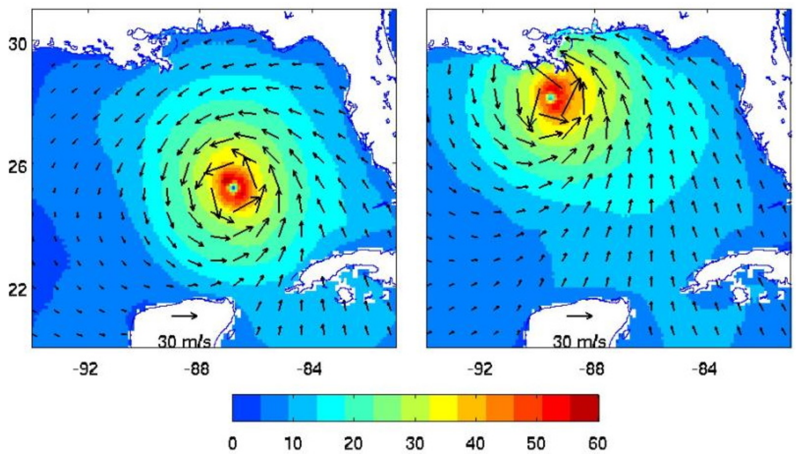
$$\varphi(x, y) = x^2 \cdot y^3 \quad \Rightarrow \quad \nabla \varphi(x, y) = \left\langle 2xy^3, 3x^2y^2 \right\rangle$$

Although it may seem reasonable to call such a mapping a "multivariable vector-valued function," we will instead call these relations **vector fields** (due to tradition, as well as their usefulness when working with electric, magnetic and gravitational fields in physics).

In this chapter, you will study two-dimensional vector fields, and in the next chapter three-dimensional vector fields.

16.1 Vector Fields in 2D

A vector field assigns a vector to each point in a domain, such as these vectors showing wind velocities at sea level for Hurricane Katrina on August 28 (below left) and August 29 (below right) in 2005:



From the U.S. Department of the Interior publication "Modeling Waves and Currents Produced by Hurricanes Katrina, Rita and Wilma" by Lie-Yauw Oey and Dong-Ping Wang.

As indicated above, a vector field may vary over time. For the most part, however, we will limit our study to **steady-state** vector fields, which are time-independent. The vectors in the Katrina example are velocities, but vectors in a vector field often represent forces.

Mathematically, we define a 2D vector field as a function with points (x, y) in the plane as inputs and vectors as outputs, where the vector components depend on the input point.

> **Definition:** A **2D vector field** is a function $\mathbf{F}$ that assigns a 2D vector $\mathbf{F}(x, y)$ to each point (x, y) in a subset $\mathcal{D}$ of the xy-plane. We write:
>
> $$\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle \quad \text{or} \quad \mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

In mathematical typesetting, the vectors $\mathbf{F}$, $\mathbf{i}$ and $\mathbf{j}$ are typically rendered in bold font, but when handwriting vectors, be sure to use arrow notation (or "hat" notation for unit vectors): $\vec{F}$, $\hat{\imath}$, $\hat{\jmath}$

We call $P(x, y)$ and $Q(x, y)$ the **component functions** of the vector field $\mathbf{F}$ and $\mathcal{D}$ the **domain** of $\mathbf{F}$.

Most of the vector fields you will be see in applications will have component functions that are differentiable (and therefore continuous and defined) at all points in the plane, or everywhere except one point (usually the origin).

Example 1. Compute the values of $\mathbf{F}(x, y) = \langle y, -x\rangle$ at the points $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 1)$, $(-1, 0)$ and $(-1, -1)$.

Solution. Substituting the given x- and y-values: $\mathbf{F}(1, 0) = \langle 0, -1\rangle$, $\mathbf{F}(1, 1) = \langle 1, -1\rangle$, $\mathbf{F}(0, 1) = \langle 1, 0\rangle$, $\mathbf{F}(-1, 1) = \langle 1, 1\rangle$, $\mathbf{F}(-1, 0) = \langle 0, 1\rangle$ and $\mathbf{F}(-1, -1) = \langle -1, 1\rangle$. ◀

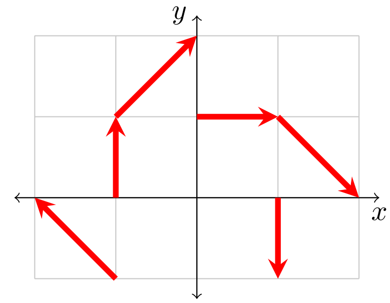
Practice 1. Compute the values of $\mathbf{G}(x, y) = \langle 1, -3\rangle$ and $\mathbf{H}(x, y) = \langle x + y, x - y\rangle$ at the points $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(0, -1)$ and $(-1, -1)$.

Graphing Vector Fields

The output of a vector field typically consists of an infinite number of vectors (one for each point in a subset $\mathcal{D}$ of the xy-plane) so we usually graph only a few of these vectors to make the pattern clear. By convention, we put the tail of the $\mathbf{F}(x, y)$ vector at the point (x, y).

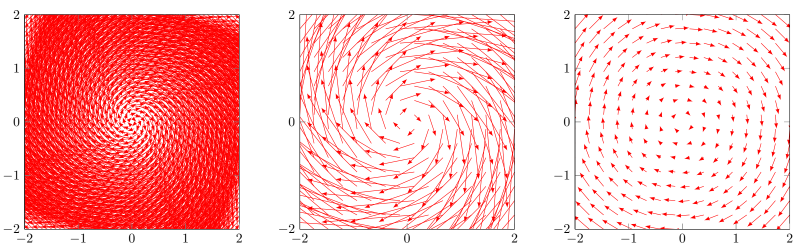
Example 2. Plot the vectors of $\mathbf{F}(x,y) = \langle y, -x\rangle$ at the points $(1,0)$, $(1,1)$, $(0,1)$, $(-1,1)$, $(-1,0)$ and $(-1,-1)$.

Solution. Using the results from Example 1 and plotting each output vector with its tail at the input point yields the graph in the margin. ◀

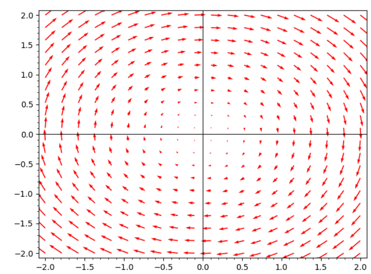


Practice 2. Plot $\mathbf{G}(x,y) = \langle 1, -3\rangle$ and $\mathbf{H}(x,y) = \langle x+y, x-y\rangle$ at the points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, $(-1,1)$, $(-1,0)$, $(0,-1)$ and $(-1,-1)$.

Graphing a few values of a vector field by hand can be instructive, but quickly becomes very tedious. A number of web-based graphing tools and computer programs do a very nice (and much faster) job. Below are three plots of $\mathbf{F}(x,y) = \langle y, -x\rangle$ from Example 2:



Here is a SageCell graph of $\mathbf{F} = \langle y, -x\rangle$:



using the code:

```
var('x,y')
plot_vector_field((y,-x),
    (x,-2,2),(y,-2,2),color="red")
```

The plot above left demonstrates the disadvantage of plotting too many vectors, while the plot in the middle reveals that plotting fewer vectors can still result in overlap, obscuring the overall pattern. The plot above right "scales" the length of the vectors to avoid overlap. (You can think of this as using different units, say cm/sec rather than mm/sec).

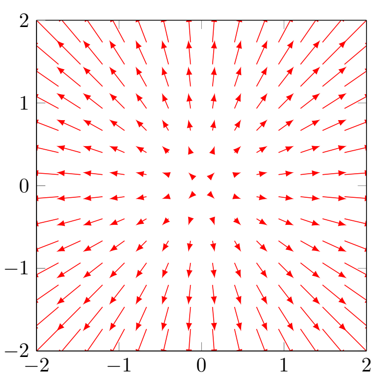
Practice 3. Plot $\mathbf{G}(x,y)$ and $\mathbf{H}(x,y)$ from Practice 2 using technology.

Gradient Fields

A common example of a vector field with which you are already familiar is a **gradient field**, the gradient of a function of two (or more) variables.

Example 3. Compute the gradient field for $\varphi(x,y) = x^2 + y^2$.

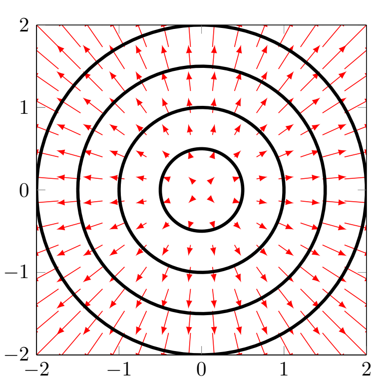
Solution. $\varphi_x = 2x$ and $\varphi_y = 2y$ so $\nabla\varphi(x,y) = \langle 2x, 2y\rangle$. A graph appears in the margin. (This is an example of a **radial vector field**, with all of the vectors pointing radially away from the origin.) ◀



Practice 4. Compute (and graph) the gradient field for $\psi(x,y) = xy$.

If a vector field $\mathbf{F}(x,y)$ is the gradient of a function $\varphi(x,y)$, we call φ a **potential function** for $\mathbf{F}$. (This term relates to potential energy, as we will see later during our study of vector fields.)

You may recall that gradient vectors of a function are normal (perpendicular) to the level curves of that function, as seen in the margin, where the gradient field $\langle 2x, 2y\rangle$ from Example 3 is graphed together with level curves of the form $x^2 + y^2 = k$ for $k = \frac{1}{2}$, 1, $\frac{3}{2}$ and 2.



In the context of vector fields, we call these level curves the **equipotentials** of the gradient field (because along these curves the potential function is constant).

Practice 5. Graph the gradient field for $\psi(x, y) = xy$ along with some of its equipotential curves.

Finding Potential Functions

Given a potential function, finding the associated gradient field is straightforward. But if you suspect that a function is a gradient field, how can you work backwards to find a potential function? Moving from the potential function to the gradient involves partial differentiation, so perhaps not surprisingly going in the opposite direction involves "partial integration" (or "partial antidifferentiation").

Example 4. Find a potential function $\varphi(x, y)$ for $\mathbf{F}(x, y) = \langle 4y, 4x + 5 \rangle$.

Solution. If $\nabla\varphi = \mathbf{F}$ for some function φ then $\left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right\rangle = \mathbf{F}$ so:

$$\frac{\partial \varphi}{\partial x} = 4y \;\Rightarrow\; \varphi(x, y) = 4xy + g(y)$$

Why $g(y)$ rather than a constant? Taking the partial derivative of $g(y)$ with respect to x yields 0 so adding any constant or any function of y to φ will not change φ_x.

Taking the partial derivative of our new candidate for φ with respect to y yields:

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left[4xy + g(y)\right] = 4x + g'(y)$$

On the other hand we need $\varphi_y = 4x + 5$ so equating these gives:

$$4x + g'(y) = 4x + 5 \;\Rightarrow\; g'(y) = 5 \;\Rightarrow\; g(y) = 5y + C$$

hence $\varphi(x, y) = 4xy + 5y + C$. Because we need *a* potential function instead of the most general one, $\varphi(x, y) = 4xy + 5y$ should work. ◀

We could have integrated first with respect to y rather than x and ended up with the same result. Try this.

Practice 6. Find a potential function for $\mathbf{G}(x, y) = \langle y \cdot e^{xy} + 2, x \cdot e^{xy} + 3 \rangle$.

Practice 7. Find a potential function for $\mathbf{F}(x, y) = \left\langle 2xy + \cos(x), x^2 \right\rangle$.

Example 5. If possible, find a potential function for $\mathbf{F}(x, y) = \langle y, -x \rangle$.

This is the field from Examples 1 and 2.

Solution. If $\mathbf{F} = \nabla\varphi$ for some function φ then $\left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right\rangle = \mathbf{F}$ so:

$$\frac{\partial \varphi}{\partial x} = y \;\Rightarrow\; \varphi(x, y) = xy + g(y)$$

Differentiating our new candidate for φ with respect to y yields:

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left[xy + g(y)\right] = x + g'(y)$$

On the other hand we need $\varphi_y = -x$ so equating these gives:

$$x + g'(y) = -x \;\Rightarrow\; g'(y) = -2x$$

but $g'(y)$, being a function of y only, cannot depend on x, thus $\mathbf{F}$ has no potential function: we conclude that $\mathbf{F}$ is not a gradient field. ◀

Example 5 shows that not all vector fields are gradient fields.

Which Vector Fields Are Gradient Fields?

In general, if a vector field $\langle P(x,y), Q(x,y)\rangle$ is a gradient field then:

$$\langle P(x,y), Q(x,y)\rangle = \nabla\varphi(x,y) = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right\rangle$$

for some function $\varphi(x,y)$. In other words:

$$P = \frac{\partial\varphi}{\partial x} \quad \text{and} \quad Q = \frac{\partial\varphi}{\partial y}$$

If these functions are all differentiable then:

$$\frac{\partial P}{\partial y} = \frac{\partial^2\varphi}{\partial y\partial x} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial^2\varphi}{\partial x\partial y}$$

The Mixed-Partials Theorem tells us that if these two second-order derivatives are continuous on some domain then they are equal on that domain, hence:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \;\Rightarrow\; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

We often call this the **mixed-partials condition** for a vector field.

Theorem: If $P(x,y)$ and $Q(x,y)$ are C^1 functions on a domain $\mathcal{D}$ and $\langle P(x,y), Q(x,y)\rangle$ is a gradient field, then:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \;\Rightarrow\; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

Notice what this theorem does and does *not* say: If a field is a gradient field, then the mixed-partials condition must be true; if the mixed-partials condition holds then (at the moment) we can say nothing about whether or not the field is a gradient field. We often make use of the following corollary (the contrapositive of the theorem above):

Corollary: If $P(x,y)$ and $Q(x,y)$ are C^1 functions on $\mathcal{D}$ and:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0$$

then $\langle P, Q\rangle$ is not a gradient field.

Example 6. What does the mixed-partials condition tell you about the vector fields $\mathbf{F}(x,y) = \langle x, y\rangle$ and $\mathbf{G}(x,y) = \langle y, -x\rangle$?

Solution. For $\mathbf{F}$ we have:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[y] - \frac{\partial}{\partial y}[x] = 0 - 0 = 0$$

so $\mathbf{F}$ *might* be a gradient field. For $\mathbf{G}$:

In fact, $\mathbf{F}$ is indeed a gradient field; you can check that $\mathbf{F} = \nabla\varphi$ where:

$$\varphi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[-x] - \frac{\partial}{\partial y}[y] = -1 - 1 = -2$$

so $\mathbf{G}$ is definitely not a gradient field. ◀

Practice 8. What does the mixed-partials condition tell you about the vector fields $\mathbf{F}(x,y) = \left\langle 5x^3y^4, 3x^2y^5\right\rangle$ and $\mathbf{G}(x,y) = \left\langle 3x^2y^5, 5x^3y^4\right\rangle$?

Flow Lines

If a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ represents the velocity of a particle at the point (x,y), what path does this particle follow?

If the particle's position is given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ at time t, then its velocity at that time is $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$. But we want this velocity to agree with the field vector at that point, so that:

$$\mathbf{r}'(t) = \mathbf{F}(x(t), y(t)) \Rightarrow \left\langle x'(t), y'(t)\right\rangle = \langle P(x(t), y(t)), Q(x(t), y(t))\rangle$$

In other words, we need:

$$\frac{dx}{dt} = P(x(t), y(t)) \quad \text{and} \quad \frac{dy}{dt} = Q(x(t), y(t))$$

This is a system of first-order differential equations, which in practice can be challenging (or even impossible) to solve. In a few special cases, however, we may be able to solve the system and determine the possible paths of the particle, which we call the **flow lines** of the vector field.

In spite of this terminology, flow lines are most often curves, rather than lines.

Example 7. Find and graph the flow lines of $\mathbf{F}(x,y) = \langle 2x, 2y\rangle$.

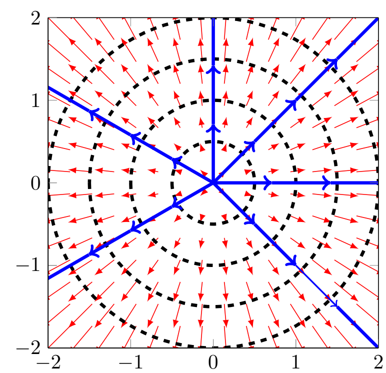
Solution. A parameter representation of the flow lines must satisfy:

$$\frac{dx}{dt} = 2x \quad \text{and} \quad \frac{dy}{dt} = 2y$$

so that $x = Ae^{2t}$ and $y = Be^{2t}$ for arbitrary constants A and B. We can rewrite this as:

$$\mathbf{r}(t) = \left\langle Ae^{2t}, Be^{2t}\right\rangle = e^{2t}\langle A, B\rangle$$

which shows that the flow lines are rays emanating from the origin, with the particles moving along those rays travelling at exponentially increasing speed. The graph in the margin shows $\mathbf{F}$ along with its flow lines (solid rays, with arrows indicating the direction of flow) and equipotential curves (dashed circles). ◀



Note in the graph accompanying Example 7 that the flow lines and the equipotential curves meet at right angles. (This is true for all gradient fields because the vectors of a gradient field are perpendicular to the level curves of the potential function.)

Example 8. Find and graph the flow lines of $\mathbf{G}(x,y) = \langle y, -x\rangle$.

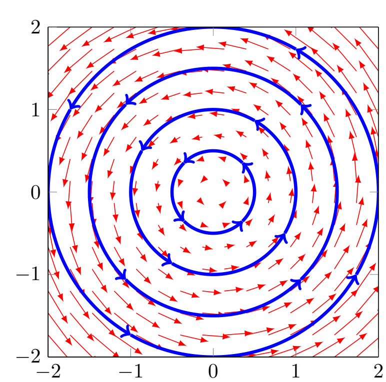
Solution. A parameter representation of the flow lines must satisfy:

$$\frac{dx}{dt} = y \quad \text{and} \quad \frac{dy}{dt} = -x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x}{y}$$

Solving this separable ODE yields:

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

or (with $K = 2C$) $x^2 + y^2 = K$, which are circles centered at the origin. The graph in the margin shows $\mathbf{G}(x,y) = \langle y, -x\rangle$ together with several of its flow lines. (We know from Example 6 that $\langle y, -x\rangle$ is not a gradient field, so it does not have equipotential curves.) ◀



Practice 9. Find the flow lines for $\mathbf{F}(x,y) = \langle x, 0\rangle$ and $\mathbf{G}(x,y) = \langle 0, x\rangle$.

16.1 Problems

In Problems 1–8, sketch a few vectors from the given vector field by hand, then use technology to create a more robust graphical representation.

1. $\langle 2, 1\rangle$
2. $\langle 1, x\rangle$
3. $\langle x, x\rangle$
4. $\langle y, y\rangle$
5. $\langle x^2, y\rangle$
6. $\langle 1, xy\rangle$
7. $\langle x, -y\rangle$
8. $\langle y, 1-x\rangle$

In Problems 9–14, find a formula for a vector field with vectors satisfying the given properties.

9. Have length exactly 2 and point in the positive x-direction.
10. Have length at least 2 and point in the negative y-direction.
11. Point toward the origin.
12. Have length 1 and point toward the origin.
13. Have length 1 and point away from the origin.
14. Are tangent to a circle centered at the origin.

In Problems 15–18, find the gradient field associated with the given function, then graph that vector field along with some level curves for the function.

15. $\varphi(x, y) = xy^2 - x^2y$
16. $\psi(x, y) = \dfrac{x}{y}$
17. $f(x, y) = x^3 + y^3$
18. $g(x, y) = x \cdot \sin(y) + e^{xy}$

In Problems 19–24, find a potential function for the given vector field, or show that the given vector field does not have a potential function.

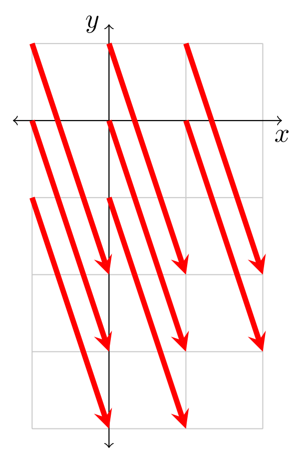
19. $\mathbf{F}(x, y) = \langle 3x^2 + 4, 6\rangle$
20. $\mathbf{G}(x, y) = \left\langle \dfrac{2}{2x+3y}, \dfrac{3}{2x+3y} + 6y^2 \right\rangle$
21. $\mathbf{H}(x, y) = \langle y \cdot \cos(xy) + x^2y, x \cdot \cos(xy) + x^3\rangle$
22. $\mathbf{F}(x, y) = \langle xy^3 + 3x^2y, 3xy^2 + x^2\rangle$
23. $\mathbf{G}(x, y) = \langle \sin(y), x \cdot \cos(y)\rangle$
24. $\mathbf{H}(x, y) = \langle 5x^4y^4 + 7y, 4x^5y^{-}7x\rangle$

In Problems 25–30, find parametric representations of the flow lines for the given vector field.

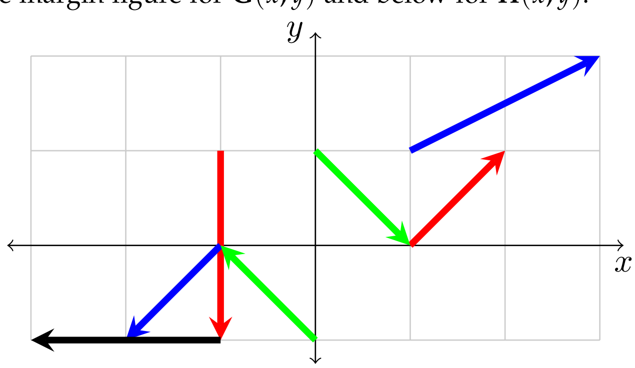
25. $\mathbf{F}(x, y) = \langle 3, -2\rangle$
26. $\mathbf{G}(x, y) = \langle x, -2\rangle$
27. $\mathbf{H}(x, y) = \langle y, 1\rangle$
28. $\mathbf{F}(x, y) = \langle y, 2y\rangle$
29. $\mathbf{G}(x, y) = \langle y, -x\rangle$
30. $\mathbf{H}(x, y) = \langle y, x\rangle$

16.1 Practice Answers

1. Substituting the given x- and y-values: $\mathbf{G}(0,0) = \langle 1, -3\rangle = \mathbf{G}(1,0) = \mathbf{G}(1,1) = \mathbf{G}(0,1) = \mathbf{G}(-1,1) = \mathbf{G}(-1,0) = \mathbf{G}(0,-1) = \mathbf{G}(-1,-1)$ while $\mathbf{H}(0,0) = \langle 0,0\rangle$, $\mathbf{H}(1,0) = \langle 1,1\rangle$, $\mathbf{H}(1,1) = \langle 2,0\rangle$, $\mathbf{H}(0,1) = \langle 1,-1\rangle$, $\mathbf{H}(-1,1) = \langle 0,-2\rangle$, $\mathbf{H}(-1,0) = \langle -1,-1\rangle$, $\mathbf{H}(1,-1) = \langle -1,1\rangle$ and $\mathbf{H}(-1,-1) = \langle -2,0\rangle$.

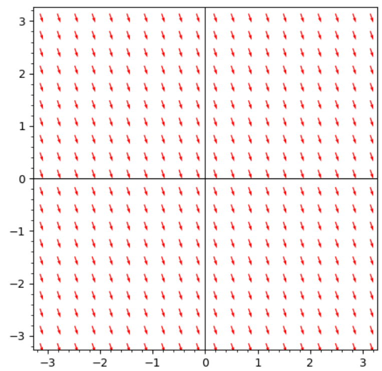


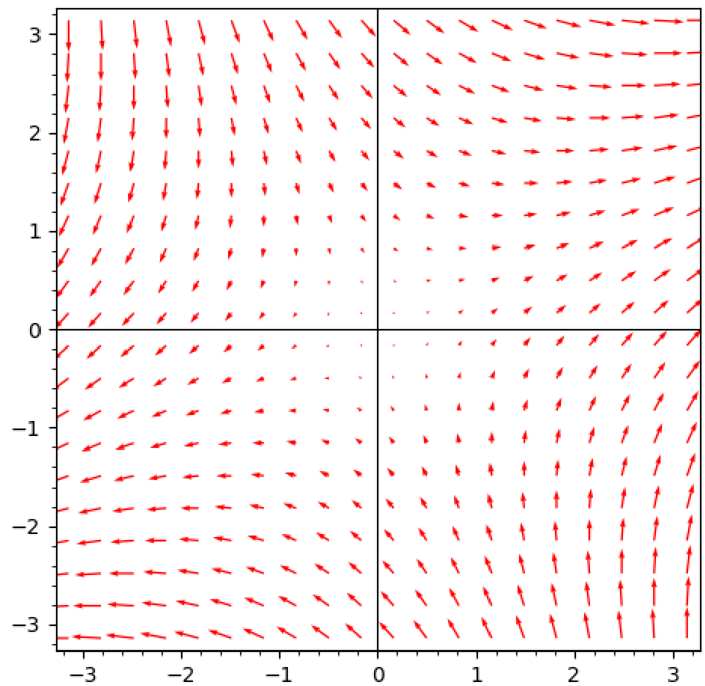
2. See the margin figure for $\mathbf{G}(x, y)$ and below for $\mathbf{H}(x, y)$:



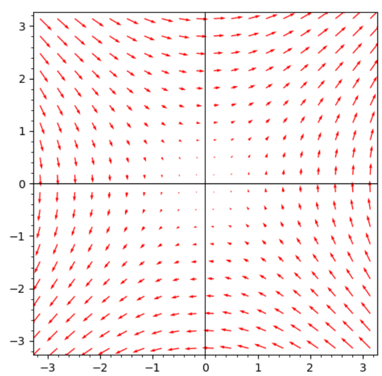
$\mathbf{H}(0,0) = \langle 0,0\rangle$ is not visible.

3. A SageMathCell graph of $\mathbf{G}(x, y)$ appears in the margin and a graph of $\mathbf{H}(x, y)$ appears below:

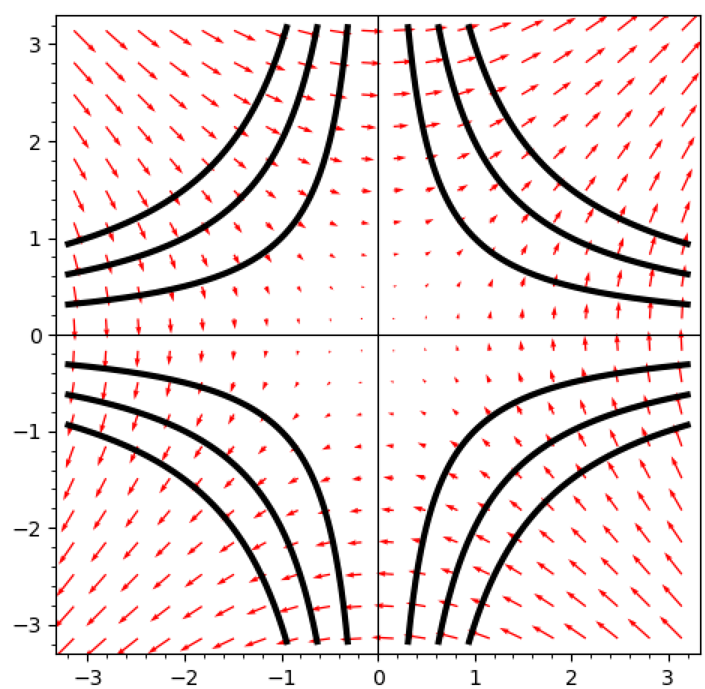




4. $\nabla\psi(x, y) = \langle y, x\rangle$. A graph appears in the margin.



5. A graph appears below:



6. If $\nabla\varphi = \mathbf{G}$ for some φ then:

$$\frac{\partial\varphi}{\partial x} = y \cdot e^{xy} + 2 \;\Rightarrow\; \varphi(x,y) = e^{xy} + 2x + g(y)$$

Differentiating this φ with respect to y yields:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\left[e^{xy} + 2x + g(y)\right] = x \cdot e^{xy} + g'(y)$$

On the other hand we need $\varphi_y = x \cdot e^{xy} + 3$ so equating these gives:

$$x \cdot e^{xy} + g'(y) = x \cdot e^{xy} + 3 \;\Rightarrow\; g'(y) = 3 \;\Rightarrow\; g(y) = 3y + C$$

hence $\varphi(x,y) = e^{xy} + 2x + 3y$ works.

7. If $\nabla\varphi = \mathbf{F}$ for some φ then:

$$\frac{\partial\varphi}{\partial x} = 2xy + \cos(x) \;\Rightarrow\; \varphi(x,y) = x^2y + \sin(x) + g(y)$$

Differentiating this φ with respect to y yields:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\left[x^2y + \sin(x) + g(y)\right] = x^2 + g'(y)$$

On the other hand we need $\varphi_y = x^2$ so equating these gives:

$$x^2 + g'(y) = x \cdot x^2 \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = C$$

hence $\varphi(x,y) = x^2y + \sin(x)$ works.

8. For $\mathbf{F}(x,y)$, with $P(x,y) = 5x^3y^4$ and $Q(x,y) = 3x^2y^5$:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left[3x^2y^5\right] - \frac{\partial}{\partial y}\left[5x^3y^4\right] = 6xy^5 - 20x^3y^3 \neq 0$$

so $\mathbf{F}$ is *definitely not* a gradient field. For $\mathbf{G}(x,y) = \left\langle 3x^2y^5, 5x^3y^4\right\rangle$:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left[5x^3y^4\right] - \frac{\partial}{\partial y}\left[3x^2y^5\right] = 15x^2y^4 - 15x^2y^4 = 0$$

so $\mathbf{G}$ *might* be a gradient field.

In fact, $\nabla\varphi = \mathbf{G}$ where $\varphi(x,y) = x^3y^5$.

9. A parameter representation of the flow lines for $\mathbf{F}(x,y) = \langle x, 0\rangle$ must satisfy:

$$\frac{dx}{dt} = x \quad\text{and}\quad \frac{dy}{dt} = 0$$

so that $x = Ae^t$ and $y = B$ for arbitrary constants A and B, hence the flow lines are horizontal half-lines, with the particles moving along those lines travelling to the right (or left, if $A < 0$) at exponentially increasing speed.

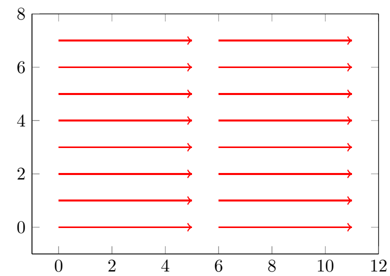
A parameter representation of the flow lines for $\mathbf{G}(x,y) = \langle 0, x\rangle$ must satisfy:

$$\frac{dx}{dt} = 0 \quad\text{and}\quad \frac{dy}{dt} = x$$

so that $x = A$, hence $y' = A \;\Rightarrow\; y = At + B$ for arbitrary constants A and B: the flow lines are vertical lines, with the particles moving along those lines travelling at linear speed.

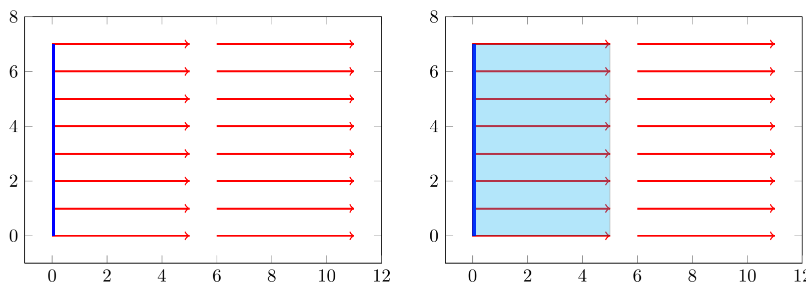
16.2 Flux

Throughout this section (and the ones that follow) imagine, if you will, a thin layer of water (or some other substance) with uniform (and very shallow) depth, flowing along a very flat surface (such as a newly paved road) so that the velocity of a very small leaf floating on the water will be given by a vector field $\mathbf{F}$ at any point (x, y).



One of the simplest such situations would involve a constant velocity field, say $\langle 5, 0 \rangle$, which might represent water flowing in the positive x-direction at a speed of 5 cm per second (as shown in the margin).

If you position a string of length 7 cm above the surface and perpendicular to the velocity field (as shown below left), how much water flows underneath the string during the course of one second?



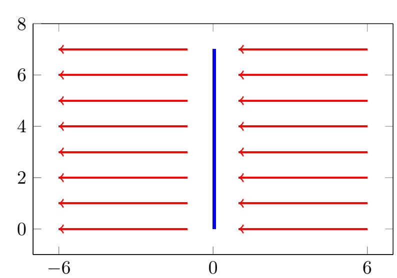
Assuming the water has constant depth (say, 1 cm), the volume of water that flows under the string in one second would be $7 \cdot 5 \cdot 1\ \text{cm}^3$. Because we are operating (for now) with 2D vector fields and assuming that the stream has a very shallow (and uniform) depth, we will define the **flux** of the vector field past this string as "area per unit of time." In this situation the flux is: $7 \cdot 5 = 35\ \text{cm}^2/\text{sec}$ (as shown above right).

The word "flux" is a fancy synonym for "flow," from the Latin word "fluxus," the past participle of the Latin word "fluere" (meaning "to flow"), from which we also get the word "fluent."

Example 1. Compute the flux of the vector field $\langle c, 0 \rangle$ under a string of length h positioned perpendicular to the vector field.

Solution. In one time unit, the amount of "stuff" flowing under the string equals $c \cdot h$, the area of a rectangle of base c and height h. ◀

Practice 1. Compute the flux of the vector field $\langle \pi, 0 \rangle$ under a string of length e parallel to the x-axis.

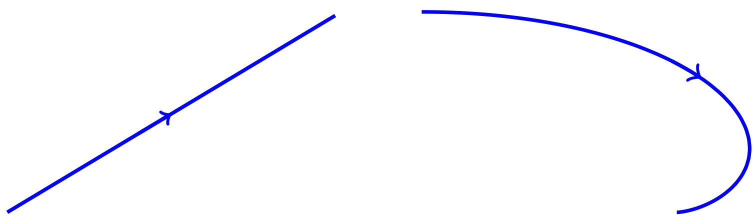


Now consider what would happen if, instead of the water (or "stuff") flowing from left to right, it flowed from right to left (as in the margin figure). Would the flux still be positive? Or would the flux be negative?

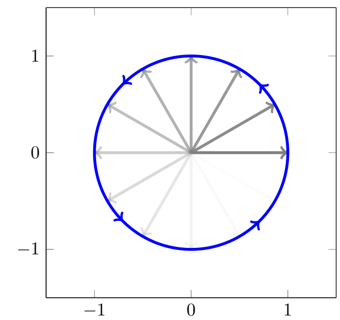
From now on, we will pick a "positive direction of flow" for each line under which our "stuff" flows and define the flux to be positive if the "stuff" is flowing in the direction defined to be positive, and negative if the "stuff" is flowing in the opposite direction.

Oriented Curves

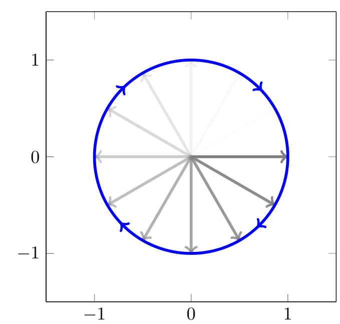
In our first examples, our "stuff" was flowing from left to right, which we assumed was the positive direction of flow, but for more general vector fields and line segments (and, eventually, curves) we need to be more precise. We call a curve **oriented** if we define a direction of travel from one endpoint to the other. We typically draw an arrow on the line (or curve) pointing in this direction, as shown below:



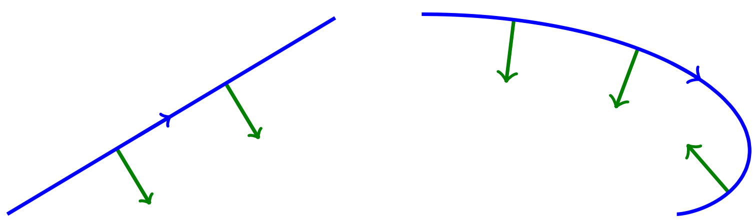
If a vector-valued function $\mathbf{r}(t)$ traces out a curve without doubling back on itself (so that $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ if $t_1 \neq t_2$) then this parameterization defines a natural orientation for the curve. For example, $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \leq t \leq 2\pi$ traces out the unit circle in the positive (counterclockwise) direction:



If you want to orient this curve in the opposite (negative, or clockwise) direction you can adjust the parameterization. For the unit circle, replacing t with $-t$ yields $\mathbf{r}(t) = \langle \cos(t), -\sin(t) \rangle$ for $0 \leq t \leq 2\pi$:



By convention, we define the positive direction of flow (or flux) to be to the **right** (90° counterclockwise from the direction of motion) as we travel along the curve, as shown here:



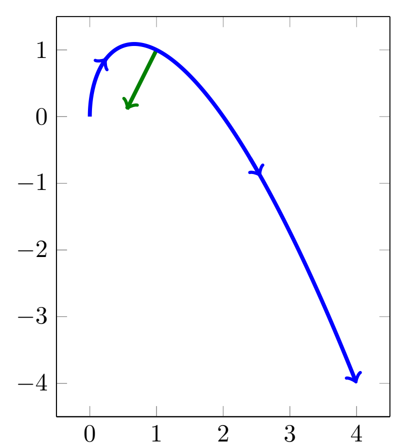
Example 2. Find the unit normal vector $\mathbf{n}$ for the line segment starting at $(0,0)$ and ending at $(3,4)$.

Solution. A direction vector for the line segment is $\langle 3,4 \rangle$. Both $\langle 4,-3 \rangle$ and $\langle -4,3 \rangle$ are normal (perpendicular) to the line's direction vector, but only $\langle 4,-3 \rangle$ points to the right when facing the direction of orientation for the line. Hence $\mathbf{n} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$ is the unit normal vector. ◀

Practice 2. Find the unit normal vector $\mathbf{n}$ for the line segment starting at $(11,7)$ and ending at $(2,7)$.

Example 3. Find the unit normal vector $\mathbf{n}$ for the curve $\mathcal{C}$ parameterized by $\mathbf{r}(t) = \left\langle t^2, 2t - t^3 \right\rangle$ for $0 \leq t \leq 2$, at the point where $t = 1$.

Solution. For any t, $\mathbf{r}'(t) = \left\langle 2t, 2 - 3t^2 \right\rangle$ is tangent to $\mathcal{C}$, so at $t = 1$, $\mathbf{r}'(1) = \langle 2,-1 \rangle$ is tangent to $\mathcal{C}$, hence $\langle -1,-2 \rangle$ will be normal to the curve, pointing rightward when facing the direction of motion. Thus $\mathbf{n} = \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$ is the desired unit normal vector (see margin). ◀



Practice 3. Find the unit normal vector $\mathbf{n}$ for the curve $\mathcal{C}$ from Example 3 at the point where $t = 1.5$.

In general, for any curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, the vector $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ will be tangent to the curve, and:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \frac{x'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}, \frac{y'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}} \right\rangle$$

will be the **unit tangent vector** at the point corresponding to time t, while the **unit normal vector** (pointing to the right when facing the direction of $\mathbf{T}(t)$) will be:

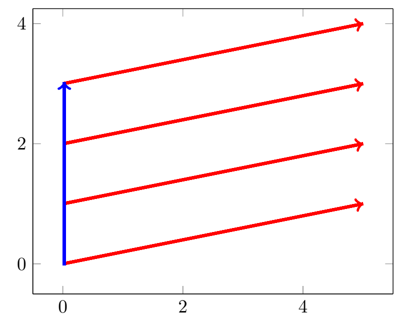
This $\mathbf{T}$ is the same unit tangent vector we encountered when studying **TNB** frames for curves, while $\mathbf{n}$ is either $\mathbf{N}$ or $-\mathbf{N}$: the $\mathbf{N}$ from the **TNB** frame points in the direction a curve is "turning," while $\mathbf{n}$ always points to the right.

$$\mathbf{n}(t) = \left\langle \frac{y'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}, \frac{-x'(t)}{\sqrt{[x'(t)]^2 + [y'(t)]^2}} \right\rangle$$

Practice 4. Compute the flux of the vector field $\langle 3, 0 \rangle$ across the oriented line segment starting at $(2, 5)$ and ending at $(2, 1)$.

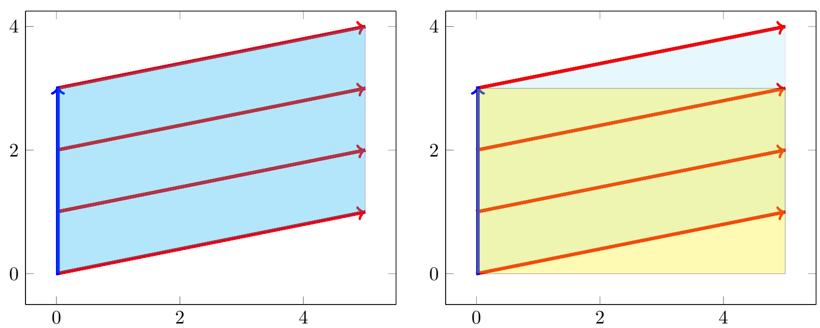
Flux Computations

So far in our flux computations, the velocity vectors have been perpendicular to the line. What if this is not the case?



Example 4. Compute the flux of the vector field $\langle 5, 1 \rangle$ under a string extending from $(0, 0)$ to $(0, 3)$ (see margin for graph).

Solution. In one unit of time, the amount of "stuff" that flows under the string is equal to the area of the parallelogram formed by the vectors $\langle 5, 1 \rangle$ and $\langle 0, 3 \rangle$, as shown below left:



This area is the same at the rectangle shown above right, which has base 5 and height 3, so the flux is $5 \cdot 3 = 15$. ◀

Can you see why the areas are equal?

Another way to view the computation in Example 4 is to think of the height of the rectangle as $\|\mathbf{L}\| = \|\langle 0, 3 \rangle\| = 3$ (where $\mathbf{L}$ is the oriented line segment) and the base of the rectangle as:

$$\mathbf{F} \cdot \mathbf{n} = \langle 5, 1 \rangle \cdot \langle 1, 0 \rangle$$

the projection of the field vectors $\mathbf{F}$ onto the unit normal vector for $\mathbf{L}$.

In general, given a constant vector field $\mathbf{F}$ and an oriented line segment $\mathbf{L}$, we can define the flux of $\mathbf{F}$ across $\mathbf{L}$ to be:

$$(\mathbf{F} \cdot \mathbf{n}) \|\mathbf{L}\|$$

where $\mathbf{n}$ is the unit normal vector for $\mathbf{L}$

Practice 5. Compute the flux of $\mathbf{F} = \langle -11, 17 \rangle$ across the oriented line segment extending from $(1, 1)$ to $(5, 9)$.

If we write $\mathbf{F} = \langle P, Q \rangle$ and $\mathbf{L} = \langle \Delta x, \Delta y \rangle$ then:

For now, P and Q are constants.

$$(\mathbf{F} \cdot \mathbf{n}) \, \|\mathbf{L}\| = \left(\langle P, Q \rangle \cdot \frac{\langle \Delta y, -\Delta x \rangle}{\sqrt{(\Delta y)^2 + (-\Delta x)^2}} \right) \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

which simplifies to:

$$(\mathbf{F} \cdot \mathbf{n}) \, \|\mathbf{L}\| = \langle P, Q \rangle \cdot \langle \Delta y, -\Delta x \rangle = P \, \Delta y \; - \; Q \, \Delta x$$

We can now use this simpler formula to compute flux.

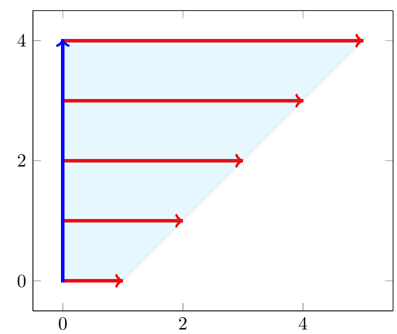
Example 5. Compute the flux of $\mathbf{F} = \langle -11, 17 \rangle$ across the oriented line segment extending from $(1, 1)$ to $(5, 9)$.

Solution. $\Delta x = 4$ and $\Delta y = 8$, so flux $= -11 \cdot 8 - 17 \cdot 4 = -156$. ◀

Does this agree with Practice 5?

Practice 6. Compute the flux of $\mathbf{F} = \langle -2, 3 \rangle$ across the oriented line segment extending from $(2, 1)$ to $(7, 1)$.

Example 6. Compute the flux of $\mathbf{F} = \langle 1 + y, 0 \rangle$ across the oriented line segment extending from $(0, 0)$ to $(0, 4)$.



Solution. We cannot use our shortcut formulas here, because $\mathbf{F}$ is not constant. A graph (see margin) indicates that the flux should equal the area of the shaded trapezoid, which is:

$$\frac{(1+5)}{2} \cdot 4 = 3 \cdot 4 = 12$$

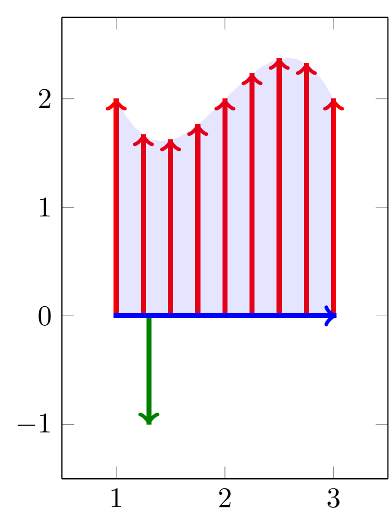
We could also compute this area (and hence the flux) using an integral:

$$\int_{y=0}^{y=4} (1 + y) \, dy = \left[y + \frac{1}{2} y^2 \right]_{y=0}^{y=4} = 4 + \frac{1}{2} \cdot 16 = 12$$

which agrees with the area of the trapezoid. ◀

Practice 7. Compute the flux of $\mathbf{F} = \langle 0, 2 + x \rangle$ across the oriented line segment extending from $(0, 0)$ to $(7, 0)$.

Example 7. Compute the flux of $\mathbf{F} = \langle 0, 8 - 11x + 6x^2 - x^3 \rangle$ across the oriented line segment extending from $(1, 0)$ to $(3, 0)$.



Solution. We cannot use geometry here but a graph (see margin) indicates the flux should correspond to the area of the shaded region, which is:

$$\int_{x=1}^{x=3} (8 - 11x + 6x^2 - x^3) \, dx = \left[8x - \frac{11}{2} x^2 + 2x^3 - \frac{1}{4} x^4 \right]_1^3 = 4$$

so the flux is -4 (with the negative sign due to the unit normal vector pointing in the opposite direction of the field vectors). ◀

Practice 8. Compute the flux of $\mathbf{F} = \langle 0, 2 + \cos(x) \rangle$ across the oriented line segment extending from $(0, 0)$ to $(2\pi, 0)$.

In Example 6, the vector field had the form $\langle P(y), 0\rangle$ and the oriented line segment pointed in the positive y-direction, while in Example 7 the vector field had the form $\langle 0, Q(x)\rangle$ and the oriented line segment pointed in the positive x-direction. The integrals we used to compute flux were, respectively, of the form:

$$\int P\,dy \qquad \text{and} \qquad -\int Q\,dx$$

This resembles the $P\,\Delta y - Q\,\Delta x$ formula for constant vector fields.

Might these integral formulas generalize to other situations? What happens if our oriented line segment becomes an oriented curve?

Flux Across Curves

Consider a general situation where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$ parameterizes an oriented curve $\mathcal{C}$ and $\mathbf{F} = \langle P(x,y), Q(x,y)\rangle$ is a vector field defined on some open set containing $\mathcal{C}$. How might we compute the flux of $\mathbf{F}$ across $\mathcal{C}$?

We know how to compute flux when the curve $\mathcal{C}$ is a line segment and $\mathbf{F}$ is constant (and have even computed flux for some special cases where $\mathbf{F}$ is not constant). To work out a method for the most general case, we follow a pattern you have seen over and over again in calculus: chop the curve $\mathcal{C}$ into many small pieces, each of which can be approximated by a line segment.

Choose a partition $\{t_0, t_1, t_2, \ldots, t_n\}$ of the interval $[a,b]$ so that $t_0 = a$ and $t_n = b$. We will assume here that $\mathbf{r}(t)$ traces out the curve $\mathcal{C}$ from one end (where $t = a$) to the other (where $t = b$) without reversing direction, and that $\mathcal{C}$ does not cross itself (in other words $\mathbf{r}(c) \neq \mathbf{r}(\gamma)$ unless $c = \gamma$), in which case we call $\mathcal{C}$ a **simple curve**, with the possible exception that $\mathbf{r}(a) = \mathbf{r}(b)$, in which case we call $\mathcal{C}$ a **closed curve**.

If n is suitably large, and each subinterval $[t_k, t_{k+1}]$ of our partition is suitably small, then the portion of $\mathcal{C}$ corresponding to $[t_k, t_{k+1}]$ will not be too different from the line segment joining its endpoints, which will correspond to the vector $\mathbf{r}(t_{k+1}) - \mathbf{r}(t_k) = \Delta\mathbf{r}_k \approx \mathbf{r}'(t_k)\cdot\Delta t_k$. The flux across the k-th piece of $\mathcal{C}$ is approximately:

Here we need $\mathbf{r}(t)$ to be differentiable. We call a curve that has a differentiable paramterization **smooth**. From now on in this chapter, we will tacitly assume that all curves are smooth, or at least **piecewise smooth**, in other words a finite union of smooth curves.

$$(\mathbf{F}(\mathbf{r}(t_k))\cdot\mathbf{n}(t_k))\,\|\mathbf{r}'(t_k)\cdot\Delta t_k\|$$

Expanding the first part in terms of the component functions P and Q:

$$\mathbf{F}\cdot\mathbf{n} = \langle P(x(t_k), y(t_k)), Q(x(t_k), y(t_k))\rangle \cdot \frac{\langle y'(t_k), -x'(t_k)\rangle}{\|\langle y'(t_k), -x'(t_k)\rangle\|}$$

Here we need $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in [a,b]$. We call such a parameterization **regular**.

while the second part becomes:

$$\|\mathbf{r}'(t_k)\cdot\Delta t_k\| = \|\langle x'(t_k), y'(t_k)\rangle\|\;\Delta t_k$$

but $\|\langle y'(t_k), -x'(t_k)\rangle\|$ and $\|\langle x'(t_k), y'(t_k)\rangle\|$ cancel, yielding:

$$\left[P(x(t_k), y(t_k))\cdot\frac{dy}{dt}(t_k) - Q(x(t_k), y(t_k))\cdot\frac{dx}{dt}(t_k)\right]\Delta t_k$$

This is because:

$$\|\langle y'(t_k), -x'(t_k)\rangle\| = \sqrt{(y'(t_k))^2 + (-x'(t_k))^2}$$

while:

$$\|\langle x'(t_k), y'(t_k)\rangle\| = \sqrt{(x'(t_k))^2 + (y'(t_k))^2}$$

so that:

$$\|\langle y'(t_k), -x'(t_k)\rangle\| = \|\langle x'(t_k), y'(t_k)\rangle\|$$

Adding up these approximate fluxes for all pieces of the partition yields a Riemann sum that converges to this definite integral:

$$\int_{t=a}^{t=b}\left[P(x,y)\cdot\frac{dy}{dt} - Q(x,y)\cdot\frac{dx}{dt}\right]dt$$

which we often write more succinctly (in **differential form**) as:

$$\int_{\mathcal{C}} [P\,dy - Q\,dx]$$

This should remind you of the formula:

$$P\,\Delta y - Q\Delta x$$

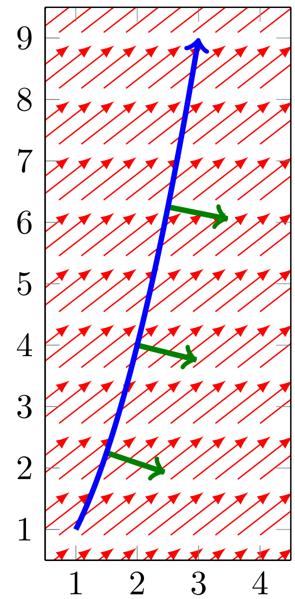
we use when P and Q are constant and $\mathcal{C}$ is a line segment.

Example 8. Compute the flux of $\mathbf{F}(x,y) = \langle 0.9, 0.7\rangle$ across $\mathcal{C}$, the portion of the parabola $y = x^2$ extending from $(1,1)$ to $(3,9)$.

Solution. Parameterizing $\mathcal{C}$ with $\mathbf{r}(t) = \langle t, t^2\rangle$ for $1 \le t \le 3$, we have $\mathbf{r}'(t) = \langle 1, 2t\rangle$ so the flux of $\mathbf{F}$ across $\mathcal{C}$ is:

$$\int_{t=1}^{t=3} [0.9 \cdot 2t - 0.7 \cdot 1]\,dt = \left[0.9t^2 - 0.7t\right]_1^3 = 5.8$$

Notice in the margin figure that the angles between the field vectors and the normal vectors to the curve appear to be acute, resulting in positive dot products, hence a positive total flux.



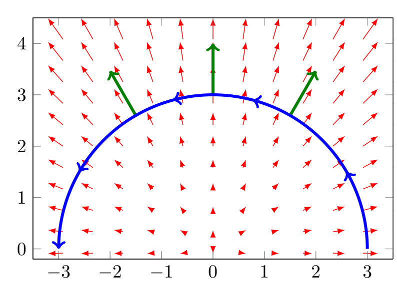
Another approach (which works with similar computations where the curve $\mathcal{C}$ is a graph of the form $y = f(x)$ or $x = g(y)$) is to write $y = x^2 \Rightarrow dy = 2x\,dx$ and compute:

$$\int_{\mathcal{C}} [P\,dy - Q\,dx] = \int_{x=1}^{x=3} [0.9 \cdot 2x\,dx - 0.7 \cdot dx] = \int_1^3 [1.8x - 0.7]\,dx$$

which also yields an answer of 5.8. ◀

Practice 9. Compute the flux of the vector field $\mathbf{G}(x,y) = \langle x^3, y^2\rangle$ across the curve $\mathcal{K}$ given by the graph of $y = x^2 + 1$ for $-1 \le x \le 1$ (oriented in the direction of increasing x).

Example 9. Compute the flux of the vector field $\mathbf{F}(x,y) = \langle x, y\rangle$ across $\mathcal{C}$, the upper half of the semicircle $x^2 + y^2 = 9$ oriented in the positive (counterclockwise) direction.



Solution. With $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le \pi$ parmaterizing $\mathcal{C}$, we have $\mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t)\rangle$ so the flux of $\mathbf{F}$ across $\mathcal{C}$ is:

$$\int_{t=0}^{t=\pi} [3\cos(t) \cdot 3\cos(t) - 3\sin(t) \cdot (-3\sin(t))]\,dt = \int_0^{\pi} 9\,dt = 9\pi$$

Notice in the margin figure that the field vectors point radially outward, as do the normal vectors for the semicircle. ◀

Practice 10. Compute the flux of the vector field $\mathbf{G}(x,y) = \langle y, -x\rangle$ across the upper half of the unit circle, oriented positively.

You might recall that for a curve parmameterized by $\mathbf{r}(t)$ the arclength from $t = a$ to an arbitrary point on the curve is:

$$s(t) = \int_a^t \|\mathbf{r}'(\tau)\|\,d\tau \;\Rightarrow\; \frac{ds}{dt} = \|\mathbf{r}'(t)\| \;\Rightarrow\; ds = \|\mathbf{r}'(t)\|\,dt$$

This last quantity turned up in our development of the integral formula(s) for flux. Replacing $\|\mathbf{r}'(t)\|\,dt$ with ds, the arclength element, we can write the flux formula in yet another way:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\,ds$$

which we interpret as "the integral of the normal component of **F** along $\mathcal{C}$ with respect to arclength."

In Example 9, at any point on the semicircle $\mathbf{n} = \left\langle \frac{x}{3}, \frac{y}{3} \right\rangle$ so that:

$$\mathbf{F} \cdot \mathbf{n} = \langle x, y \rangle \cdot \left\langle \frac{x}{3}, \frac{y}{3} \right\rangle = \frac{1}{3}\left(x^2 + y^2\right) = \frac{1}{3} \cdot 9 = 3$$

(using the fact that everywhere on the semicircle $x^2 + y^2 = 9$) so that:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{\mathcal{C}} 3\, ds = 3 \cdot 3\pi = 9\pi$$

because the arclength of the semicircle is 3π, yielding the same answer as in our original solution to Example 9.

Here we have made use of geometry to avoid parameterizing the curve (or the need to work out an antiderivtarive).

16.2 Problems

In Problems 1–14, compute the flux of the given vector field **F** across the specified oriented curve $\mathcal{C}$.

1. $\mathbf{F} = \langle 5, 12 \rangle$, $\mathcal{C}$ the line segment from $(3,2)$ to $(3,7)$
2. $\mathbf{F} = \langle 5, 12 \rangle$, $\mathcal{C}$ the line segment from $(3,2)$ to $(8,2)$
3. $\mathbf{F} = \langle 5, 12 \rangle$, $\mathcal{C}$ the line segment from $(8,2)$ to $(3,2)$
4. $\mathbf{F} = \langle 5, 12 \rangle$, $\mathcal{C}$ the line segment from $(3,7)$ to $(3,2)$
5. $\mathbf{F} = \langle -7, 10 \rangle$, $\mathcal{C}$ is the portion of the parabola $y = 3 + x + x^2$ from $(0,3)$ to $(2,9)$
6. $\mathbf{F} = \langle 8, 11 \rangle$, $\mathcal{C}$ is the portion of the parabola $x = 1 + 2y + y^2$ from $(0,-1)$ to $(4,1)$
7. $\mathbf{F} = \langle 2x, 3y \rangle$, $\mathcal{C}$ is the portion of the parabola $x = 2 + y + y^2$ from $(4,-2)$ to $(4,1)$
8. $\mathbf{F} = \langle 2y, 3x \rangle$, $\mathcal{C}$ is the portion of the parabola $y = 2 - x^2$ from $(-1,1)$ to $(1,1)$
9. $\mathbf{F} = \langle -y, x \rangle$, $\mathcal{C}$ is the curve parameterized by $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \frac{\pi}{2}$
10. $\mathbf{F} = \langle -y, x \rangle$, $\mathcal{C}$ is the curve parameterized by $\mathbf{r}(t) = \left\langle \cos\left(\frac{\pi}{2}t\right), \sin\left(\frac{\pi}{2}t\right) \right\rangle$ for $0 \le t \le \frac{\pi}{2}$
11. $\mathbf{F} = \left\langle 2xy^2, 2x^2y \right\rangle$, $\mathcal{C}$ is the curve parameterized by $\mathbf{r}(t) = \left\langle t^2, t^3 \right\rangle$ for $1 \le t \le 3$
12. $\mathbf{F} = \langle 3x, 5y \rangle$, $\mathcal{C}$ is the unit circle, with positive (counterclockwise) orientation
13. $\mathbf{F} = \nabla\varphi$ where $\varphi(x,y) = x^3y + 4xy^2$, $\mathcal{C}$ is the line segment from $(0,0)$ to $(2,4)$
14. $\mathbf{F} = \nabla\varphi$ where $\varphi(x,y) = x^3y + 4xy^2$, $\mathcal{C}$ is the part of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$

In Problems 15–22, evaluate each flux integral over the specified curve.

15. $\mathcal{C}$ is the line segment from $(-2,-2)$ to $(3,3)$:
$$\int_{\mathcal{C}} [5\,dy \;-\; 3\,dx]$$
16. $\mathcal{C}$ is the line segment from $(-2,-2)$ to $(3,3)$:
$$\int_{\mathcal{C}} [7\,dy \;-\; 5\,dx]$$
17. $\mathcal{C}$ is the line segment from $(-2,-2)$ to $(3,3)$:
$$\int_{\mathcal{C}} [7\,dy \;+\; 5\,dx]$$
18. $\mathcal{C}$ is the line segment from $(-1,-2)$ to $(3,6)$:
$$\int_{\mathcal{C}} [5\,dy \;+\; 9\,dx]$$
19. $\mathcal{C}$ is the line segment from $(0,0)$ to $(4,7)$:
$$\int_{\mathcal{C}} [x\,dy \;-\; y\,dx]$$
20. $\mathcal{C}$ is the line segment from $(0,0)$ to $(3,-6)$:
$$\int_{\mathcal{C}} [y\,dy \;-\; x\,dx]$$
21. $\mathcal{C}$ is the portion of the parabola the portion of the parabola $y = x^2 + 7$ from $(-1,8)$ to $(1,8)$:
$$\int_{\mathcal{C}} \left[x^2\,dy \;-\; y^2\,dx\right]$$
22. $\mathcal{C}$ is the portion of the parabola the portion of the curve $y = 5 - x^3$ from $(-2,13)$ to $(2,0)$:
$$\int_{\mathcal{C}} \left[xy^2\,dy \;-\; x^2y\,dx\right]$$

23. Compute the flux of the vector field $\langle 2,5\rangle$ along the line segment from $(0,0)$ to $(3,6)$ using:
    (a) $\mathbf{r}_1(t) = \langle 3t, 6t\rangle$ for $0 \le t \le 1$
    (b) $\mathbf{r}_2(t) = \langle 3t^2, 6t^2\rangle$ for $0 \le t \le 1$

    and verify that each results in the same answer.

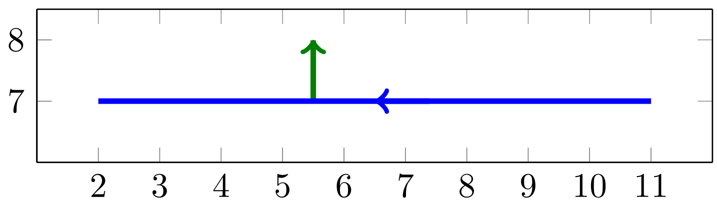
24. Compute the flux of the vector field $\langle 2,5\rangle$ along the line segment from:
    (a) $(0,0)$ to $(3,6)$
    (b) $(3,6)$ to $(0,0)$

    and interpret these results geometrically.

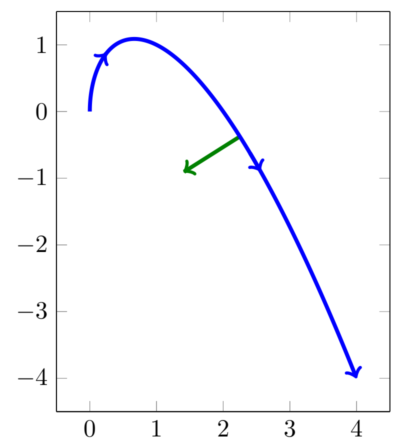
16.2 Practice Answers

1. Using the result of Example 1: $\pi \cdot e$

2. A direction vector for the line segment is $\langle -9,0\rangle$. Both $\langle 0,1\rangle$ and $\langle 0,-1\rangle$ are normal to the this direction vector, but only $\langle 0,1\rangle$ points to the right when facing the direction of orientation for the line segment, hence $\mathbf{n} = \langle 0,1\rangle$ is the unit normal vector:



3. $\mathbf{r}'(t) = \langle 2t, 2-3t^2\rangle$, so $\mathbf{r}'(1.5) = \langle 3, -4.75\rangle$ is tangent to $\mathcal{C}$, hence $\langle -4.75, 3\rangle$ will be normal to the curve, pointing rightward when facing the direction of motion. Because $|\langle -4.75,3\rangle| = \sqrt{31.5625}$, $\mathbf{n} = \left\langle -\frac{4.75}{\sqrt{31.5625}}, \frac{3}{\sqrt{31.5625}}\right\rangle \approx \langle -0.845, 0.534\rangle$ (see margin).



4. A direction vector for the line segment is $\langle 0,-4\rangle$, so $\mathbf{n} = \langle -1,0\rangle$ is the unit normal vector, which points in the opposite direction of the field vectors, hence the flux is $-4 \cdot 3 = -12$.

5. $\mathbf{L} = \langle 4,8\rangle$ so $\mathbf{n} = \frac{\langle 8,-4\rangle}{\sqrt{8^2+(-4)^2}} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$ and:

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n}\,(|\mathbf{L}|) &= \langle -11,17\rangle \cdot \frac{\langle 8,-4\rangle}{\sqrt{8^2+(-4)^2}}\left(\sqrt{4^2+8^2}\right)\\ &= \langle -11,17\rangle\cdot\langle 8,-4\rangle = -88-68 = -156\end{aligned}$$

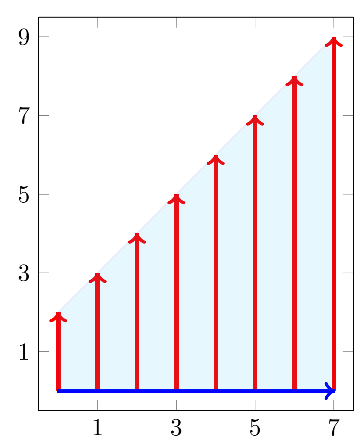
6. $P\Delta y - Q\Delta x = -2\cdot 0 + 3\cdot 5 = 15$

7. A graph (see margin) indicates that the flux should equal the area of the shaded trapezoid, which is:

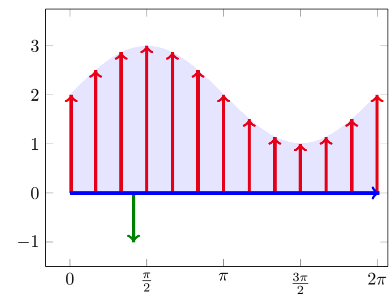
$$\frac{(2+9)}{2}\cdot 7 = 5.5\cdot 7 = 38.5$$

We could also compute this area (and flux) using an integral:

$$\int_{x=0}^{x=7} (2+x)\,dx = \left[2x + \frac{1}{2}x^2\right]_{x=0}^{x=7} = 14 + \frac{1}{2}\cdot 49 = 38.5$$



8. A graph (see margin) indicates that the flux should equal (negative) the area of the shaded region:



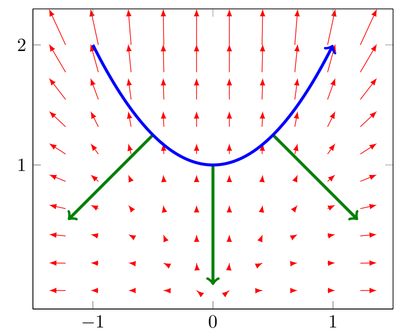
$$-\int_{x=0}^{x=2\pi} (2+\cos(x))\ dx = -\left[2x+\sin(x)\right]_{x=0}^{x=2\pi} = -4\pi$$

with the negative sign due to the field vectors pointing in the opposite direction of the unit normal vector for the oriented line segment. (You can also determine this flux without integrating—how?)

9. Parameterizing $\mathcal{K}$ with $\mathbf{r}(t) = \langle t, t^2+1\rangle$ for $-1 \le t \le 1$, we have $\mathbf{r}'(t) = \langle 1, 2t\rangle$ so the flux of $\mathbf{G}$ across $\mathcal{K}$ is:

$$\int_{t=-1}^{t=1} \left[t^3 \cdot 2t\ -\ \left(t^2+1\right)^2 \cdot 1\right]\ dt$$

which becomes:



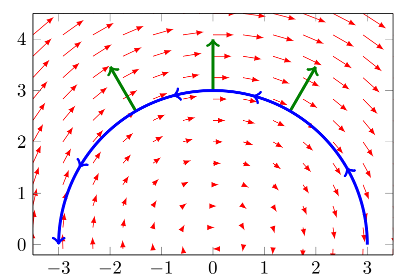
$$\int_{t=-1}^{t=1} \left[t^4 - 2t^2 - 1\right]\ dt = \left[\frac{1}{5}t^5 - \frac{2}{3}t^3 - t\right]_{t=-1}^{t=1} = -\frac{44}{15}$$

Notice (see margin) that the angles between the field vectors and the normal vectors generally appear to be obtuse, resulting in negative dot products, hence a negative total flux. You could also write:

$$\int_{\mathcal{K}} [P\,dy\ -\ Q\,dx] = \int_{x=-1}^{x=1} \left[x^3\,dy - y^2 \cdot dx\right]$$
$$= \int_{-1}^{3} \left[x^3 \cdot 2x\,dx - \left(x^2+1\right)^2\right]\ dx$$

which yields the same numerical result.

10. With $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le \pi$ parmaterizing the semicircle, as in Example 9, $\mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t)\rangle$ so the flux of $\mathbf{G}$ across the curve is:

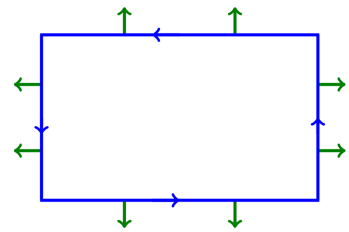


$$\int_{t=0}^{t=\pi} [3\sin(t) \cdot 3\cos(t)\ -\ (-3\cos(t))\,(-3\sin(t))]\ dt = \int_0^{\pi} 0\,dt = 0$$

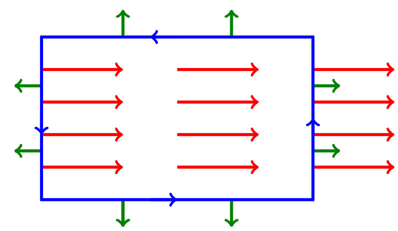
Notice (see margin) that the field vectors are tangent to the semicircle, while the normal vectors are (by definition) normal to semicircle, and hence to the field vectors, resulting in dot products of 0.

16.3 Divergence

In this section we continue to think of vector fields as velocity fields of a thin layer of water (or "stuff") flowing along a very flat surface, and compute the net flux of the "stuff" flowing through a closed curve, such as a rectangle or circle. By default we assume such curves have a positive (counterclockwise) orientation so that the corresponding unit normal vector at any point on the curve points *away from* (or out of) the region enclosed by the curve (as shown in the margin).



Example 1. Compute the total flux of the vector field $\mathbf{F} = \langle 3, 0 \rangle$ through the rectangle shown in the margin, with base length b and height h.



Solution. Consider the flux of $\mathbf{F}$ across each of the four sides of the rectangle separately. The flux of $\langle 3, 0 \rangle$ across the bottom of the rectangle is 0 because $\mathbf{F} \cdot \mathbf{n} = \langle 3, 0 \rangle \cdot \langle 0, -1 \rangle = 0$ there, and likewise the flux is 0 across the top because $\mathbf{F} \cdot \mathbf{n} = \langle 3, 0 \rangle \cdot \langle 0, 1 \rangle = 0$ on that part of the rectangle. The flux across the right side of the rectangle is:

$$(\langle 3, 0 \rangle \cdot \langle 1, 0 \rangle)(h) = 3h$$

while the flux across the left side is:

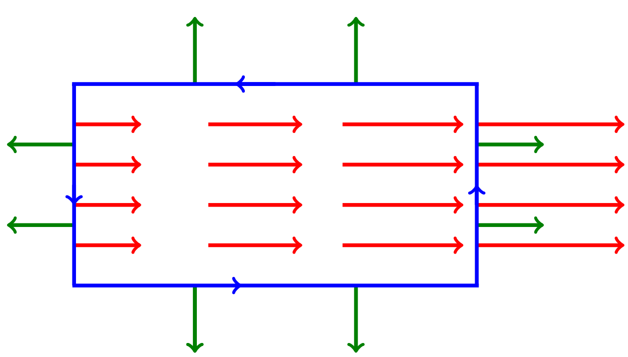
$$(\langle 3, 0 \rangle \cdot \langle -1, 0 \rangle)(h) = -3h$$

so the total flux across the rectangle is $0 + 3h + 0 - 3h = 0$. ◀

The amount of "stuff" flowing into the rectangle per unit of time equals the amount flowing out per unit of time.

Practice 1. Compute the total flux of the vector field $\mathbf{G} = \langle 0, 4 \rangle$ through the rectangle from Example 1.

Example 2. Compute the total outward flux of $\mathbf{F}(x, y) = \langle 1 + 0.2x, 0 \rangle$ through the rectangle $\mathcal{R}$ with opposing vertices at $(0, 0)$ and $(6, 3)$.



Less "stuff" is flowing into the rectangle on the left than is flowing out on the right (and nothing is flowing through the top of bottom), so you might well predict that the net (outward) flux is positive. How can this happen in our shallow stream example? Imagine a steady rainfall that adds water to the stream as it flows from left to right. (If the next flux were negative, we could imagine an array of small drains at the bottom of the stream.)

Solution. Consider the flux of $\mathbf{F}$ across each of the four sides of the rectangle separately. Along the bottom of the rectangle:

$$\mathbf{F} = \langle 1 + 0.2x, 0 \rangle \cdot \langle 0, -1 \rangle = 0$$

so the flux across the bottom is 0 and along the top:

$$\mathbf{F} = \langle 1 + 0.2x, 0 \rangle \cdot \langle 0, 1 \rangle = 0$$

so the flux is likewise 0 there. Along the left side, $x = 0$ so that:

$$\mathbf{F}(0, y) = \langle 1 + 0.2(0), 0 \rangle = \langle 1, 0 \rangle \;\Rightarrow\; \mathbf{F} \cdot \mathbf{n} = \langle 1, 0 \rangle \cdot \langle -1, 0 \rangle = -1$$

and hence the flux across the left side is $(-1)(3) = -3$. Along the right side, $x = 6$ so that:

$$\mathbf{F}(6, y) = \langle 1 + 0.2(6), 0 \rangle = \langle 2.2, 0 \rangle \;\Rightarrow\; \mathbf{F} \cdot \mathbf{n} = \langle 2.2, 0 \rangle \cdot \langle 1, 0 \rangle = 2.2$$

hence the flux across the right side is $(2.2)(3) = 6.6$ and the total flux across $\mathcal{R}$ is $0 + 6.6 + 0 - 3 = 3.6$. ◀

The net flux is positive, as predicted.

Practice 2. Compute the total flux of $\mathbf{G}(x, y) = \langle 0, 3 - 0.5y \rangle$ through the rectangle from Example 1.

If we rework Example 2 with a more generic vector field $\mathbf{F}(x, y) = \langle c + mx, 0 \rangle$ and a more generic rectangle with opposing vertices at $(0, 0)$ and (b, h), the net flux turns out to be:

$$(\langle c, 0 \rangle \cdot \langle -1, 0 \rangle)\, h + (\langle c + mb, 0 \rangle \cdot \langle 1, 0 \rangle)\, h = -ch + (c + mb)h = mbh$$

Likewise, the flux of $\mathbf{G}(x, y) = \langle 0, k + ny \rangle$ across the same rectangle is:

$$(\langle 0, k \rangle \cdot \langle 0, -1 \rangle)\, b + (\langle 0, k + nh \rangle \cdot \langle 0, 1 \rangle)\, b = -kb + (k + nh)b = nbh$$

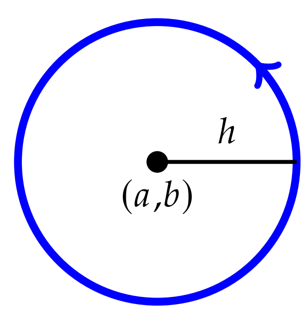
so the flux of $\langle c + mx, k + ny \rangle$ across the rectangle is:

$$mbh + nbh = (m + n)bh = (m + n)(\text{area of rectangle})$$

Can we generalize this further? Is there anything special about a rectangle or our choice of coordinate system?

Net Flux Across a Circle

Let $\mathcal{C}_h$ be the circle of radius h centered at the point (a, b), oriented positively. To find the net flux of the vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ across this (closed) circle we need to compute:



$$\int_{\mathcal{C}_h} \mathbf{F} \cdot \mathbf{n}\, ds$$

If we parameterize the circle using $\mathbf{r}(t) = \langle a + h\cos(t), b + h\sin(t) \rangle$, then an (outward) unit normal vector is $\mathbf{n}(t) = \langle \cos(t), \sin(t) \rangle$ and the arclength element is $ds = \|\mathbf{r}'(t)\|\, dt = h\, dt$ so the integral becomes:

$$\int_{\mathcal{C}_h} \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \mathbf{F}\left(a + h\cos(t), b + h\sin(t)\right) \cdot \langle h\cos(t), h\sin(t) \rangle\, dt$$

For small values of $h > 0$, the point (x, y) will be close to (a, b), so we can approximate the component functions of the vector field using:

$$P(x, y) \approx P(a, b) + P_x(a, b) \cdot (x - a) + P_y(a, b) \cdot (y - b)$$
$$Q(x, y) \approx Q(a, b) + Q_x(a, b) \cdot (x - a) + Q_y(a, b) \cdot (y - b)$$

These are the linearizations of P and Q.

Putting $x = a + h\cos(t)$ and $y = b + h\sin(t)$ this becomes:

$$\begin{aligned} P(a + h\cos(t), b + h\sin(t)) &\approx P(a,b) + P_x(a,b)\cdot h\cos(t) + P_y(a,b)\cdot h\sin(t) \\ Q(a + h\cos(t), b + h\sin(t)) &\approx Q(a,b) + Q_x(a,b)\cdot h\cos(t) + Q_y(a,b)\cdot h\sin(t) \end{aligned}$$

so that $\mathbf{F}\,(a + h\cos(t), b + h\sin(t)) \cdot \langle h\cos(t), h\sin(t)\rangle$ is (approximately):

$$\begin{aligned} &P(a,b)\cdot h\cos(t) + P_x(a,b)\cdot h^2\cos^2(t) + P_y(a,b)\cdot h^2\sin(t)\cos(t) \\ &+ Q(a,b)\cdot h\sin(t) + Q_x(a,b)\cdot h^2\cos(t)\sin(t) + Q_y(a,b)\cdot h^2\sin^2(t) \end{aligned}$$

Integrating this quantity from 0 to 2π yields:

$$\int_{C_h} \mathbf{F}\cdot\mathbf{n}\, ds \approx \pi h^2\left[P_x(a,b) + Q_y(a,b)\right]$$

Here we use these facts:

$$\int_0^{2\pi}\cos(t)\,dt = \int_0^{2\pi}\sin(t)\,dt = 0$$

$$\int_0^{2\pi}\sin(t)\cos(t)\,dt = 0$$

$$\int_0^{2\pi}\cos^2(t)\,dt = \int_0^{2\pi}\sin^2(t)\,dt = \pi$$

This tells us that the total flux across the circle is equal to the area enclosed by the circle times the expression $P_x(a,b) + Q_y(a,b)$, which we now define to be the **divergence** of $\mathbf{F}$ at the point (a,b), writing:

$$\mathrm{div}(\mathbf{F})\Big|_{(a,b)} = P_x(a,b) + Q_y(a,b) = \lim_{h\to 0^+}\frac{1}{\pi h^2}\cdot\int_{C_h}\mathbf{F}\cdot\mathbf{n}\,ds$$

For a circle of small radius, we can interpret the divergence as "flux per unit of area." Might this interpretation extend to other regions?

Computing Divergence

Finding the divergence of a vector field is one of the most straightforward computations you will encounter in vector calculus:

$$\mathrm{div}\left(\langle P(x,y), Q(x,y)\rangle\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

Simply compute the x-derivative of the x-component and the y-derivative of the y-component, then add them together.

Example 3. Compute the divergence of the vector fields $\mathbf{F}(x,y) = \langle 3, 4\rangle$, $\mathbf{G}(x,y) = \langle 3 - 2x, 4 + 7y\rangle$ and $\mathbf{B}(x,y) = \left\langle x^2y^5, x^3y\right\rangle$.

Solution. $\mathrm{div}\,(\mathbf{F}) = \partial_x(3) + \partial_y(4) = 0 + 0 = 0$, while:

$$\mathrm{div}\,(\mathbf{G}) = \frac{\partial}{\partial x}(3 - 2x) + \frac{\partial}{\partial x}(4 + 7y) = -2 + 7 = 5$$

and $\mathrm{div}\,(\mathbf{B}) = \partial_x(x^2y^5) + \partial_y(x^3y) = 2xy^5 + x^3$. ◀

Practice 3. Compute the divergence of $\mathbf{F}(x,y) = \left\langle \pi^2, e^3\right\rangle$, $\mathbf{G}(x,y) = \langle 13 + 2x - 6y, 14 + 8x - 7y\rangle$ and $\mathbf{B}(x,y) = \langle x\cdot\cos(xy), x\cdot\sin(xy)\rangle$.

Interpreting Divergence

For a circle C of small radius, we know that the divergence of a vector field $\mathbf{F}$ near C is (approximately) the net flux of $\mathbf{F}$ across C divided by the area enclosed by C.

In the discussion following Practice 2, we saw that the flux of $\langle c + mx, k + ny\rangle$, which has divergence $m + n$, across a rectangle is

$(m+n)$(area of rectangle), so that here too the divergence of the vector field equals the net flux of the vector field across the rectangle, divided by the area of the rectangle. Furthermore, we can show that any vector field of the form $\mathbf{F}(x,y) = \langle c+mx+\mu y, k+\nu x+ny\rangle$ (with linear component functions) and any curve $\mathcal{C}$ that is a circle or rectangle:

See Problem 25.

We are assuming here that these rectangles have sides parallel to the coordinate axes, although we will shall soon see that the assertion is true for any rectangle.

$$\text{flux of } \mathbf{F} \text{ across } \mathcal{C} = \text{div}(\mathbf{F}) \cdot (\text{area enclosed by } \mathcal{C})$$

If $\mathcal{C}$ is a circle or rectangle with sufficiently small dimensions centered at (a,b) and, more generally, $\mathbf{F} = \langle P,Q\rangle$ has continuously differentiable component functions near (a,b) (so that $\text{div}(\mathbf{F}) = P_x + Q_y$ is continuous on a region containing $\mathcal{C}$) then $\text{div}(\mathbf{F})(x,y)$ will be close to $\text{div}(\mathbf{F})(a,b)$ on this region and:

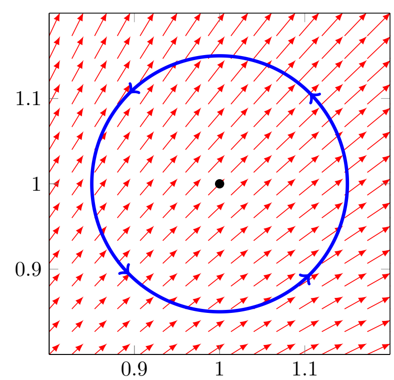
$$\text{flux of } \mathbf{F} \text{ across } \mathcal{C} \approx \text{div}(\mathbf{F})(a,b) \cdot (\text{area enclosed by } \mathcal{C})$$

Because any area is always positive, this tells us that the divergence of $\mathbf{F}$ at (a,b) and the flux of $\mathbf{F}$ across a circle of small radius centered at (a,b) must have the same sign.

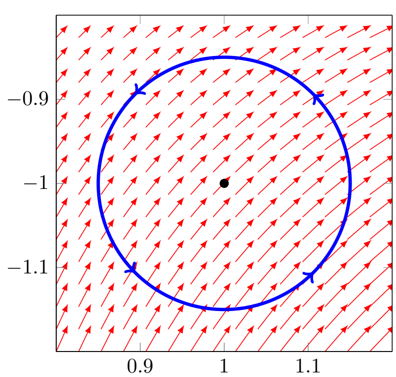
If the net flux is positive, then "stuff' is "diverging" from the small circle, hence the term "divergence."

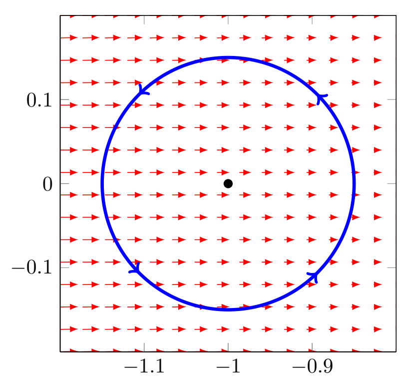
Example 4. Compute the divergence of $\mathbf{F}(x,y) = \langle x^2, y^2\rangle$ at $(1,1)$, $(1,-1)$ and $(-1,0)$, then interpret these values with the aid of a graph.

Solution. $\text{div}(\mathbf{F})(x,y) = \partial_x(x^2) + \partial_y(y^2) = 2x+2y \Rightarrow \text{div}(\mathbf{F})(1,1) = 4$, $\text{div}(\mathbf{F})(1,-1) = 0$ and $\text{div}(\mathbf{F})(-1,0) = -2$. A graph of $\langle x^2, y^2\rangle$ together a small circle centered at $(1,1)$ (see margin) reveals that near $(1,1)$ there appears to be more "stuff" flowing *out of* the circle than flowing in, which corresponds to the fact that $\text{div}(\mathbf{F})(1,1) > 0$.



Similarly, near $(1,-1)$ there seems to be about the same amount of "stuff" flowing in and out of the circle (see graph below left), which corresponds to the fact that $\text{div}(\mathbf{F})(1,-1) = 0$.





And near $(-1,0)$ (above right) there appears to be more "stuff" flowing in than out, which agrees with the fact that $\text{div}(\mathbf{F})(-1,0) < 0$. ◀

Practice 4. If $\mathbf{G}(x,y) = \langle x^2, 2y\rangle$, compute $\text{div}(\mathbf{G})$ at $(1,0)$, $(-1.5,0)$ and $(-1,0)$, then interpret these values with the aid of a graph.

16.3 Problems

In Problems 1–12, compute the divergence of the given vector field.

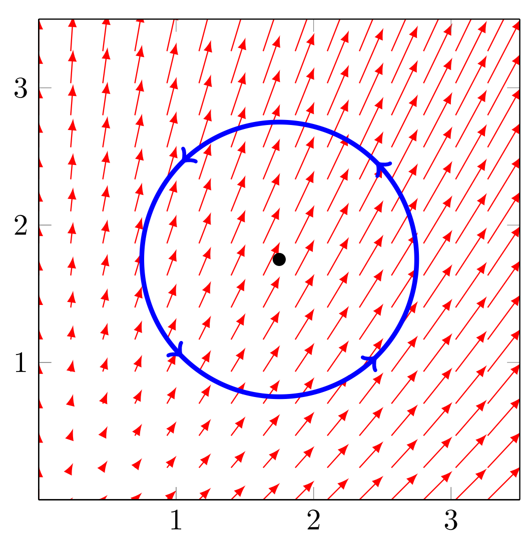
1. $\mathbf{F}(x,y) = \langle x, y\rangle$
2. $\mathbf{G}(x,y) = \langle y, -x\rangle$
3. $\mathbf{F}(x,y) = \langle -y, x\rangle$
4. $\mathbf{G}(x,y) = \langle 4, 9\rangle$
5. $\mathbf{F}(x,y) = \langle -1+3x, 7-4y\rangle$
6. $\mathbf{G}(x,y) = \left\langle -1+3x^2, 7-4y^2\right\rangle$
7. $\mathbf{F}(x,y) = \langle -1+3y, 7-4x\rangle$
8. $\mathbf{G}(x,y) = \left\langle -1+3y^2, 7-4x^2\right\rangle$
9. $\mathbf{F}(x,y) = \left\langle 2-x^3, \pi^4+y^5\right\rangle$
10. $\mathbf{G}(x,y) = \langle \sin(xy), \cos(xy)\rangle$
11. $\mathbf{F}(x,y) = \left\langle x^3y^2+\arctan(y), x^2y^3-\ln(x^2+10)\right\rangle$
12. $\mathbf{G}(x,y) = \left\langle (x+y)^5, (x-y)^5\right\rangle$

In Problems 13–18, compute the divergence of the given vector field and evaluate it at the given points.

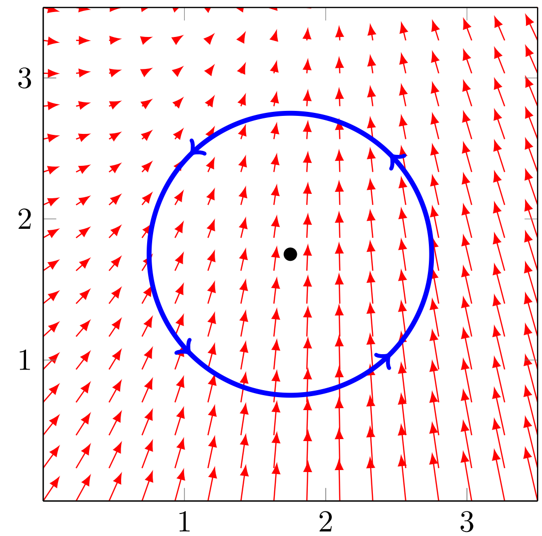
13. $\mathbf{F} = \left\langle x^2+3y, 2y+x\right\rangle$ at $(1,1)$, $(2,-1)$ and $(1,3)$
14. $\mathbf{G} = \left\langle xy^2, x^2y+3\right\rangle$ at $(3,2)$, $(0,3)$ and $(1,4)$
15. $\mathbf{F} = \langle 5x-3y, x+2y\rangle$ at $(3,2)$, $(0,3)$ and $(1,4)$
16. $\mathbf{G} = \left\langle x^2-y^2, x^2+y^2\right\rangle$ at $(2,3)$, $(-2,2)$ and $(3,1)$
17. $\mathbf{F} = \langle -3y, x\cdot y\rangle$ at $(3,2)$, $(0,3)$ and $(1,4)$
18. $\mathbf{G} = \left\langle e^3, \pi^2\right\rangle$ at $(2,3)$, $(-2,2)$ and $(3,1)$

In Problems 19–24, estimate whether the divergence of the vector field at the indicated point is positive, negative or approximately zero.

19.



20.



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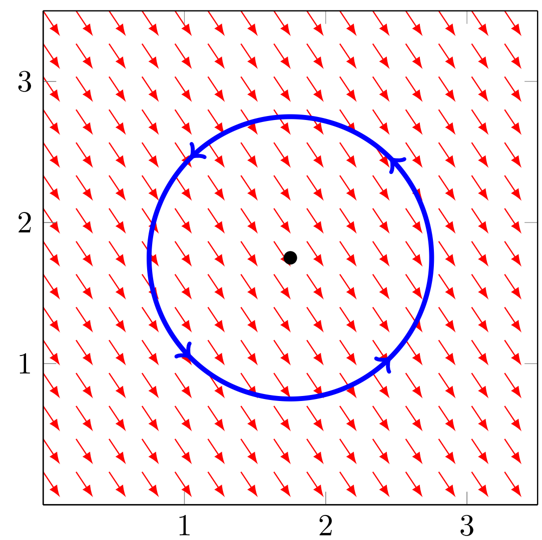

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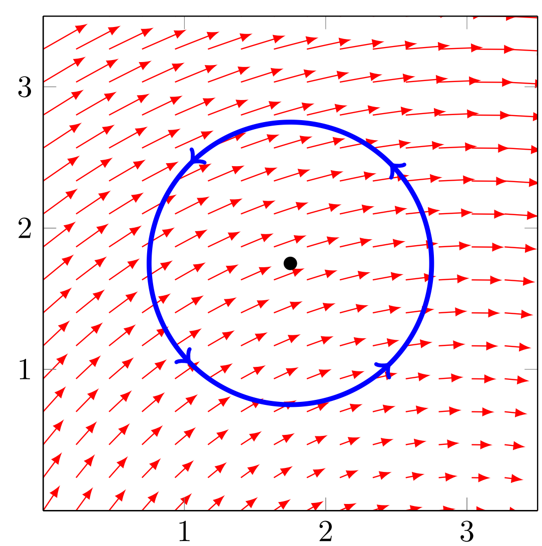
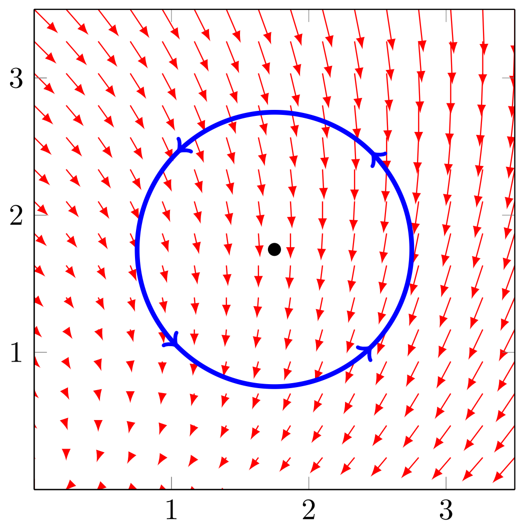
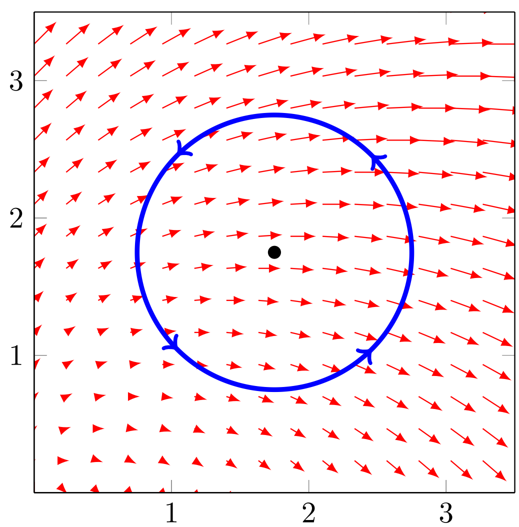


23. 

24. 

25. Show that, for constants a, b, c, α, β and γ, and the vector field:

$$\mathbf{F}(x,y) = \langle c + ax + by, \gamma + \alpha x + \beta y \rangle$$

the flux of $\mathbf{F}$ across any rectangle $\mathcal{R}$ (with sides parallel to the coordinate axes) equals:

$$\text{div}(\mathbf{F}) \cdot (\text{area enclosed by } \mathcal{R})$$

What must be true about the constants a, b, c, α, β and γ if the flux equals 0?

In 26–30, compute the divergence of the vector field, assuming f, g, φ and ψ are all differentiable.

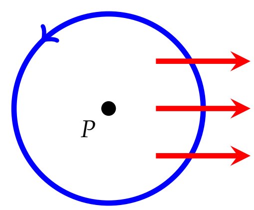
26. $\mathbf{F}(x,y) = \langle f(x), g(y) \rangle$
27. $\mathbf{G}(x,y) = \langle \varphi(y), \psi(x) \rangle$
28. $\mathbf{F}(x,y) = \langle f(x) \cdot \varphi(y), g(x) \cdot \psi(y) \rangle$
29. $\mathbf{G}(x,y) = \langle f(x) + \varphi(y), g(x) + \psi(y) \rangle$
30. $\mathbf{G}(x,y) = \langle f(x+y), g(x-y) \rangle$

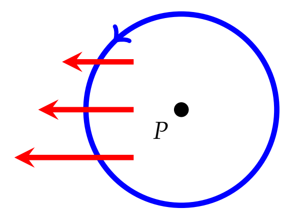
In Problems 31–36, a few vectors of a vector field $\mathbf{F}$ are shown near a point P. In each problem, draw additional vectors so that:

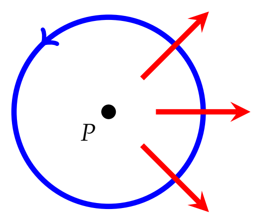
(a) $\text{div}(\mathbf{F})(P) > 0$

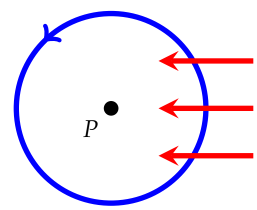
(b) $\text{div}(\mathbf{F})(P) < 0$

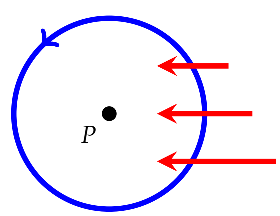
(c) $\text{div}(\mathbf{F})(P) \approx 0$

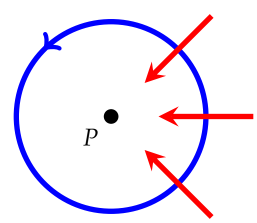
31. 

32. 

33. 

34. 

35. 

36. 

37. Given a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ and a circle $\mathcal{C}$ with center $A = (a,b)$ and small radius, consider what would happen if you imposed a different coordinate system (u,v). Would the value of the divergence of the vector field at A change? What about the value of the flux of the vector field across the circle?

38. Given a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ and a circle $\mathcal{C}$ with center $A = (a,b)$ and small radius, consider what would happen if you measured distances in mm instead of cm. How would the value of the divergence of the vector field at A change? What about the value of the flux of the vector field across the circle?

16.3 Practice Answers

1. On the right side of the rectangle $\mathbf{G} \cdot \mathbf{n} = \langle 0, 4 \rangle \cdot \langle 1, 0 \rangle = 0$ and on the left side $\mathbf{G} \cdot \mathbf{n} = \langle 0, 4 \rangle \cdot \langle -1, 0 \rangle = 0$, so the flux through these sides is each 0. On the top side $\mathbf{G} \cdot \mathbf{n} = \langle 0, 4 \rangle \cdot \langle 0, 1 \rangle = 4$ so the flux is $4b$ and on the bottom side $\mathbf{G} \cdot \mathbf{n} = \langle 0, 4 \rangle \cdot \langle 0, -1 \rangle = -4$ so the flux is $-4b$. The net flux is $0 + 4b + 0 - 4b = 0$.

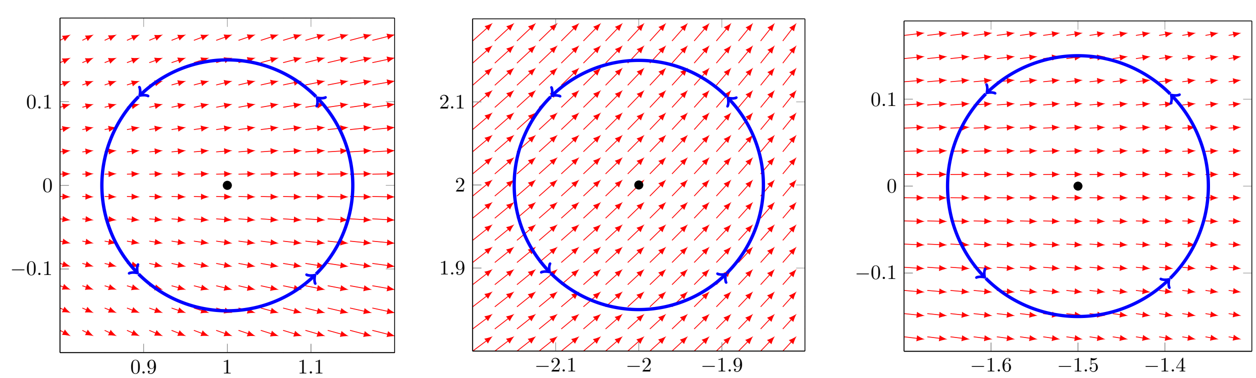
2. On the right side $\mathbf{G} \cdot \mathbf{n} = \langle 0, 3 - 0.5y \rangle \cdot \langle 1, 0 \rangle = 0$ and on the left $\mathbf{G} \cdot \mathbf{n} = \langle 0, 3 - 0.5y \rangle \cdot \langle -1, 0 \rangle = 0$, so the flux through each of these sides is 0. If the bottom of the rectangle is at $y = \beta$ then the top is at $y = \beta + h$ so on the top side $\mathbf{G} \cdot \mathbf{n} = \langle 0, 3 - 0.5(\beta + h) \rangle \cdot \langle 0, 1 \rangle = 3 - 0.5(\beta + h)$, hence the flux is $(3 - 0.5(\beta + h))\, b$. Finally, on the bottom $\mathbf{G} \cdot \mathbf{n} = \langle 0, 3 - 0.5\beta \rangle \cdot \langle 0, -1 \rangle = -(3 - 0.5\beta)$ so the flux is $-(3 - 0.5\beta)\, b$, yielding a total net flux of:

$$0 + (3 - 0.5(\beta + h))\, b + 0 - (3 - 0.5\beta)\, b = -0.5bh$$

3. $\operatorname{div}(\mathbf{F}) = 0 + 0 = 0$, $\operatorname{div}(\mathbf{G}) = 2 - 7 = -5$ and:

$$\begin{aligned} \operatorname{div}(\mathbf{B}) &= -xy\sin(xy) + \cos(xy) + x^2\cos(xy) \\ &= \left(1 + x^2\right)\cos(xy) - xy\sin(xy) \end{aligned}$$

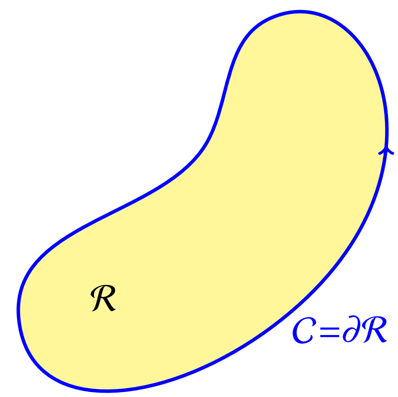
4. $\operatorname{div}(\mathbf{G}) = 2x + 2$ so $\operatorname{div}(\mathbf{G})(1, 0) = 4 > 0$, $\operatorname{div}(\mathbf{G})(-1.5, 0) = -1 < 0$ and $\operatorname{div}(\mathbf{G})(-1, 0) = 0$. These signs agree with the behavior of $\mathbf{G}(x, y)$ near each of these points:



16.4 2D Divergence Theorem

We have seen that, for a vector field with linear component functions (hence constant divergence), the net flux of that vector field across a circle or certain rectangles was equal to the divergence of the vector field times the area enclosed by the rectangle or circle. In other words, for vector fields $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ with constant divergence and certain closed (oriented) curves $\mathcal{C}$ with interior region $\mathcal{R}$:

Our goal is to extend this result to more general vector fields over more general 2D regions $\mathcal{R}$ with (closed, oriented) boundary curves $\mathcal{C}$, such as this one:



We will often use the notation $\partial\mathcal{R}$ for the boundary curve of the region $\mathcal{R}$. (There is a reason for this. Stay tuned.)

$$\text{flux of } \mathbf{F} \text{ across } \mathcal{C} = \text{div}(\mathbf{F}) \cdot (\text{area of } \mathcal{R})$$

or, writing the area as a double integral:

$$\int_{\mathcal{C}} [P\,dy \;-\; Q\,dx] = \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] \cdot \iint_{\mathcal{R}} 1\,dA$$

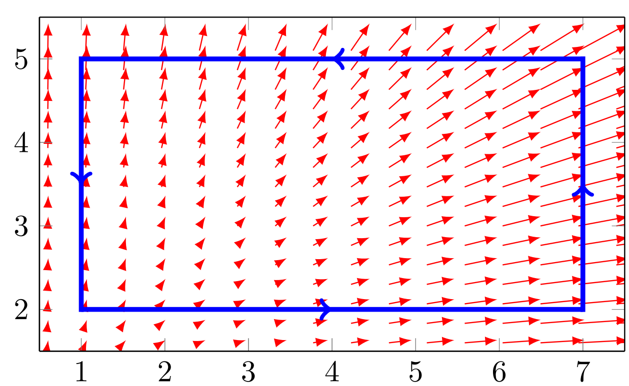
Moving the (constant) divergence inside the integral, this becomes:

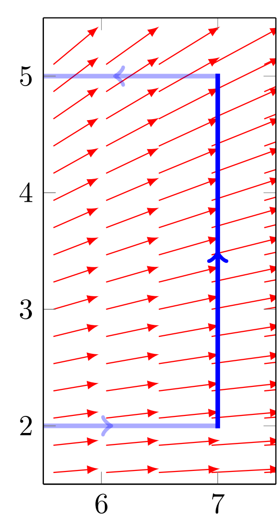
$$\int_{\mathcal{C}} [P\,dy \;-\; Q\,dx] = \iint_{\mathcal{R}} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA$$

Might this hold true for more general vector fields (even ones without constant divergence)? Might it hold for more general curves? Let's investigate by considering a specific example.

Example 1. Compute the flux of $\mathbf{F}(x,y) = \langle x^2, y^2\rangle$ across the rectangle with opposing vertices at $(1,2)$ and $(7,5)$ with positive (counterclockwise) orientation, then integrate $\text{div}(\mathbf{F})$ over the region in the xy-plane with this rectangle as its boundary.

Solution. To compute the flux of $\mathbf{F}$ across the rectangle, we will compute the flux across each of its four sides separately:





On the right side of the rectangle (see margin), where $x = 7$ (so that $dx = 0$), the flux is:

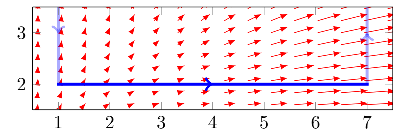
$$\int_{y=2}^{y=5} \left[7^2\,dy \;-\; y^2 \cdot 0\right] = 49 \cdot 3 = 147$$

On the left side of the rectangle, where $x = 1 \;\Rightarrow\; dx = 0$, the flux is:

$$\int_{y=5}^{y=2} \left[1^2\,dy \;-\; y^2 \cdot 0\right] = -3$$

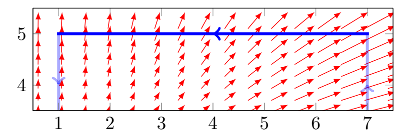
On the bottom of the rectangle, where $y = 2 \Rightarrow dy = 0$, the flux is:

$$\int_{x=1}^{x=7} \left[x^2 \cdot 0 \; - \; 2^2\, dx\right] = -24$$



And on the top of the rectangle, where $y = 5 \Rightarrow dy = 0$, the flux is:

$$\int_{x=7}^{x=1} \left[x^2 \cdot 0 \; - \; 5^2\, dx\right] = 25 \cdot 5 = 150$$



Summing this up, the total flux is $147 - 3 - 24 + 150 = 270$.

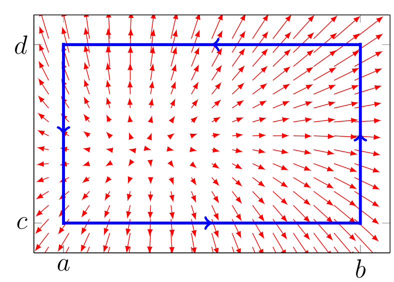
On the other hand, $\text{div}(\mathbf{F}) = 2x + 2y$, so integrating this over the rectangular region yields:

$$\int_{y=2}^{y=5} \int_{x=1}^{x=7} [2x + 2y]\; dx\, dy = \int_{y=2}^{y=5} \left[x^2 + 2xy\right]_{x=1}^{x=7} dy$$
$$= \int_{y=2}^{y=5} [48 + 12y]\; dy = \left[48y + 6y^2\right]_{y=2}^{y=5} = 270$$

It worked! This does not prove that the result holds for all vector fields (or all regions), but our hypothesis remains a possibility. ◀

Practice 1. Find the flux of $\mathbf{G}(x, y) = \langle xy, x^2 y\rangle$ across the (positively oriented) rectangle with opposing vertices at $(-2, -1)$ and $(4, 3)$, then integrate $\text{div}(\mathbf{G})$ over the region with this rectangle as its boundary.

Consider now a generic vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ and a more generic rectangular region $\mathcal{R}$ bounded by the lines $x = a$, $x = b$, $y = c$ and $y = d$ (see margin). We want to show that:



$$\int_{\partial\mathcal{R}} [P\, dy \; - \; Q\, dx] = \iint_{\mathcal{R}} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA$$

where $\partial\mathcal{R}$ is the (closed, positively oriented) boundary curve (a rectangle) for the rectangular region $\mathcal{R}$. Starting with the double integral on the right side of this equality and splitting it in two we get:

$$\int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} \frac{\partial P}{\partial x}\, dx\right] dy \; + \; \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} \frac{\partial Q}{\partial y}\, dy\right] dx$$

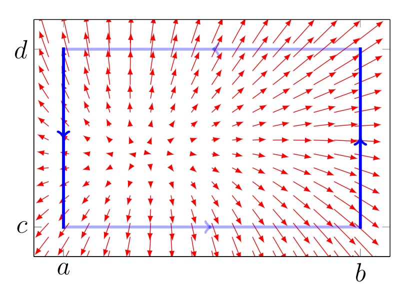
Notice that these integrals are iterated differently, so that we can more easily find a "partial antiderivative" in each case.

The first of these new integrals becomes:

$$\int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} \frac{\partial P}{\partial x}\, dx\right] dy = \int_{y=c}^{y=d} \Big[P(x, y)\Big]_{x=a}^{x=b} dy$$

and evaluating $P(x, y)$ at the x-endpoints yields:

$$\int_{y=c}^{y=d} [P(b, y) - P(a, y)]\; dy = \int_{y=c}^{y=d} P(b, y)\, dy + \int_{y=d}^{y=c} P(a, y)\, dy$$



The last two integrals here are integrals of $P\, dy$ up the right side of the rectangle and down the left side of the rectangle. But along the top of the rectangle (where $y = d$) and the bottom (where $y = c$), $dy = 0$, so the integrals of $P\,dy$ along the top and the bottom of the rectangle equal 0, hence we can "add 0" to the previous result to get:

$$\int_{y=c}^{y=d}\left[\int_{x=a}^{x=b}\frac{\partial P}{\partial x}\,dx\right]\,dy=\int_{\partial\mathcal{R}}P\,dy$$

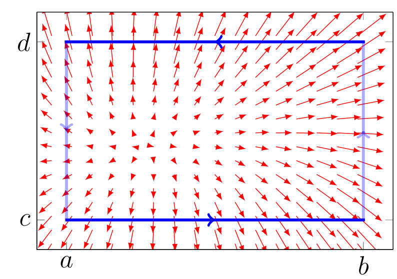
A similar process yields:

$$\int_{x=a}^{x=b}\left[\int_{y=c}^{y=d}\frac{\partial Q}{\partial y}\,dy\right]\,dx=\int_{x=a}^{x=b}\left[Q(x,d)-Q(x,c)\right]\,dx$$

This last integral then becomes:

$$-\int_{x=b}^{x=a}Q(x,d)\,dx-\int_{x=a}^{x=b}Q(x,c)\,dx$$

These two integrals integrate $Q\,dx$ along the top and bottom (respectively) of the rectangle (moving counterclockwise). On the left and right sides x is constant to $dx=0$, hence the integral of $Q\,dx$ is 0 there. "Adding zero" to the two negative integrals tells us that:



$$\int_{x=a}^{x=b}\left[\int_{y=c}^{y=d}\frac{\partial Q}{\partial y}\,dy\right]\,dx=-\int_{\partial\mathcal{R}}Q\,dx$$

Combining this with the result about the integral of $P\,dy$ yields:

$$\iint_{\mathcal{R}}\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right]\,dA=\int_{\partial\mathcal{R}}\left[P\,dy\;-\;Q\,dx\right]$$

which is exactly what we wanted to show. We state a somewhat more general version of this as our first big theorem of this chapter.

2D Divergence Theorem:

If: $\mathcal{R}$ is a "nice" closed, bounded region in the xy-plane with $\partial\mathcal{R}$ a simple, closed, positively oriented curve, and $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ with P and Q both C^1 functions on an open region containing $\mathcal{R}$,

then:

$$\int_{\partial\mathcal{R}}\left[P\,dy\;-\;Q\,dx\right]=\iint_{\mathcal{R}}\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right]\,dA$$

or, equivalently:

$$\int_{\partial\mathcal{R}}\mathbf{F}\cdot\mathbf{n}\,ds=\iint_{\mathcal{R}}\text{div}(\mathbf{F})\,dA$$

Remember that a C^1 function is differentiable and its derivatives are continuous. Also remember that we are tacitly assuming that our simple, closed, positively oriented boundary curve $\partial\mathcal{R}$ is piecewise smooth.

Some comments about this result are in order.

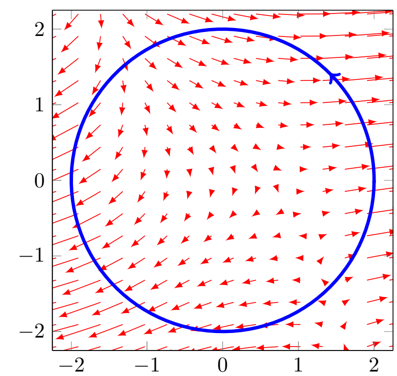
- We have not yet defined what a "nice" region is.
- The boundary curve $\mathcal{R}$ *must* be closed for this theorem to apply.
- So far we have only proved this theorem for regions with rectangles as boundary curves, so there is more work to do.

"Nice" will encompass pretty much any region you encounter in practice.

The theorem applies to the unit circle, $x^2+y^2=1$ (which is closed), but not the top half of that circle, $y=\sqrt{1-x^2}$.

- We will often use this theorem to find flux when the flux integral is difficult to compute but the corresponding double integral of the divergence of the vector field is easier to set up or easier to work out.
- In almost every other textbook you will encounter, this result is called the "flux-divergence form of Green's Theorem," but "2D Divergence Theorem" is a much better name for a variety of reasons.

Example 2. Compute the flux of $\mathbf{F}(x,y) = \langle x^3 + 2y, x-1 \rangle$ across the (positively oriented) circle $\mathcal{C}$ given by $x^2 + y^2 = 4$.



Solution. Applying the 2D Divergence Theorem, $\text{div}\,(\mathbf{F})\,(x,y) = 3x^2$ so the flux across the circle is:

$$\int_{\mathcal{C}} [P\,dy \;-\; Q\,dx] = \iint_{\mathcal{D}} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA = \iint_{\mathcal{D}} 3x^2\,dA$$

where $\mathcal{D}$ is the disk $x^2 + y^2 \leq 4$. Using polar coordinates, this becomes:

$$\iint_{\mathcal{D}} 3x^2\,dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} 3\,(r\cos(\theta))^2 \cdot r\,dr\,d\theta$$

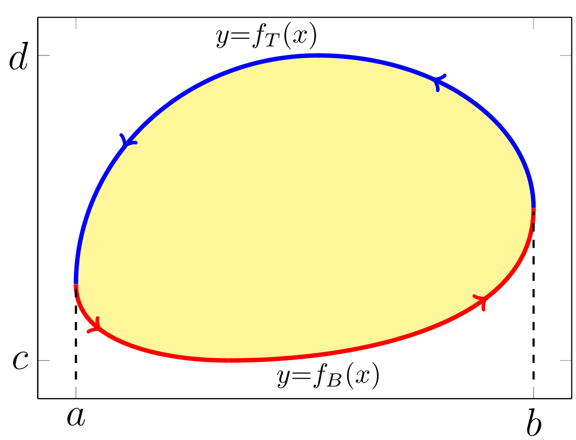
which evaluates to 12π. ◀

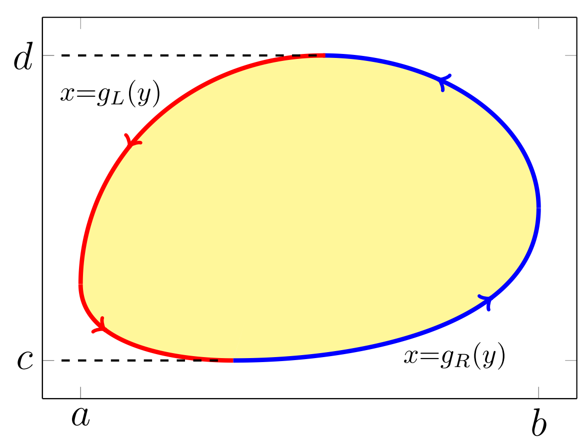
Practice 2. If $\mathbf{G}(x,y) = \langle x^2 + \arctan(y^3), y^4 - \ln(1+x^5) \rangle$ and $\mathcal{R}$ is the rectangle with opposing vertices at $(1,2)$ and $(3,7)$, compute the flux of $\mathbf{G}$ across $\mathcal{R}$.

Practice 3. Compute the flux of $\mathbf{B}(x,y) = \langle y^3 + 17, x^5 - 9 \rangle$ across the circle of radius $\sqrt{17}$ centered at (π^4, e^2).

Simple Regions

We call a region $\mathcal{R}$ **simple** if it is both "Type V" and "Type H" (using terminology from our study of double integrals). That is, $\mathcal{R}$ can described both as $f_B(x) \leq y \leq f_T(x)$ for $a \leq x \leq b$ (as shown below left) and as $g_L(y) \leq x \leq g_R(y)$ for $c \leq y \leq d$ (below right).





Rectangles and circles are special cases of simple regions (as is the region shown above).

Given a C^1 vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ and a simple region $\mathcal{R}$ we want to show that:

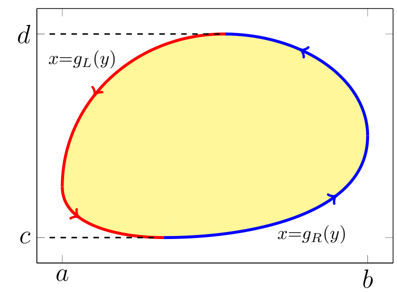
$$\int_{\partial\mathcal{R}} [P\,dy - Q\,dx] = \iint_{\mathcal{R}} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA$$

Following our method for rectangles, rewrite the double integral as:

$$\int_{y=c}^{y=d} \left[\int_{x=g_L(y)}^{x=g_R(y)} \frac{\partial P}{\partial x}\,dx\right] dy + \int_{x=a}^{x=b} \left[\int_{y=f_B(x)}^{y=f_T(x)} \frac{\partial Q}{\partial y}\,dy\right] dx$$

The first of these new double integrals becomes:

$$\int_{y=c}^{y=d} \Big[P(x,y)\Big]_{x=g_L(y)}^{x=g_R(y)} dy = \int_{y=c}^{y=d} [P(R(y),y) - P(L(y),y)]\,dy$$

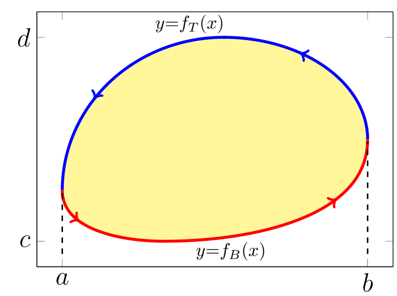


Splitting this last integral into two pieces, we can reverse the order of integration on the second piece (so that we are traversing the left side using the proper orientation, see margin) to get:

$$\int_{y=c}^{y=d} P(R(y),y)\,dy + \int_{y=d}^{y=c} P(L(y),y)\,dy = \int_{\partial\mathcal{R}} P\,dy$$

Proceeding similarly, the second of the double integrals becomes:

$$\int_{x=a}^{x=b} [Q(x,y)]_{y=f_B(x)}^{y=f_T(x)}\,dx = \int_{x=a}^{x=b} [Q(x,T(x)) - Q(x,B(x))]\,dx$$

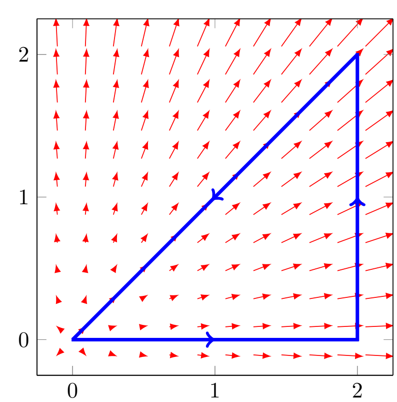


Now split this last integral into two pieces, and reverse the order of integration on the first piece (so that we are traversing the top side using the proper orientation, see margin) to get:

$$-\int_{x=b}^{x=a} Q(x,T(x))\,dx - \int_{x=a}^{x=b} Q(x,B(x))\,dx = -\int_{\partial\mathcal{R}} Q\,dx$$

Adding together these two results proves the 2D Divergence Theorem for simple regions.

Example 3. Compute the flux of $\langle x,y\rangle$ across the (positively oriented) triangle with vertices at $(0,0)$, $(2,0)$ and $(2,2)$.

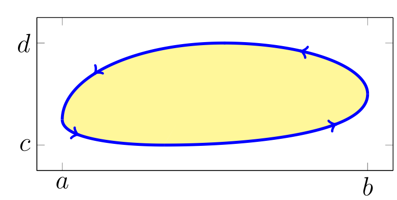


Solution. We could compute this flux directly by parameterizing the three sides of the triangle (see margin) and computing three flux integrals (see Practice 4), or we could apply the 2D Divergence Theorem. If $\mathcal{T}$ is the region inside the triangle, then the flux across the triangle is:

$$\int_{\partial\mathcal{T}} [P\,dy - Q\,dx] = \iint_{\mathcal{T}} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA = \iint_{\mathcal{T}} 2\,dA = 4$$

Because the divergence is constant, instead of integrating you can just multiply the divergence (2) by the area of the triangle (2) to get 4. ◀

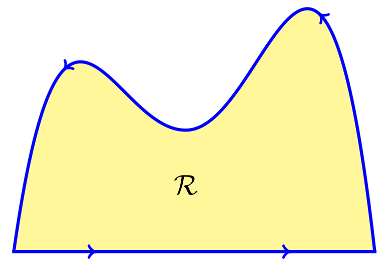
Practice 4. Verify the 2D Divergence Theorem in Example 3 by computing three flux integrals.

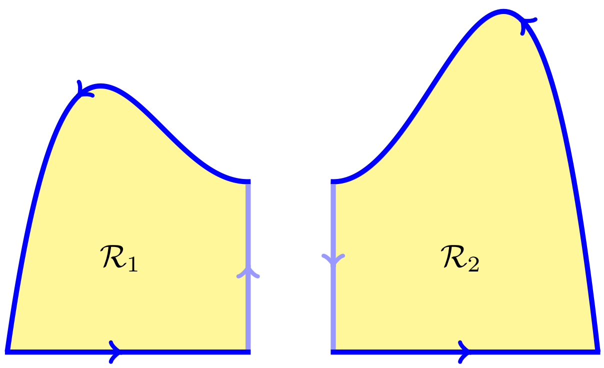


Practice 5. Evaluate $\displaystyle\int_{\partial\mathcal{R}} \left[x^2y\,dy + xy^2\,dx\right]$ (with $\mathcal{R}$ as in the margin).

More General Regions

What about still more general regions that are not simple? The region $\mathcal{R}$ in the margin is not a simple region, but we can cut it into two pieces, $\mathcal{R}_1$ and $\mathcal{R}_2$, each of which is a simple region, as shown below:





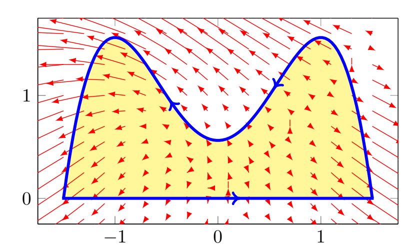
If we compute the flux across the boundaries of $\mathcal{R}_1$ and $\mathcal{R}_2$ then add the results, the fluxes across the common boundary will cancel ("stuff" exiting $\mathcal{R}_1$ on the right, resulting in positive flux, will immediately enter $\mathcal{R}_2$ on the left, resulting in negative flux of the same magnitude — and vice versa). So the sum of the fluxes across the boundaries of the smaller regions will equal the flux across the boundary of the original larger region.

Applying this idea repeatedly, we can extend the 2D Divergence Theorem to any region in the xy-plane that is a finite union of simple regions, and we can now retroactively define "nice" in the statement of the 2D Divergence Theorem accordingly.

Example 4. Compute the flux of $\mathbf{F}(x,y) = \left\langle x^3 - y^3, y^2 - x^2 \right\rangle$ across the (positively oriented) curve C consisting of the portion of the graph of $y = 1.5625 - \left(x^2 - 1\right)^2$ above the x-axis, along with the portion of the x-axis connecting its endpoints.

Solution. The region $\mathcal{R}$ enclosed by $\mathcal{C}$ is not a simple region (see margin) but it is the union of two simple regions (split $\mathcal{R}$ along the y-axis), so we can apply the 2D Divergence Theorem

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\, dA = \iint_{\mathcal{R}} \left[3x^2 + 2y\right] dA$$



Iterating the double integral, this becomes:

$$\int_{x=-1.5}^{x=1.5} \int_{y=0}^{y=1.5625-\left(x^2-1\right)^2} \left[3x^2 + 2y\right] dy\, dx = \frac{891}{80} = 11.1375$$

You should work out the details of evaluating this double integral.

where the x-limits come from solving $0 = 1.5625 - \left(x^2 - 1\right)^2$. ◀

Practice 6. Compute the flux of $\mathbf{F}(x,y) = \left\langle x^3 - y^3, y^2 - x^2 \right\rangle$ across the (positively oriented) curve $\mathcal{K}$ consisting of the portion of the graph of $y = 1.5625 - \left(x^2 - 1\right)^2$ above the x-axis.

16.4 Problems

In Problems 1–14 evaluate the given flux integral over the specified curve (assuming positive orientation). (Suggestion: Use the 2D Divergence Theorem.)

1. $\mathcal{C}$ is the unit circle:
$$\int_{\mathcal{C}} \left[y^2\,dy \;-\; x^2\,dx\right]$$
2. $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$:
$$\int_{\mathcal{C}} \left[y^3\,dy \;-\; x^3\,dx\right]$$
3. $\mathcal{C}$ is the unit circle:
$$\int_{\mathcal{C}} [5x\,dy \;-\; 8y\,dx]$$
4. $\mathcal{C}$ is the rectangle with opposing vertices at $(0,0)$ and $(10,12)$:
$$\int_{\mathcal{C}} [(5x - 17y)\;dy \;-\; (11x + 8y)\;dx]$$
5. $\mathcal{C}$ is the rectangle with vertices at $(-3,3)$, $(10,3)$, $(10,12)$ and $(-3,12)$:
$$\int_{\mathcal{C}} [(17x - 5y)\;dy \;-\; (8x + 11y)\;dx]$$
6. $\mathcal{C}$ is the rectangle with opposing vertices at $(0,0)$ and (L,H):
$$\int_{\mathcal{C}} [(\alpha x + \beta y)\;dy \;-\; (\gamma x + \delta y)\;dx]$$
7. $\mathcal{C}$ is the circle $x^2 + y^2 = 16$:
$$\int_{\mathcal{C}} \left[x^3\,dy \;-\; y^3\,dx\right]$$
8. $\mathcal{C}$ is the circle $x^2 + y^2 = 16$:
$$\int_{\mathcal{C}} \left[x^4\,dy \;-\; y^4\,dx\right]$$
9. $\mathcal{R}$ is the region in the first quadrant below the line $x + y = 1$:
$$\int_{\partial\mathcal{R}} \left[2x^3\,dy \;-\; 4y^2\,dx\right]$$
10. $\mathcal{R}$ is the region in the first quadrant below the line $x + y = 10$:
$$\int_{\partial\mathcal{R}} \left[3x^2\,dy \;-\; 5y^3\,dx\right]$$
11. $\mathcal{R}$ is the region below the parabola $y = 9 - x^2$ and above the x-axis:
$$\int_{\partial\mathcal{R}} \left[x^2y\,dy \;-\; xy^2\,dx\right]$$
12. $\mathcal{R}$ is the region below the parabola $y = 9 - x^2$ and above the x-axis:
$$\int_{\partial\mathcal{R}} \left[3xy^2\,dy \;-\; \left(5y - y^3\right)\,dx\right]$$
13. $\mathcal{R}$ is the region below the parabola $y = 9 - x^2$ and above the x-axis:
$$\int_{\partial\mathcal{R}} \left[x^2y\,dy \;-\; y\,dx\right]$$
14. $\mathcal{R}$ is the region below the parabola $y = 25 - x^2$ and above the x-axis:
$$\int_{\partial\mathcal{R}} \left[xy^2\,dy \;-\; 5y\,dx\right]$$
15. Compute the flux of the vector field $\langle x, y\rangle$ across the unit circle (oriented positively).
16. Compute the flux of $\left\langle 2x + y^3, x^3 + 5y\right\rangle$ across the circle $x^2 + y^2 = 7$ (oriented positively).
17. Compute the flux of $\left\langle x^3 - 7y, 8x + y^3\right\rangle$ across the unit circle (oriented positively).
18. Compute the flux of $\left\langle 17x^3 - 8\sin(y), 13e^x + 7y^3\right\rangle$ across the circle $x^2 + y^2 = 5$ (oriented positively).
19. Compute the flux of $\langle 2 - 5x + 7y, 8 - 11x + 14y\rangle$ across $\partial\mathcal{R}$ if $\mathcal{R}$ is a simple region with area $100\pi^2$.
20. If $\mathbf{F} = \left\langle 12 + 11x + 7y \cdot \sin(y), 14 - 11e^{2x} + 19y\right\rangle$ and $\mathcal{R}$ is a simple region with area $4\sqrt{2}$, compute the flux of $\mathbf{F}$ across $\partial\mathcal{R}$.

In Problems 21–24, use the given information to compute $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\,ds$ (assume $\mathcal{C}$ is positively oriented).

21. $\operatorname{div}(\mathbf{F}) = 64$, $\mathcal{C}$ is the circle $x^2 + y^2 = 10$
22. $\operatorname{div}(\mathbf{F}) = 6$, $\mathcal{C}$ is the circle $x^2 + y^2 = 100$
23. $\operatorname{div}(\mathbf{F}) = y$, $\mathcal{C}$ is $x^2 + y^2 = 4$
24. $\operatorname{div}(\mathbf{F}) = x$, $\mathcal{C}$ is $x^2 + y^2 = 9$
25. $\operatorname{div}(\mathbf{F}) = x$, $\mathcal{C}$ is $(x - 2)^2 + y^2 = 16$
26. $\operatorname{div}(\mathbf{F}) = y$, $\mathcal{C}$ is $x^2 + (y + 7)^2 = 25$

16.4 Practice Answers

1. Along the bottom side (BS) of the rectangle, $y = -1 \Rightarrow dy = 0$ so:

$$\int_{BS} \left[xy\,dy \;-\; x^2y\,dx \right] = \int_{x=-2}^{x=4} \left[-x \cdot 0 + x^2 \right] \, dx = 24$$

Along the top side (TS) of the rectangle, $y = 3 \Rightarrow dy = 0$ so:

$$\int_{TS} \left[xy\,dy \;-\; x^2y\,dx \right] = \int_{x=4}^{x=-2} \left[3x \cdot 0 - 3x^2 \right] \, dx = 72$$

Along the right side (RS) of the rectangle, $x = 4 \Rightarrow dx = 0$ so:

$$\int_{RS} \left[xy\,dy \;-\; x^2y\,dx \right] = \int_{y=-1}^{y=3} \left[4y - 16y \cdot 0 \right] \, dy = 16$$

And along the left side (LS) of the rectangle, $x = -2 \Rightarrow dx = 0$ so:

$$\int_{LS} \left[xy\,dy \;-\; x^2y\,dx \right] = \int_{y=3}^{y=-1} \left[-2y - 4y \cdot 0 \right] \, dy = 8$$

hence the total flux is $24 + 72 + 16 + 8 = 120$. On the other hand, $\text{div}\,(\mathbf{G}) = y + x^2$ so:

$$\int_{x=-2}^{x=4} \int_{y=-1}^{y=3} \left[y + x^2 \right] \, dy\,dx = \int_{x=-2}^{x=4} \left[\frac{1}{2}y + x^2y \right]_{y=-1}^{y=3} \, dx$$

$$= \int_{-2}^{4} \left[4 + 4x^2 \right] \, dx = \left[4x + \frac{4}{2}x^3 \right]_{-2}^{4} = 120$$

results in the same answer.

2. $\text{div}\,(\mathbf{G}) = 2x + 4y^3$ so the flux is:

$$\int_{x=1}^{x=2} \int_{y=3}^{y=7} \left[2x + 4y^3 \right] \, dy\,dx = \int_{x=1}^{x=2} \left[8x + 2320 \right]_{y=3}^{y=7} \, dx = 2332$$

3. $\text{div}\,(\mathbf{B}) = 0$ so the flux is 0.

4. Along the bottom side (BS) of the triangle, $y = 0 \Rightarrow dy = 0$ so:

$$\int_{BS} \left[x\,dy \;-\; y\,dx \right] = \int_{x=0}^{x=2} \left[x \cdot 0\,dy - 0\,dx \right] = 0$$

Along the right side (RS) of the triangle, $x = 2 \Rightarrow dx = 0$ so:

$$\int_{RS} \left[x\,dy \;-\; y\,dx \right] = \int_{y=0}^{y=2} \left[2\,dy - 0\,dx \right] = \left[2y \right]_0^2 = 4$$

And along the hypotenuse (H) of the triangle, $y = x \Rightarrow dy = dx$ so:

$$\int_{H} \left[x\,dy \;-\; y\,dx \right] = \int_{x=2}^{x=0} \left[x\,dx - x\,dx \right] \, dy = 0$$

hence the total flux is $0 + 4 + 0 = 4$.

Can you see from they graph why the flux across the bottom side and the hypotenuse should be 0, while the flux across the right side should be positive?

5. Applying the 2D Divergence Theorem:

$$\int_{\partial\mathcal{R}} \left[x^2y\,dy \;+\; xy^2\right]\,dA = \int_{\partial\mathcal{R}} \left[x^2y\,dy \;-\; \left(-xy^2\right)\,dx\right]$$
$$= \iint_{\mathcal{R}} \left[\frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}\left(-xy^2\right)\right]\,dA = \iint_{\mathcal{R}} 0\,dA = 0$$

no matter what region $\mathcal{R}$ is.

6. We cannot apply the 2D Divergence Theorem directly because $\mathcal{K}$ is not a closed curve. However, $\mathcal{K}$ together with the line segment $\mathcal{L}$ along the x-axis from $(-1.5, 0)$ to $(1.5, 0)$ is the curve $\mathcal{C}$ from Example 4, so:

$$\text{flux across } \mathcal{K} + \text{flux across } \mathcal{L} = \text{flux across } \mathcal{C} = \frac{891}{80}$$

Along $\mathcal{L}$, $y = 0 \;\Rightarrow\; dy = 0$ so the flux across $\mathcal{L}$ is:

$$\int_{\mathcal{L}} \left[(x^3 - y^3)\,dy - (y^2 - x^2)\,dx\right] = \int_{-1.5}^{1.5} x^2\,dx = \frac{9}{4}$$

Inserting this result into the equation above and solving for the desired flux yields:

$$\text{flux across } \mathcal{K} = \frac{891}{80} - \frac{9}{4} = \frac{711}{80} = 11.1375$$

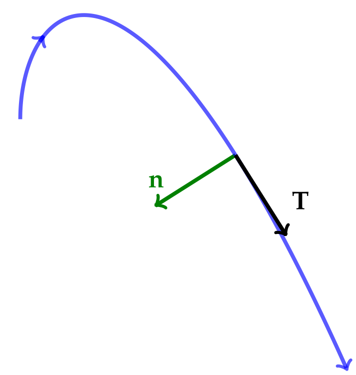
16.5 Line Integrals

You have now computed a number of "flux integrals," which accumulate the total flux of a vector field **F** across an oriented curve $\mathcal{C}$ by integrating the normal component of the field, $\mathbf{F}\cdot\mathbf{n}$, along $\mathcal{C}$. What if, instead, we integrated the tangential component, $\mathbf{F}\cdot\mathbf{T}$, along $\mathcal{C}$? What would this represent and how can we compute it in practice?

Thinking of the vector field **F** as a velocity field, we motivated the flux integral by interpreting it as the amount of "stuff" flowing perpendicularly to the the curve $\mathcal{C}$ during one unit of time. Interpreting the quantity $\mathbf{F}\cdot\mathbf{T}$ might measure something about the "stuff" flowing along the curve at one point, and this integral:

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds$$

would then measure something about the "stuff" flowing along $\mathcal{C}$ overall. We sometimes call this integral the **flow** of **F** along $\mathcal{C}$, and if $\mathcal{C}$ happens to be a closed curve we might also call it the **circulation** of **F** along $\mathcal{C}$. In general we call of this type of integral a **line integral**.



Physical interpreting flow and circulation is more complicated than the interpretation of flux.

Although we will sometimes compute line integrals along actual line segments, **curve integral** would be a much better name. Alas.

Evaluating Line Integrals

Evaluating line integrals turns out to be remarkably similar to—and, in at least one sense, even easier than—computing flux integrals.

If $\mathbf{r}(t)$ for $a \le t \le b$ paramterizes a smooth, oriented, simple curve $\mathcal{C}$ then $\mathbf{r}'(t)$ gives us a tangent vector at each point so:

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\,\|\mathbf{r}'(t)\|\,dt = \int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

Recall that the unit tangent vector is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

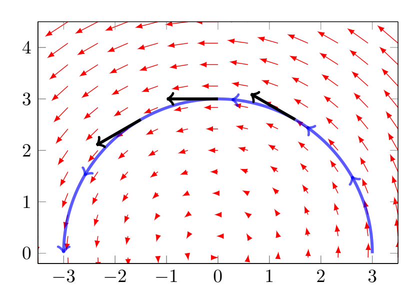
and that the arclength element is:

$$ds = \|\mathbf{r}'(t)\|\,dt$$

Example 1. Compute the flow of $\mathbf{F}(x,y) = \langle -y, x\rangle$ along the upper half of the (positively oriented) circle $x^2+y^2=9$.

Solution. $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le \pi$ parameterizes the semicircle, so with $\mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t)\rangle$ our line integral becomes:

$$\int_{t=0}^{t=\pi} \langle -3\sin(t), 3\cos(t)\rangle\cdot\langle -3\sin(t), 3\cos(t)\rangle\,dt = \int_0^{\pi} 9\,dt = 9\pi$$



Alternatively, we could note that $\langle -y, x\rangle$ is always tangent to the circle (see margin) and pointing in the positive rotational direction, so $\mathbf{F}\cdot\mathbf{T} = \|\mathbf{F}\|\cdot\|\mathbf{T}\|\cos(0) = \|\mathbf{F}\| = \sqrt{y^2+x^2} = 3$ and the line integral becomes 3 times the arclength integral for $\mathcal{C}$, which has length 3π. ◀

Practice 1. Compute the circulation of $\mathbf{G}(x,y) = \langle x, y\rangle$ along the (positively oriented) unit circle. Could you have predicted the value of the line integral based on a graph?

Practice 2. Compute the circulation of $\mathbf{F}(x,y) = \langle -y, x\rangle$ along the portion of the parabola $y = 9 - x^2$ from $x = 3$ to $x = -3$.

Work

So far we have been envisioning most vectors fields as velocity fields, but another very common use of vector fields is to think of each field vector as a force vector.

You have previously learned how to compute work in the special case where the displacement of an object and the force acting on that object happened to be in the same direction. And using the definition of work from physics, $W = \mathbf{F} \cdot \mathbf{d}$, you computed work when the force, $\mathbf{F}$, and the displacement, $\mathbf{d}$, were constant (but not necessarily pointing in the same direction). We now have the tools to tackle much more general work problems.

First in 2D, with 3D problems in the next chapter.

If $\mathcal{C}$ is a smooth curve given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$ and $\mathbf{F}(x, y)$ is a vector field representing the force applied to an object at the point (x, y), we want to compute the work done by the force field in moving the object from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. If $\mathbf{F}$ is constant and $\mathcal{C}$ is a line segment, then the answer would be:

$$\mathbf{F} \cdot (\mathbf{r}(b) - \mathbf{r}(a))$$

but otherwise we can partition $\mathcal{C}$ into small pieces corresponding to small increments in time Δt. On the k-th piece of the curve, starting at position $\left(x_{t_k}, y_{t_k}\right)$, we can approximate:

$$\mathbf{F}(x, y) \approx \mathbf{F}\left(x_{t_k}, y_{t_k}\right) = \mathbf{F}\left(\mathbf{r}\left(t_k\right)\right)$$

and approximate the displacement along this part of the curve by:

$$\mathbf{r}\left(t_k + \Delta t\right) - \mathbf{r}\left(t_k\right) \approx \mathbf{r}'\left(t_k\right) \cdot \Delta t$$

so the work done moving the object along this piece of the curve is:

$$W_k \approx \mathbf{F}\left(\mathbf{r}\left(t_k\right)\right) \cdot \mathbf{r}'\left(t_k\right) \cdot \Delta t$$

and adding up all of these approximate work values along the curve yields a Riemann sum:

$$\sum_{k=1}^{n} W_k = \sum_{k=1}^{n} \mathbf{F}\left(\mathbf{r}\left(t_k\right)\right) \cdot \mathbf{r}'\left(t_k\right) \cdot \Delta t \longrightarrow \int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral should look familiar!

which converges to the definite integral above right.

Example 2. Compute the work done by the force field $\mathbf{F}(x, y) = \langle y, x \rangle$ moving an object on the circle $x^2 + y^2 = 2$ from $(\sqrt{2}, 0)$ to $(1, 1)$.

Solution. $\mathbf{r}(t) = \left\langle \sqrt{2}\cos(t), \sqrt{2}\sin(t) \right\rangle$ for $0 \leq t \leq \frac{\pi}{4}$ parameterizes the circle in the desired direction, so with $\mathbf{r}'(t) = \left\langle -\sqrt{2}\sin(t), \sqrt{2}\cos(t) \right\rangle$ our work integral becomes:

$$\int_{t=0}^{t=\frac{\pi}{4}} \left\langle \sqrt{2}\sin(t), \sqrt{2}\cos(t) \right\rangle \cdot \left\langle -\sqrt{2}\sin(t), \sqrt{2}\cos(t) \right\rangle dt$$

$$= \int_0^{\frac{\pi}{4}} \left[2\cos^2(t) - 2\sin^2(t)\right] dt = \int_0^{\frac{\pi}{4}} 2\cos(2t)\, dt = 1$$

If forces are measured in Newtons and distances in meters, then the work done by the force field would be 1 N-m, or 1 J. ◀

Practice 3. Compute the work done by the force field $\mathbf{G}(x, y) = \langle 2xy, x^2 \rangle$ moving an object along $y = x^2$ from $(1, 1)$ to $(3, 9)$.

It is important to note that, in Example 2 and Practice 3, the path along which the object moves is *not* the path along which the particle would move if subjected only to the forces from $\mathbf{F}$. How could this be? Imagine a thin wire bent in the shape of the curve $\mathcal{C}$, with the object being a small bead with hole drilled through the center so that it is constrained to move (with minimal friction) along the curve $\mathcal{C}$.

In such a situation, only the component of the force vectors in the curve's tangential direction, $\mathbf{F} \cdot \mathbf{T}$, contribute to the object moving along the curve. The force applied to the bead by the wire at each point cancels out the normal component of the force vectors.

Example 3. Compute the work done by the force field $\mathbf{F}(x, y)$ moving an object of mass m in such a way that the object's position is given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ from time $t = 0$ to time $t = 1$.

Solution. Here we do not have a formula for $\mathbf{F}$, but we do know that $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2, 6t \rangle$ so the work done by $\mathbf{F}$ is:

$$\int_{t=0}^{t=1} m \langle 2, 6t \rangle \cdot \left\langle 2t, 3t^2 \right\rangle \, dt = m \int_0^1 \left[4t + 18t^3 \right] \, dt = m \left[2t^2 + 4.5t^4 \right]_0^1 = 6.5m$$

If forces are measured in N, mass in kg and distances in m, then the work done by the force field would be $6.5m$ J. ◀

Practice 4. Compute the work done by the force field $\mathbf{G}(x, y)$ moving an object of mass m in such a way that the object's position is given by $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t) \rangle$ from time $t = 0$ to time $t = \pi$.

It is also important to note that our definition of work computes the work done *by* the force field $\mathbf{F}$ moving an object along a curve. If you want to compute the work *you* do pushing an object *against* a force field (say, lifting a heavy textbook from the floor to a table, working against gravity) then you would need to multiply the answer by -1 because all of the forces you need to apply are in the opposite direction of the forces in the force field.

Properties of Line Integrals

In Example 2, $\mathbf{r}(t) = \left\langle \sqrt{2}\cos(t), \sqrt{2}\sin(t) \right\rangle$ was not the only parameterization we could have used for the arc of $x^2 + y^2 = 2$. If instead we use $\mathbf{r}(t) = \left\langle t, \sqrt{2 - t^2} \right\rangle$ for $1 \le t \le \sqrt{2}$ then the work integral becomes:

$$\int_{t=\sqrt{2}}^{t=1} \left\langle \sqrt{2 - t^2}, t \right\rangle \cdot \left\langle 1, -\frac{t}{\sqrt{2 - t^2}} \right\rangle \, dt = \int_{\sqrt{2}}^{1} \frac{2 - 2t^2}{\sqrt{2 - t^2}} \, dt$$

Here we integrate from $t = \sqrt{2}$ to $t = 1$ rather than the other way around so that the object moves from $(\sqrt{2}, 0)$ to $(1, 1)$, as specified in Example 2.

The trig substitution $t = \sqrt{2}\sin(\theta)$ turns this (improper) integral into:

$$\int_{\theta=\frac{\pi}{4}}^{\theta=0} \frac{2 - 4\sin^2(\theta)}{\sqrt{2}\cos(\theta)} \cdot \sqrt{2}\cos(\theta) \, d\theta = \int_0^{\frac{\pi}{4}} 2\cos(2\theta) \, d\theta$$

which is the same integral as in Example 2. This is very much what we should expect to be true: the value of a line integral of a field **F** across a curve $\mathcal{C}$ should not depend on our choice of parameterization for $\mathcal{C}$.

If $\mathbf{r}_1(t)$ for $a \leq t \leq b$ and $\mathbf{r}_2(t)$ for $\alpha \leq \tau \leq \beta$ are both one-to-one, C^1 parameter representations of a smooth, oriented curve $\mathcal{C}$, with $\mathbf{r}_1(a) = \mathbf{r}_2(\alpha)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(\beta)$, and if $\psi(\tau)$ is some differentiable function with $\mathbf{r}_2(\tau) = \mathbf{r}_1(\psi(\tau))$ for $\alpha \leq \tau \leq \beta$, then computing the line integral of **F** along $\mathcal{C}$ using $\mathbf{r}_2$ yields:

We need the assumption that these parameter functions are one-to-one to assure that they do not "double back" as they trace out the curve.

$$\int_{\tau=\alpha}^{\tau=\beta} \mathbf{F}\left(\mathbf{r}_2(\tau)\right) \cdot \mathbf{r}_2'(\tau)\, d\tau = \int_{\tau=\alpha}^{\tau=\beta} \mathbf{F}\left(\mathbf{r}_1(\psi(\tau))\right) \cdot \mathbf{r}_1'(\psi(\tau)) \cdot \psi'(\tau)\, d\tau$$

but the substitution $t = \psi(\tau) \;\Rightarrow\; dt = \psi'(\tau)\, d\tau$ turns this into:

$$\int_{t=a}^{t=b} \mathbf{F}\left(\mathbf{r}_1(t)\right) \cdot \mathbf{r}_1'(t)\, dt$$

which the the line integral of **F** along $\mathcal{C}$ when computed using $\mathbf{r}_1$.

Theorem: $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$ does not depend on the parameterization of $\mathcal{C}$.

The general proof hinges on the existence of ψ in the preceding discussion. This is guaranteed by the **Implicit Function Theorem**, which you may not have studied. The result also holds if two parameterizations of a closed curve do not have the same endpoints, for example the unit circle traced counterclockwise from $(1,0)$ to $(1,0)$, or from $(0,1)$ to $(0,1)$.

In our computations with line integrals, $\mathcal{C}$ is an *oriented* curve. What happens if we reverse the orientation?

Definition: If $\mathcal{C}$ is an oriented curve going from A to B then $-\mathcal{C}$ is the oriented curve that traces out the same path but from B to A.

At each point along $-\mathcal{C}$, **T** will point in the opposite direction as it does at the same point on $\mathcal{C}$, so:

Theorem: $\int_{-\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = -\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$

So far our line integral computations have involved smooth curves. If instead our curve is piecewise smooth (consisting of finitely many smooth pieces), say $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, meaning $\mathcal{C}$ consists of $\mathcal{C}_1$ followed by $\mathcal{C}_2$ where the ending point of $\mathcal{C}_1$ is the starting point of $\mathcal{C}_2$, then:

$$\int_{\mathcal{C}_1 \cup \mathcal{C}_2} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{T}\, ds$$

This extends to any finite number of smooth curves pieced together.

Differential Forms and Other Notation

We originally defined a line integral as the integral of $\mathbf{F} \cdot \mathbf{T}$ over a curve $\mathcal{C}$, which we compute using:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{t=a}^{t=b} \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\, dt$$

where $\mathbf{r}(t)$ for $a \leq t \leq b$ parameterizes $\mathcal{C}$. Using differential notation:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) \;\Rightarrow\; d\mathbf{r} = \mathbf{r}'(t)\, dt$$

we rewrite the line integral as $\int_C \mathbf{F} \cdot d\mathbf{r}$ or sometimes $\int_C \mathbf{F} \cdot d\mathbf{s}$. If we write $\mathbf{F} = \langle P, Q \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ then:

$$\int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{t=a}^{t=b} \langle P, Q \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$
$$= \int_{t=a}^{t=b} \left[P \cdot \frac{dx}{dt} + Q \cdot \frac{dy}{dt} \right] dt = \int_C [P\, dx \,+\, Q\, dy]$$

This differential form notation can be useful.

Compare these line-integral notations:

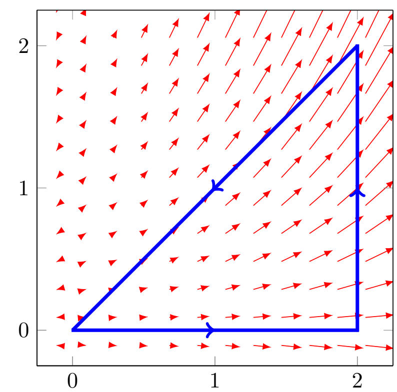
$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C [P\, dx \,+\, Q\, dy]$$

with our notations for flux integrals:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C [P\, dy \,-\, Q\, dx]$$

Example 4. Compute the circulation of $\mathbf{F}(x, y) = \langle x, xy \rangle$ along the x-axis from the origin to the point $(2, 0)$, then vertically up to the point $(2, 2)$, and finally back to the origin along the line $y = x$.

Solution. The triangular curve here is piecewise smooth, so we need to split our work integral into three pieces (see margin). Along the first line segment, $\mathcal{C}_1$, from $(0, 0)$ to $(2, 0)$, $y = 0 \Rightarrow dy = 0$ so:



$$\int_{\mathcal{C}_1} [P\, dx \,+\, Q\, dy] = \int_{x=0}^{x=2} [x\, dx + xy \cdot 0] = \left[\frac{1}{2} x^2 \right]_0^2 = 2$$

On the second segment, $\mathcal{C}_2$, from $(2, 0)$ to $(2, 2)$, $x = 2 \Rightarrow dx = 0$ so:

$$\int_{\mathcal{C}_2} [P\, dx \,+\, Q\, dy] = \int_{y=0}^{y=2} [2 \cdot 0 + 2y\, dy] = \left[y^2 \right]_0^2 = 4$$

On the third segment, $\mathcal{C}_3$, from $(2, 2)$ to $(0, 0)$, $y = x \Rightarrow dy = dx$ so:

$$\int_{\mathcal{C}_3} [P\, dx \,+\, Q\, dy] = \int_{x=2}^{x=0} \left[x\, dx + x^2\, dx \right] = \left[\frac{1}{2} x^2 + \frac{1}{3} x^3 \right]_2^0 = -\frac{14}{3}$$

Adding up these values: $2 + 4 - \frac{14}{3} = \frac{4}{3}$. Notice that along $\mathcal{C}_1$, the field vectors $\mathbf{F}$ point in the same direction as the unit tangent vectors $\mathbf{T} = \langle 1, 0 \rangle$, so $\mathbf{F} \cdot \mathbf{T} > 0$ everywhere on $\mathcal{C}_1$: the circulation of $\mathbf{F}$ along $\mathcal{C}_1$ should be positive (as indeed it is). Likewise, along $\mathcal{C}_2$ the angle between each field vector and the unit tangent vector $\langle 0, 1 \rangle$ is acute, so that $\mathbf{F} \cdot \mathbf{T} > 0$ here as well. Finally, along $\mathcal{C}_3$ the field vectors generally point in the opposite direction from the unit tangent vectors $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ so that $\mathbf{F} \cdot \mathbf{T} < 0$ (resulting in a negative circulation). ◀

Practice 5. Compute the flow of $\mathbf{G}(x, y) = \langle x, xy \rangle$ along the portion of the parabola $x = -y^2 + 5y - 4$ in the first quadrant, oriented in the direction of increasing y.

Practice 6. Compute the flow of $\mathbf{F}(x, y) = \langle -y, x \rangle$ along the portion of the cycloid given by $\mathbf{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$ for $0 \le t \le 2\pi$.

Scalar Line Integrals

A line integral involves integrating the tangential component of a force, $\mathbf{F} \cdot \mathbf{T}$, along a curve $\mathcal{C}$. The function $\mathbf{F} \cdot \mathbf{T}$ is a scalar function defined at every point along this curve. If, somehow, $\mathbf{F} \cdot \mathbf{T} = 1$ at all points on $\mathcal{C}$ this line integral becomes:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C 1\, ds$$

which is simply the arclength of $\mathcal{C}$. If $f(x,y) = \mathbf{F} \cdot \mathbf{T}(x,y)$ is some function other than 1, the line integral $\int_{\mathcal{C}} f\, ds$ may represent other physical quantities, such as mass (if f is a density function) or area (if f represents the height of a "fence" sitting above $\mathcal{C}$).

Example 5. A wire sits in the xy-plane along the curve $y = x^3$ for $1 \le x \le 2$ (with distances measured in cm). If the wire's density at any point is $\delta(x,y) = 2 + 0.1y$ g/cm, compute the mass of the wire.

Solution. To compute the mass, we can use a line integral to integrate the density function along the curve, which we designate $\mathcal{C}$. Parameterizing $\mathcal{C}$ using $\mathbf{r}(t) = \langle t, t^3 \rangle$ for $1 \le t \le 2$, we have:

$$\mathbf{r}(t) = \left\langle t, t^3 \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 1, 3t^2 \right\rangle \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{1+9t^4}$$

so that:

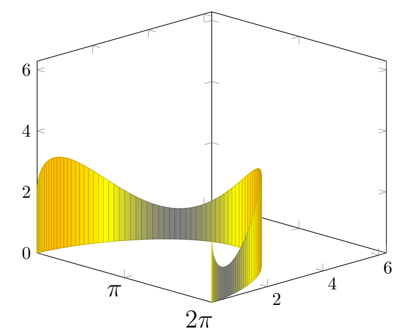
$$ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{1+9t^4}\, dt$$

and hence the mass is given by:

$$\int_{\mathcal{C}} [2+0.1y]\, ds = \int_{t=1}^{t=2} \left[2+0.1t^3\right] \sqrt{1+9t^4}\, dt \approx 17.34$$

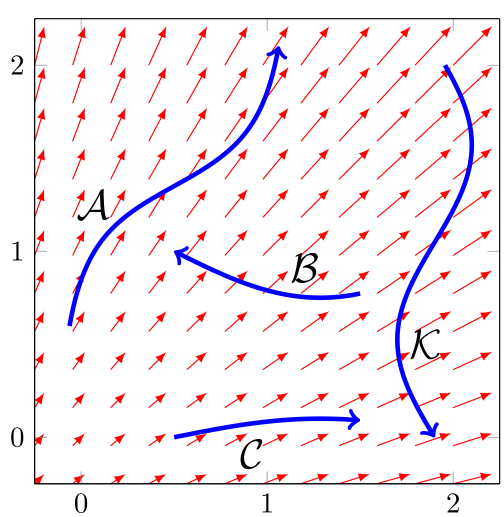
The mass of the wire is (approximately) 17.34 g. ◀

Practice 7. You build a fence of variable height above the curve in the xy-plane given by $\mathbf{r}(t) = \langle t - \sin(t), 2 - 2\cos(t) \rangle$ for $0 \le t \le 2\pi$ such that the height of the fence is $h(t) = 2 + \sin(2t)$ (with all distances measured in meters). Find the surface area of (one side of) this fence.



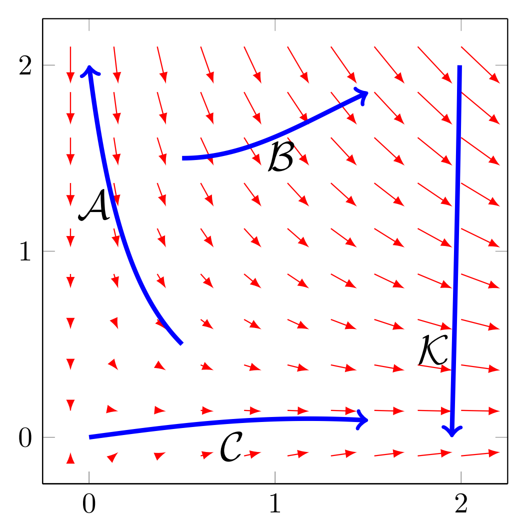
16.5 Problems

In Problems 1–4 estimate whether the work done by the vector field in the graph below along the indicated path is positive, negative or zero.



1. $\mathcal{A}$
2. $\mathcal{B}$
3. $\mathcal{C}$
4. $\mathcal{K}$

In Problems 5–8 estimate whether the work done by the vector field in the graph below along the indicated path is positive, negative or zero.



5. $\mathcal{A}$
6. $\mathcal{B}$
7. $\mathcal{C}$
8. $\mathcal{K}$

9. Calculate the work done by the force field $\mathbf{F}(x,y) = \langle x, x+y \rangle$ to move an object along the path $\mathbf{r}(t) = \langle 2+3t, 4t \rangle$ for $0 \le t \le 3$.
10. Calculate the work done by the force field $\mathbf{G}(x,y) = \langle -y, x \rangle$ to move an object along the path $\mathbf{r}(t) = \langle t^2, t \rangle$ for $0 \le t \le 2$.
11. Calculate the work done by the force field $\mathbf{F}(x,y) = \langle x, x \rangle$ to move an object along the line segment from $(1,3)$ to $(2,7)$.
12. Calculate the work done by the force field $\mathbf{F}(x,y) = \langle y, 2-x \rangle$ to move an object along the line segment from $(1,3)$ to $(2,7)$.

In Problems 13–16 compute the work done by the radial force field:

$$\mathbf{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

moving a particle along the indicated oriented curve.

13. The portion of the circle $x^2+y^2=9$ in the first quadrant (in the positive direction).
14. The line segment from $(1,1)$ to $(3,3)$.
15. The line segment from $(2,2)$ to $(1,1)$.
16. The portion of the circle $x^2+y^2=9$ in the first quadrant in the clockwise (negative) direction.

In Problems 17–22 compute the work done by the "swirl" force field:

$$\mathbf{G}(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$$

moving a particle along the indicated oriented curve.

17. The portion of $x^2+y^2=9$ in the first quadrant (oriented positively).
18. The line segment from $(1,1)$ to $(3,3)$.
19. The line segment from $(2,2)$ to $(1,1)$.
20. The portion of $x^2+y^2=9$ in the first quadrant (oriented negatively).
21. The circle $x^2+y^2=16$ (oriented positively).
22. The circle $x^2+y^2=25$ (oriented negatively).

In Problems 23–36 evaluate each line integral over the specified (positively oriented) curve.

23. $\mathcal{C}$ is the square with opposing vertices at $(0,0)$ and $(1,1)$:
$$\int_{\mathcal{C}} [5y\,dx \;+\; 3y\,dy]$$
24. $\mathcal{C}$ is the square with opposing vertices at $(0,0)$ and $(1,1)$:
$$\int_{\mathcal{C}} [5x\,dx \;+\; 3y\,dy]$$
25. $\mathcal{C}$ is the portion of the parabola $y=1-x^2$ from $(1,0)$ to $(-1,0)$, followed by the line segment from $(-1,0)$ to $(1,0)$:
$$\int_{\mathcal{C}} [5y\,dx \;+\; 3x\,dy]$$
26. $\mathcal{C}$ is the portion of the parabola $y=1-x^2$ from $(1,0)$ to $(-1,0)$, followed by the line segment from $(-1,0)$ to $(1,0)$:
$$\int_{\mathcal{C}} [5x\,dx \;+\; 3x\,dy]$$
27. $\mathcal{C}$ is the ellipse $4x^2+9y^2=36$:
$$\int_{\mathcal{C}} \left[y^3\,dx \;+\; x^3\,dy\right]$$
28. $\mathcal{C}$ is the unit circle:
$$\int_{\mathcal{C}} [5x\,dx \;+\; 8y\,dy]$$
29. $\mathcal{C}$ is the rectangle with opposing vertices at $(0,0)$ and $(10,12)$:
$$\int_{\mathcal{C}} [(5x-17y)\;dx \;+\; (11x+8y)\;dy]$$
30. $\mathcal{C}$ is the rectangle with vertices at $(-3,3)$, $(10,3)$, $(10,12)$ and $(-3,12)$:
$$\int_{\mathcal{C}} [(17x-5y)\;dx \;+\; (8x+11y)\;dy]$$
31. $\mathcal{C}$ is the rectangle with opposing vertices at $(0,0)$ and (L,H):
$$\int_{\mathcal{C}} [(\alpha x+\beta y)\;dx \;+\; (\gamma x+\delta y)\;dy]$$
32. $\mathcal{C}$ is the circle of radius R centered at the origin:
$$\int_{\mathcal{C}} [(\alpha x+\beta y)\;dx \;+\; (\gamma x+\delta y)\;dy]$$
33. $\mathcal{C}$ is the circle $x^2+y^2=16$:
$$\int_{\mathcal{C}} \left[x^3\,dx \;+\; y^3\,dy\right]$$
34. $\mathcal{C}$ is the circle $x^2+y^2=16$:
$$\int_{\mathcal{C}} \left[x^4\,dx \;+\; y^4\,dy\right]$$

35. $\mathcal{C}$ is the circle $x^2 + y^2 = 16$:

$$\int_{\mathcal{C}} \left[y^3\,dx \;+\; x^3\,dy \right]$$

36. $\mathcal{C}$ is the circle $x^2 + y^2 = 16$:

$$\int_{\mathcal{C}} \left[y^4\,dx \;+\; x^4\,dy \right]$$

37. Compute the work done to move an object through the vector field $\langle x^3, x + y^3 \rangle$ along the triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$ (oriented positively).

38. Compute the work done to move an object through the vector field $\langle 5 + x^3, 6x + y^3 \rangle$ along the triangle with vertices at $(0,0)$, $(7,0)$ and $(7,3)$ (oriented positively).

39. Find the mass of a wire with density $\delta(x,y) = 1 + x$ bent in the shape of the parabola $y = x^2$ for $1 \le x \le 2$.

40. Find the mass of a wire with density $\delta(x,y) = 1 + y$ bent in the shape of the parabola $y = x^2$ for $1 \le x \le 2$.

41. Find the mass of a wire with density $\delta(x,y) = 1 + y$ bent in the shape of the curve $y = x^3$ for $0 \le x \le 2$.

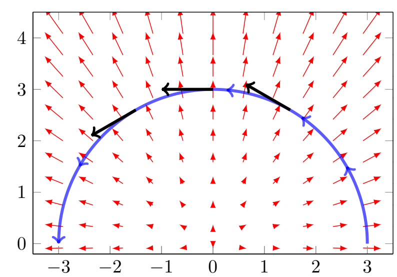
42. Find the mass of a wire with density $\delta(x,y) = 1 + x$ bent in the shape of the curve $y = x^3$ for $0 \le x \le 2$.

16.5 Practice Answers

1. As in Example 1, $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t) \rangle$ for $0 \le t \le \pi$ parameterizes the semicircle, so with $\mathbf{r}'(t) = \langle -3\sin(t), \cos(t) \rangle$ our line integral becomes:

$$\int_{t=0}^{t=2\pi} \langle 3\cos(t), 3\sin(t) \rangle \cdot \langle -3\sin(t), 3\cos(t) \rangle \; dt = \int_0^{2\pi} 0\,dt = 0$$

Alternatively, we could note that $\langle x, y \rangle$ is always normal to the circle (see margin) and therefore perpendicular to $\mathbf{T}$, so $\mathbf{F} \cdot \mathbf{T} = 0$ everywhere along the semicircle.



2. Using $\mathbf{r}(t) = \langle t, 9 - t^2 \rangle$ for t ranging from -3 to 3 the line integral becomes:

$$\int_{t=3}^{t=-3\pi} \left\langle -9 + t^2, t \right\rangle \cdot \langle 1, -2t \rangle \; dt = \int_3^{-3} \left[-9 - t^2 \right] \; dt = 72$$

3. Using $\mathbf{r}(t) = \langle t, t^2 \rangle$ for t ranging from 1 to 3 the line integral becomes:

$$\int_{t=1}^{t=3} \left\langle 2t^3, t^2 \right\rangle \cdot \langle 1, 2t \rangle \; dt = \int_1^3 \left[4t^3 \right] \; dt = 80$$

4. $\mathbf{F}(t) = m\mathbf{a}(t) = m\mathbf{r}''(t) = \langle -3m\cos(t), -3m\sin(t) \rangle$ so the work done is:

$$\int_{t=0}^{t=\pi} \langle -3m\cos(t), -3m\sin(t) \rangle \cdot \langle \cos(t), \sin(t) \rangle \; dt = -3m\pi$$

5. The curve is in the first quadrant when $y > 0$ and $x = -y^2 + 5y - 4 > 0 \Rightarrow (y-1)(y-4) < 0 \Rightarrow 1 < y < 4$ so with $dx = (-2y + 5)\;dy$ the flow is given by:

$$\int_{y=1}^{y=4} \left(-y^2 + 5y - 4 \right) (-2y + 5) \; dy \;+\; \left(-y^2 + 5y - 4 \right) y\,dy = 11.25$$

16.6 *The Fundamental Theorem of Line Integrals*

You now know how to compute the line integral of a vector field $\mathbf{F}(x, y)$ along a curve $\mathcal{C}$ parameterized by $\mathbf{r}(t)$ for $a \le t \le b$:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\, dt$$

If $\mathbf{F} = \nabla\varphi$ (so that $\mathbf{F}$ is a gradient field) then:

$$\int_{\mathcal{C}} \nabla\varphi \cdot d\mathbf{r} = \int_{t=a}^{t=b} \nabla\varphi\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\, dt$$

This integrand should look familiar (from the Chain Rule for Paths):

$$\frac{d}{dt}\left[\varphi\left(\mathbf{r}(t)\right)\right] = \nabla\varphi\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)$$

So the line integral of $\nabla\varphi$ along $\mathcal{C}$ becomes (using the Fundamental Theorem of Calculus):

$$\int_{t=a}^{t=b} \nabla\varphi\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\, dt = \int_{t=a}^{t=b} \frac{d}{dt}\left[\varphi\left(\mathbf{r}(t)\right)\right]\, dt = \varphi\left(\mathbf{r}(b)\right) - \varphi\left(\mathbf{r}(a)\right)$$

This says that the line integral of a gradient field along a curve can be computed by evaluating the potential function for that gradient field at the endpoints of the curve and taking the difference.

Fundamental Theorem of Line Integrals (FTLI)

If: $\mathcal{C}$ is a piecewise-smooth, oriented curve starting at A and ending at B, and φ is a C^1 function on some open set containing $\mathcal{C}$

then:

$$\int_{\mathcal{C}} \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

Example 1. If $\varphi(x, y) = x^2 + y^3$, compute the line integral of $\nabla\varphi$ along the line segment $\mathcal{L}$ from $(0, 0)$ to $(2, 4)$.

Solution. Applying the Fundamental Theorem of Line Integrals:

$$\int_{\mathcal{C}} \nabla\varphi \cdot d\mathbf{r} = \varphi(2, 4) - \varphi(0, 0) = [4 + 64] - [0 + 0] = 68$$

Using our old method with $\mathbf{r}(t) = \langle t, 2t\rangle$ for $0 \le t \le 2$:

$$\int_{\mathcal{C}} \nabla\varphi \cdot d\mathbf{r} = \int_{t=0}^{t=2} \left\langle 2t, 3 \cdot 4t^2 \right\rangle \cdot \langle 1, 2\rangle\, dt = \int_0^2 \left[2t + 24t^2\right]\, dt$$

Note that: $\nabla\varphi(x, y) = \langle 2x, 3y^2\rangle$

which also evaluates to 68, as expected. Which way was easier? ◀

Practice 1. If $\varphi(x, y) = x^2 + y^3$, compute the line integral of $\nabla\varphi$ along the portion of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.

Practice 2. If $\psi(x, y) = x^4y^5$, compute the line integral of $\nabla\psi$ along the unit circle, oriented positively.

Closed Curves

For a line integral around a closed curve we sometimes use a small circle on the integral sign to emphasize that the curve is closed:

- $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ indicates $\mathcal{C}$ is closed
- $\ointctrclockwise_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ indicates $\mathcal{C}$ is closed and positively oriented
- $\ointclockwise_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ indicates $\mathcal{C}$ is closed and negatively oriented

If $\mathbf{F} = \nabla \varphi$ then the FTLI says that:

$$\oint_{\mathcal{C}} \nabla \varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A) = 0$$

because $A = B$ for any closed curve. If $\mathbf{F}$ is not a gradient field this may or may not be true.

See Example 4 from Section 16.5 for the line integral of a vector field around a closed curve that is not 0. What can you conclude about that vector field?

Example 2. If $\mathbf{F}(x, y) = \left\langle 4x^3y^7 - \sin(e^x), 7x^4y^6 + \arctan\left(y^3\right) \right\rangle$, compute the circulation of $\mathbf{F}$ along $\mathcal{E}$, the ellipse $9(x-1)^2 + 11(y+\pi)^2 = 1$.

Solution. Applying the Mixed-Partials Test:

$$\frac{\partial}{\partial x}\left[7x^4y^6 + \arctan\left(y^3\right)\right] - \frac{\partial}{\partial y}\left[4x^3y^7\right] = 0$$

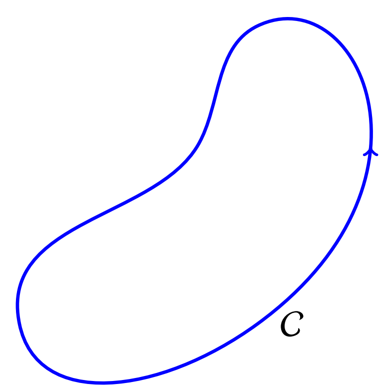
so $\mathbf{F}$ might be a gradient field, and in fact:

$$\mathbf{F}(x, y) = \nabla\left(x^4y^7 - \int_0^x \sin(e^u)\, du + \int_0^y \arctan\left(v^3\right)\, dv\right)$$

hence $\oint_{\mathcal{E}} \mathbf{F} \cdot d\mathbf{r} = 0$, because $\mathcal{E}$ is a closed curve. ◀

Example 2 did not specify the orientation of the closed curve $\mathcal{E}$, but here it did not matter because the value of the line integral turned out to be 0. Henceforth, if the orientation of a closed curve is not specified, we will assume that it is oriented positively (counterclockwise).

Practice 3. If $\mathbf{G}(x, y) = \left\langle 2x\cos\left(x^2y^3\right), 3y^2\cos\left(x^2y^3\right)\right\rangle$, compute the circulation of $\mathbf{G}$ along the curve $\mathcal{C}$ shown in the margin figure.

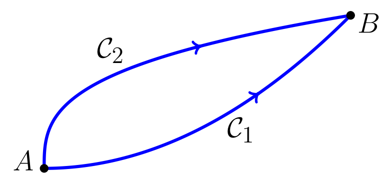


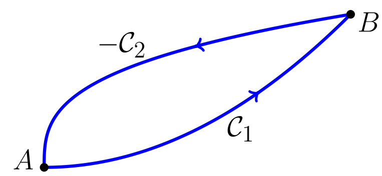
Path-Independent Vector Fields

We now know that any gradient field is **path-independent**. That is, the value of line integral of any gradient field along a piecewise-smooth curve depends only on the value of the potential function at the endpoints, not the path the curve takes between those endpoints. Consequently, the line integral of a gradient field along any closed path must be 0. We also know that there are vector fields that do not possess the path-independence property and have line integrals along certain closed paths that are not 0. It turns out that gradient fields are the *only* vectors fields that have path-independence property.

Theorem: If $\mathbf{F}(x,y)$ is a C^1 vector field on an open, path-connected set $\mathcal{D}$ then the following are equivalent:

- $\mathbf{F} = \nabla\varphi$ for some function $\varphi(x,y)$
- $\mathbf{F}$ is path-independent
- $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 0$ for any piecewise-smooth, closed curve $\mathcal{C}$





Proof. If the third condition holds, for any two distinct points A and B in the xy-plane let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two paths from A to B. Then $\mathcal{C}_1 - \mathcal{C}_2 = \mathcal{C}_1 \cup \{-\mathcal{C}_2\}$ is a closed curve, so:

$$0 = \oint_{\mathcal{C}_1-\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} + \int_{-\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}$$

hence the line integrals of $\mathbf{F}$ along these two paths must be equal.

If the second condition holds, pick any two distinct points A and B on an arbitrary closed curve $\mathcal{C}$ and call one of the two resulting curves from A to B $\mathcal{C}_1$ and the other $\mathcal{C}_2$. Then:

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} \;\Rightarrow\; 0 = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

hence the second condition implies the third.

If $\mathbf{F} = \nabla\varphi$ then the FTLI says $\mathbf{F}$ is path-independent, so the first condition implies the second. To prove the converse, assume that $\mathbf{F} = \langle P, Q\rangle$ is path-independent. We need to show that $\langle P, Q\rangle = \nabla\varphi = \langle \varphi_x, \varphi_y\rangle$ for some function φ. Choose any point $(a,b) \in D$ and define:

$$\varphi(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}$$

This definition does not specify a path for the line integral, but that is unnecessary because $\mathbf{F}$ is path-independent! Now compute:

However we do need to be able to find *some* path from (a,b) to an arbitrary point (x,y) in $\mathcal{D}$ that resides entirely in $\mathcal{D}$. For this reason we need the set $\mathcal{D}$ to be **path-connected**, which we define to mean sets for which any two points can be joined by a path that sits entirely in that set.

$$\frac{\partial\varphi}{\partial x}(x,y) = \lim_{h\to 0} \frac{\varphi(x+h,y) - \varphi(x,y)}{h}$$

The numerator of this limit is:

$$\varphi(x+h,y) - \varphi(x,y) = \int_{(a,b)}^{(x+h,y)} \mathbf{F}\cdot d\mathbf{r} - \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x,y)}^{(x+h,y)} \mathbf{F}\cdot d\mathbf{r}$$

Using the parameter representation $\mathbf{r}(t) = \langle x+t, y\rangle$ for $0 \le t \le h$ (so that $\mathbf{r}'(t) = \langle 1, 0\rangle$) for the path from (x,y) to $(x+h,y)$ this becomes:

$$\int_{t=0}^{t=h} \langle P(x+t,y), Q(x+t,y)\rangle \cdot \langle 1,0\rangle \; dt = \int_{t=0}^{t=h} P(x+t,y)\,dt$$

Finally, applying L'Hôpital's Rule, the Fundamental Theorem of Calculus and the continuity of P:

$$\frac{\partial\varphi}{\partial x}(x,y) = \lim_{h\to 0} \frac{\int_{t=0}^{t=h} P(x+t,y)\,dt}{h} = \lim_{h\to 0} \frac{P(x+h,y)}{1} = P(x,y)$$

as desired. The proof that $\varphi_y = Q$ is left to you (as Problem 21). □

Conservation of Energy

If a force field **F** acts on a particle with mass m and position $\mathbf{r}(t)$, then:

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{a}(t) \ \Rightarrow \ \mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

so the work done by **F** moving the particle from point A (at time t_A) to point B (at time t_B) is:

$$\int_{t_A}^{t_B} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_{t_A}^{t_B} m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\,dt = \left[\frac{1}{2}m\left\|\mathbf{r}'(t)\right\|^2\right]_{t_A}^{t_B}$$

Recall from Chapter 11 that:

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}' + \mathbf{v}'\cdot\mathbf{v} = 2\mathbf{v}\cdot\mathbf{v}'$$

$$\Rightarrow \mathbf{v}\cdot\mathbf{v}' = \frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}\|^2\right)$$

This evaluates to:

$$\frac{1}{2}m\left\|\mathbf{r}'(t_B)\right\|^2 - \frac{1}{2}m\left\|\mathbf{r}'(t_A)\right\|^2 = \text{KE}(B) - \text{KE}(A)$$

where KE denotes kinetic energy.

On the other hand, if **F** is a gradient field with potential function φ then the work done by **F** moving the particle from A to B is:

$$\int_{t_A}^{t_B} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_{t_A}^{t_B} \nabla\varphi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = [\varphi(t)]_{t_A}^{t_B} = \varphi(B) - \varphi(A)$$

and equating these two expressions for work:

$$\text{KE}(B) - \text{KE}(A) = \varphi(B) - \varphi(A) \ \Rightarrow \ \text{KE}(B) - \varphi(B) = \text{KE}(A) - \varphi(A)$$

If we call $-\varphi$ the **potential energy** of the particle, PE, then:

$$\text{KE}(B) + \text{PE}(B) = \text{KE}(A) + \text{PE}(A)$$

so that the total energy (kinetic plus potential) at each point is the same. In other words, energy is conserved. This is why we call gradient fields **conservative** and why we call the "antigradient" φ the **potential function**. (This also explains why physicists write $\mathbf{F} = -\nabla\psi$ and use $-\psi$ for a potential function rather than $+\varphi$.)

Gravitational, electric and magnetic fields are examples of conservative force fields.

16.6 Problems

1. Compute $\int_{\mathcal{C}} \nabla\varphi \cdot d\mathbf{r}$ if $\varphi(x,y) = 3x^2 + 4xy - 5y^2$ and $\mathcal{C}$ is the line segment from $(1,2)$ to $(7,6)$.
2. Given $\psi(x,y) = (5x-2y)^3 + \ln(2+x+3y)$, compute $\int_{\mathcal{K}} \nabla\psi \cdot d\mathbf{r}$ if $\mathcal{K}$ is the portion of the unit circle in the first quadrant.
3. Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F}(x,y) = \left\langle 2x^3, 5\right\rangle$ and $\mathcal{C}$ is the portion of $y = 3x^2$ from $(1,3)$ to $(3,27)$.
4. Compute $\int_{\mathcal{K}} \mathbf{G} \cdot d\mathbf{r}$ if $\mathbf{G}(x,y) = \langle 2x+y, x\rangle$ and $\mathcal{K}$ is the portion of the parabola $x = y^2 + 2y + 7$ from $(6,-1)$ to $(10,1)$.
5. Compute $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\,ds$ if $\mathbf{F}(x,y) = \langle 7, 4x\rangle$ and $\mathcal{C}$ is the line segment from $(10,10)$ to $(20,30)$.
6. Compute $\int_{\mathcal{K}} \mathbf{G} \cdot \mathbf{T}\,ds$ if $\mathbf{G}(x,y) = \left\langle 3x^2, 4y^3\right\rangle$ and $\mathcal{K}$ is the portion of the circle $x^2 + y^2 = 7$ in the third quadrant (oriented positively).

In Problems 7–12 compute the work done by the given force field $\mathbf{F}(x,y)$ moving a particle from A to B. (If the field is not conservative, find two paths for which the work done is different on each path.)

7. $\mathbf{F}(x,y) = \langle 2x, 2y\rangle$, $A = (1,2)$, $B = (5,1)$
8. $\mathbf{F}(x,y) = \langle y, x\rangle$, $A = (1,2)$, $B = (5,5)$
9. $\mathbf{F}(x,y) = \langle x, x\rangle$, $A = (0,2)$, $B = (3,6)$
10. $\mathbf{F}(x,y) = \left\langle 3x^2y, x^2\right\rangle$, $A = (1,0)$, $B = (3,1)$

11. $\mathbf{F}(x,y) = \left\langle 3x^2y, x^3 \right\rangle$, $A = (1,2)$, $B = (5,1)$
12. $\mathbf{F}(x,y) = \langle 2x+y, x+2y \rangle$, $A = (0,0)$, $B = (3,4)$

In Problems 13–16 evaluate each line integral over the specified (positively oriented) curve.

13. $\mathcal{E}$ is the ellipse $3x^2 + 7y^2 = 21$:

$$\int_{\mathcal{E}} \left[4x^3\,dx \ + \ 5y^4\,dy\right]$$

14. $\mathcal{C}$ is the portion of the ellipse $3x^2 + 7y^2 = 21$ in the first quadrant:

$$\int_{\mathcal{E}} \left[4x^3\,dx \ + \ 5y^4\,dy\right]$$

15. $\mathcal{K}$ is any smooth curve from $(1,\pi)$ to $(9,\pi)$

$$\int_{\mathcal{K}} \left[2x \cdot \cos(y)\,dx \ - \ x^2 \cdot \sin(y)\,dy\right]$$

16. $\mathcal{C}$ is the curve $x^4 + y^4 = 1$:

$$\int_{\mathcal{C}} \left[2x \cdot \cos(y)\,dx \ - \ x^2 \cdot \sin(y)\,dy\right]$$

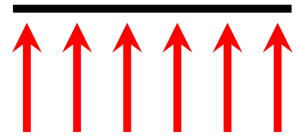
In Problems 17–20 verify that the underlying vector field is path-independent (so that the notation makes sense), then evaluate the line integral.

17. $\displaystyle\int_{(1,2)}^{(3,5)} \left[6x^2y^2\,dx \ + \ 4x^3y\,dy\right]$
18. $\displaystyle\int_{(0,0)}^{(\frac{\pi}{2},3)} \left[y \cdot \sin(xy)\,dx \ + \ x \cdot \sin(xy)\,dy\right]$
19. $\displaystyle\int_{(-1,1)}^{(\sqrt{2},\sqrt{2})} \left[(4x+5y)\ dx \ + \ \left(5x+4y^3\right)\ dy\right]$
20. $\displaystyle\int_{(-2,-2)}^{(3,3)} \left[21\,(7x-4y)^2\ dx \ - \ 12\,(7x-4y)^2\ dy\right]$
21. Complete the proof of the theorem on page 1645.

16.6 Practice Answers

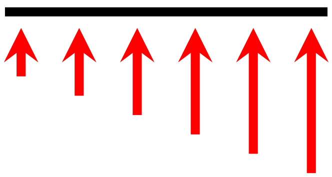
1. Using FTLI: $\varphi(2,4) - \varphi(0,0) = \left(2^2 + 4^3\right) - \left(0^2 + 0^3\right) = 68$
2. The problem does not specify a starting and ending point, but they must be the same point because the unit circle is a closed curve. If this point is (a,b) then using the FTLI the line integral equals $\psi(a,b) - \psi(a,b) - 0$.
3. $\mathbf{G} = \nabla\varphi$ with $\varphi(x,y) = \sin(x^2y^3)$, so the circulation around $\mathcal{C}$ must be 0 because the unit circle is a closed curve.

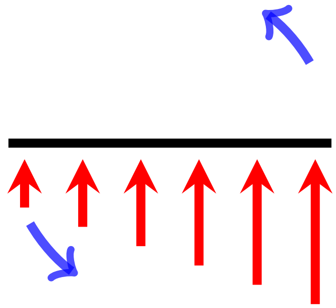
16.7 2D Curl

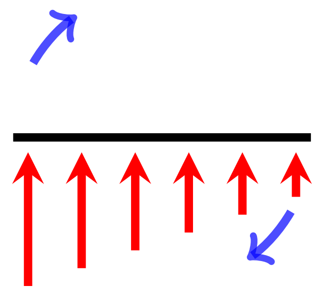


Think again of a vector field that represents the velocity of water flowing in a shallow stream. If a pine needle falls into the stream (see margin) with the velocity vectors all equal and perpendicular to the pine needle, the needle will flow in the upward (positive-y) direction.

If instead the velocity vectors are unequal and as shown below left:

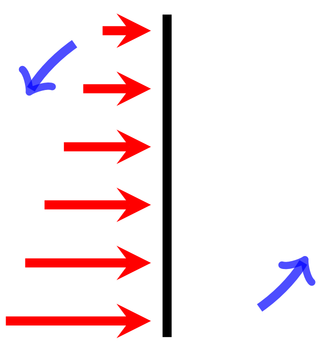


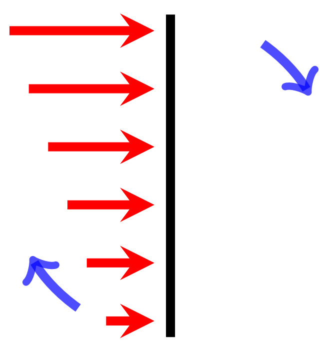




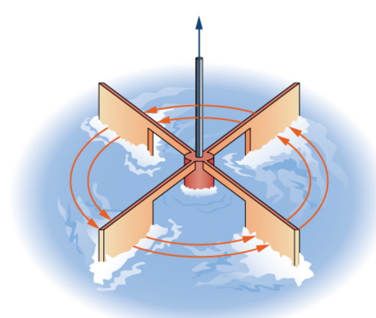
the pine needle will flow in the upward (positive-y) direction but at the same time it will tend to rotate in the positive (counterclockwise) direction (as shown above right) because the fluid is moving faster on the right than the left, inducing a positive rotation. Similarly, the pine needle depicted in the margin will tend to rotate in the negative (clockwise) direction (while flowing upward). In both of these examples, the vector field has the form $\langle 0, Q\rangle$. In the example with positive "spin," the lengths of the vectors are increasing as x increases, so $\dfrac{\partial Q}{\partial x} > 0$. In the example with negative spin, $\dfrac{\partial Q}{\partial x} < 0$.

If instead we consider vector fields of the form $\langle P, 0\rangle$, as depicted here:





a positive spin is associated with P *decreasing* as y increases (above left), so that $\dfrac{\partial P}{\partial y} < 0$, and a negative spin is associated with P *increasing* as y decreases (above right), so that $\dfrac{\partial P}{\partial y} > 0$

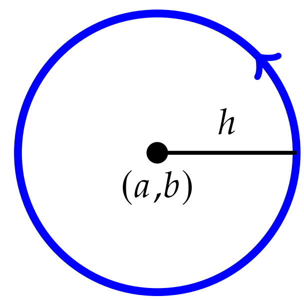


Based on these examples, you might suspect that the tendency of a pine needle (or a paddlewheel, such as the one shown in the margin) to rotate in the positive direction in a vector field $\langle P, Q\rangle$ has something to do with $\frac{\partial Q}{\partial x}$ and $-\frac{\partial P}{\partial y}$ being positive, while the tendency to rotate in the negative direction has something to with these quantities being negative. To investigate further, we will consider a more general situation.

Circulation Around a Circle

Now imagine a paddlewheel with many, many "paddles." A force vector $\mathbf{F}$ acting at the tip of one of these paddles will exert a force of magnitude $\mathbf{F} \cdot \mathbf{T}$ in the tangential direction, contributing to the tendency of the paddlewheel to rotate (in the positive direction if $\mathbf{F} \cdot \mathbf{T} > 0$ and in the negative direction if $\mathbf{F} \cdot \mathbf{T} < 0$). Summing up these $\mathbf{F} \cdot \mathbf{T}$ values for each paddle leads to a Riemann sum for the line integral of $\mathbf{F} \cdot \mathbf{T}$ around the circular edge of the paddlewheel (the circulation).

Let $\mathcal{C}_h$ be the (positively oriented) circle of radius $h > 0$ centered at the point (a, b). To find the circulation of the vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ along this circle we need to compute:



$$\int_{\mathcal{C}_h} \mathbf{F} \cdot \mathbf{T}\, ds$$

If we parameterize the circle using $\mathbf{r}(t) = \langle a + h\cos(t), b + h\sin(t) \rangle$ for $0 \le t \le 2\pi$, then $\mathbf{r}'(t) = \langle -h\sin(t), h\cos(t) \rangle$ so the integral becomes:

$$\int_0^{2\pi} \mathbf{F}\left(a + h\cos(t), b + h\sin(t)\right) \cdot \langle -h\sin(t), h\cos(t) \rangle\, dt$$

For small values of $h > 0$, the point (x, y) will be close to (a, b), so we can approximate the component functions of the vector field using:

$$\begin{aligned} P(x,y) &\approx P(a,b) + P_x(a,b) \cdot (x-a) + P_y(a,b) \cdot (y-b) \\ Q(x,y) &\approx Q(a,b) + Q_x(a,b) \cdot (x-a) + Q_y(a,b) \cdot (y-b) \end{aligned}$$

Putting $x = a + h\cos(t)$ and $y = b + h\sin(t)$ this becomes:

$$\begin{aligned} P(a+h\cos(t), b+h\sin(t)) &\approx P(a,b) + P_x(a,b) \cdot h\cos(t)) + P_y(a,b) \cdot h\sin(t) \\ Q(a+h\cos(t), b+h\sin(t)) &\approx Q(a,b) + Q_x(a,b) \cdot h\cos(t)) + Q_y(a,b) \cdot h\sin(t) \end{aligned}$$

so $\mathbf{F}\left(a + h\cos(t), b + h\sin(t)\right) \cdot \langle -h\sin(t), h\cos(t) \rangle$ is (approximately):

$$\begin{aligned} &- P(a,b) \cdot h\sin(t) - P_x(a,b) \cdot h^2\cos(t)\sin(t) - P_y(a,b) \cdot h^2\sin^2(t) \\ &+ Q(a,b) \cdot h\cos(t) + Q_x(a,b) \cdot h^2\cos^2(t) + Q_y(a,b) \cdot h^2\sin(t)\cos(t) \end{aligned}$$

Integrating this quantity from 0 to 2π yields:

$$\int_{\mathcal{C}_h} \mathbf{F} \cdot \mathbf{T}\, ds \approx \pi h^2 \left[Q_x(a,b) - P_y(a,b) \right]$$

Here we use these facts:

$$\int_0^{2\pi} \cos(t)\, dt = \int_0^{2\pi} \sin(t)\, dt = 0$$

$$\int_0^{2\pi} \sin(t)\cos(t)\, dt = 0$$

$$\int_0^{2\pi} \cos^2(t)\, dt = \int_0^{2\pi} \sin^2(t)\, dt = \pi$$

The circulation around the circle $\mathcal{C}_h$ is (approximately) equal to the area enclosed by the circle times the expression $Q_x(a, b) - P_y(a, b)$, which we now define to be the **2D curl** of $\mathbf{F}$ at the point (a, b), writing:

$$\operatorname{curl}_{2D}(\mathbf{F})\Big|_{(a,b)} = Q_x(a,b) - P_y(a,b) = \lim_{h \to 0^+} \frac{1}{\pi h^2} \cdot \int_{\mathcal{C}_h} \mathbf{F} \cdot \mathbf{T}\, ds$$

For a circle of small radius, we can interpret the 2D curl as "circulation per unit of area." (Might this interpretation extend to other regions?)

Computing 2D Curl

Computing the curl of a 2D vector field is straightforward if you remember the formula — which should look familiar, as it is the same quantity we computed in the Mixed Partials Test for gradient fields!

More on this soon (in the next section).

Example 1. If $\mathbf{F}(x,y) = \langle xy, x+y \rangle$, compute $\text{curl}_{2D}(\mathbf{F})$ and evaluate it at $(1,1)$, $(1.8,-1)$ and $(-1,-1)$.

Solution. Applying the 2D-curl formula:

$$\text{curl}_{2D}(\mathbf{F})(x,y) = \frac{\partial}{\partial x}[x+y] - \frac{\partial}{\partial y}[xy] = 1 - x$$

and evaluating yields $\text{curl}_{2D}(\mathbf{F})(1,1) = 0$, $\text{curl}_{2D}(\mathbf{F})(1.8,-1) = -0.8$ and $\text{curl}_{2D}(\mathbf{F})(-1,-1) = 2$. ◀

What can you surmise about the behavior of the vector field **F** from its 2D curl near each of these points?

Practice 1. Compute $\text{curl}_{2D}(\mathbf{G})$ and $\text{curl}_{2D}(\mathbf{H})$, evaluating each at $(1,1)$, $(1,0)$ and $(-1,-1)$, if $\mathbf{G}(x,y) = \left\langle x^2, y^2 \right\rangle$ and $\mathbf{H}(x,y) = \left\langle y^2, x^2 \right\rangle$.

Interpreting 2D Curl

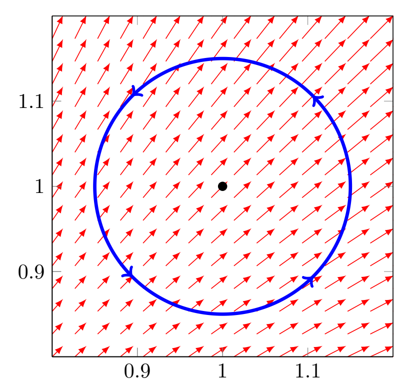
If $\mathcal{C}$ is a small circle centered at (a,b) and $\mathbf{F} = \langle P, Q \rangle$ has continuously differentiable component functions near (a,b) then:

$$\text{circulation of } \mathbf{F} \text{ around } \mathcal{C} \approx \text{curl}_{2D}(\mathbf{F})(a,b) \cdot (\text{area enclosed by } \mathcal{C})$$

Because any area is always positive, this tells us that the curl of **F** at (a,b) and the circulation of **F** along a circle of small radius centered at (a,b) must have the same sign.

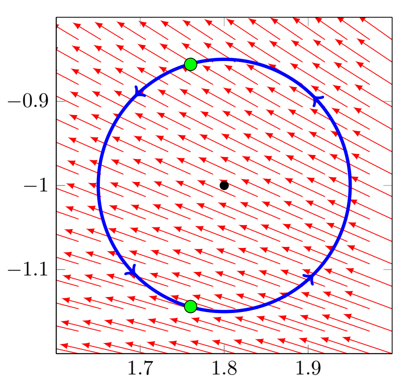
This implies that a paddlewheel of sufficiently small radius with its center at (a,b) will rotate in the positive direction if $\text{curl}_{2D}(\mathbf{F})(a,b) > 0$ (because, on average, $\mathbf{F} \cdot \mathbf{T} > 0$ along $\mathcal{C}$), and in the negative direction if $\text{curl}_{2D}(\mathbf{F})(a,b) < 0$ (because, on average, $\mathbf{F} \cdot \mathbf{T} < 0$ along $\mathcal{C}$).

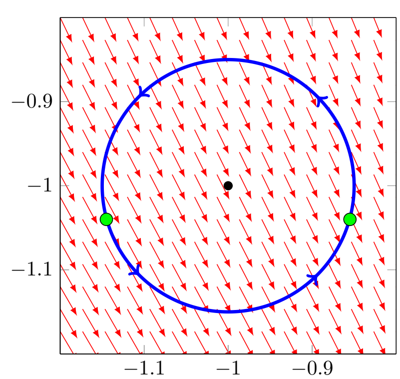
Example 2. Interpret the results of Example 1 with the aid of a graph.



Solution. A graph of $\langle xy, x+y \rangle$ together with a small circle centered at $(1,1)$ (see margin) reveals that, at any point on the circle, $\mathbf{F} \cdot \mathbf{T}$ appears to be of equal value but opposite sign compared with $\mathbf{F} \cdot \mathbf{T}$ at a symmetric point on the circle, so these values cancel: a paddlewheel would not rotate. This agrees with the result that $\text{curl}_{2D}(\mathbf{F})(1,1) = 0$.

Near $(1.8,-1)$ (see graph below left), $|\mathbf{F} \cdot \mathbf{T}|$ appears to be greater at points where $\mathbf{F} \cdot \mathbf{T} < 0$ than at corresponding points where $\mathbf{F} \cdot \mathbf{T} > 0$:





which agrees with the fact that $\text{curl}_{2D}(\mathbf{F})(1.8, -1) < 0$: a paddlewheel would rotate in the negative (clockwise) direction. And near $(-1, -1)$ (above right), the opposite appears to be true, which agrees with the fact that $\text{curl}_{2D}(\mathbf{F})(-1, -1) > 0$: a paddlewheel would rotate in the positive (counterclockwise) direction. ◀

These observations are not clearcut, which demonstrates the value of computing the curl to determine the direction of rotation (if any) of a paddlewheel.

Practice 2. Compute the 2D curl of the radial field $\mathbf{G}(x, y) = \langle x, y\rangle$ and the "swirl" field $\mathbf{H}(x, y) = \langle -y, x\rangle$, then interpret these values with the aid of a graph.

16.7 Problems

In Problems 1–12, compute the 2D curl of the given vector field.

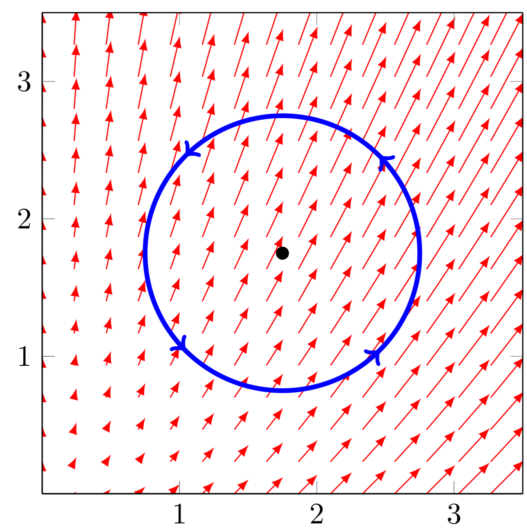
1. $\mathbf{F}(x, y) = \langle x, y\rangle$
2. $\mathbf{G}(x, y) = \langle y, -x\rangle$
3. $\mathbf{F}(x, y) = \langle y, -2x\rangle$
4. $\mathbf{G}(x, y) = \langle 4, 9\rangle$
5. $\mathbf{F}(x, y) = \langle -1 + 3x, 7 - 4y\rangle$
6. $\mathbf{G}(x, y) = \langle -1 + 3x^2, 7 - 4y^2\rangle$
7. $\mathbf{F}(x, y) = \langle -1 + 3y, 7 - 4x\rangle$
8. $\mathbf{G}(x, y) = \langle -1 + 3y^2, 7 - 4x^2\rangle$
9. $\mathbf{F}(x, y) = \langle 2 - y^3, \pi^4 + x^5\rangle$
10. $\mathbf{G}(x, y) = \langle \sin(xy), \cos(xy)\rangle$
11. $\mathbf{F}(x, y) = \langle x^3y^2 + \arctan(x), x^2y^3 - \ln(y^2 + 10)\rangle$
12. $\mathbf{G}(x, y) = \left\langle (x + y)^5, (x - y)^5\right\rangle$

In Problems 13–18, compute the 2D curl of the given vector field and evaluate it at the given points.

13. $\mathbf{F} = \langle x^2 + 3y, 2y + x\rangle$ at $(1, 1)$, $(2, -1)$ and $(1, 3)$
14. $\mathbf{G} = \langle xy^2, x^2y + 3\rangle$ at $(3, 2)$, $(0, 3)$ and $(1, 4)$
15. $\mathbf{F} = \langle 5x - 3y, x + 2y\rangle$ at $(3, 2)$, $(0, 3)$ and $(1, 4)$
16. $\mathbf{G} = \langle x^2 - y^2, x^2 + y^2\rangle$ at $(2, 3)$, $(-2, 2)$ and $(3, 1)$
17. $\mathbf{F} = \langle -3y, x \cdot y\rangle$ at $(3, 2)$, $(0, 3)$ and $(1, 4)$
18. $\mathbf{G} = \langle e^3, \pi^2\rangle$ at $(2, 3)$, $(-2, 2)$ and $(3, 1)$

In Problems 19–24, estimate whether the 2D curl of the vector field at the indicated point is positive, negative or approximately zero.

19.



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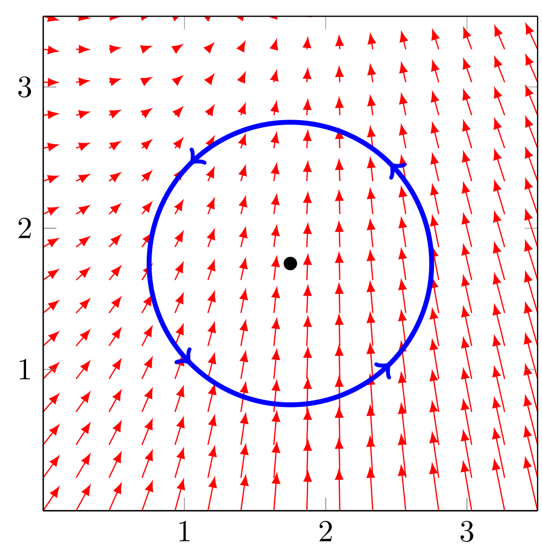

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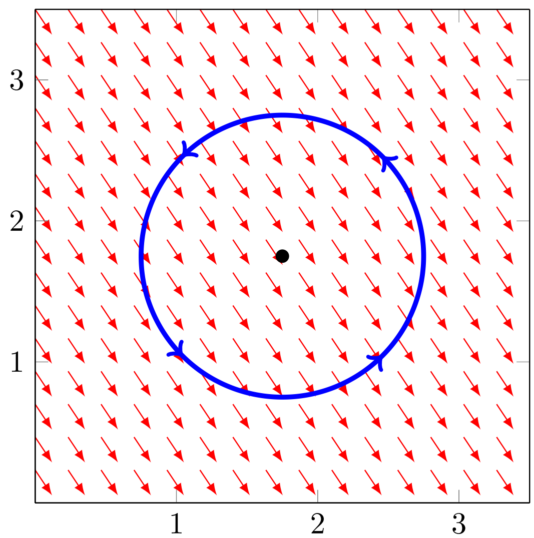

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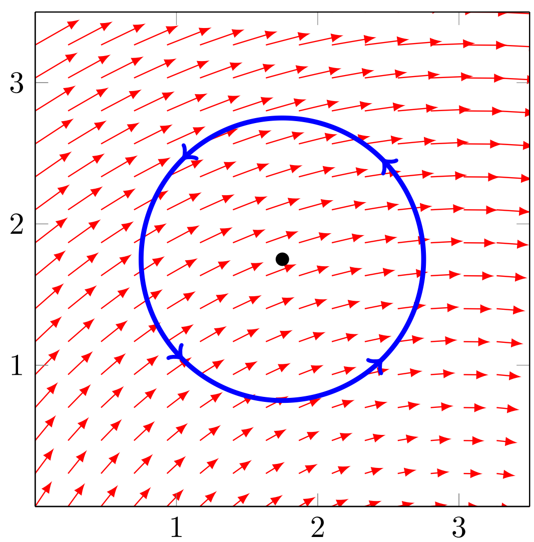

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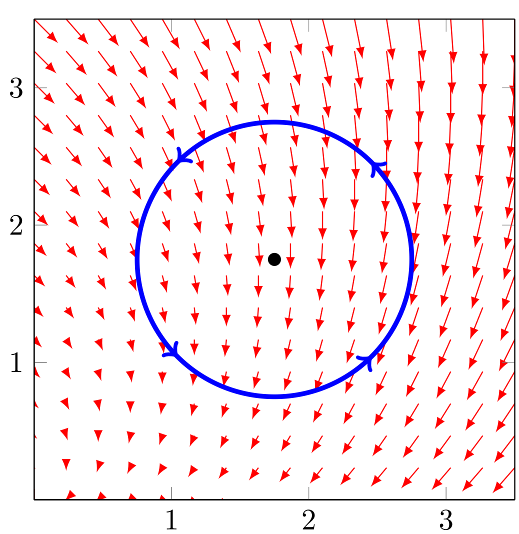

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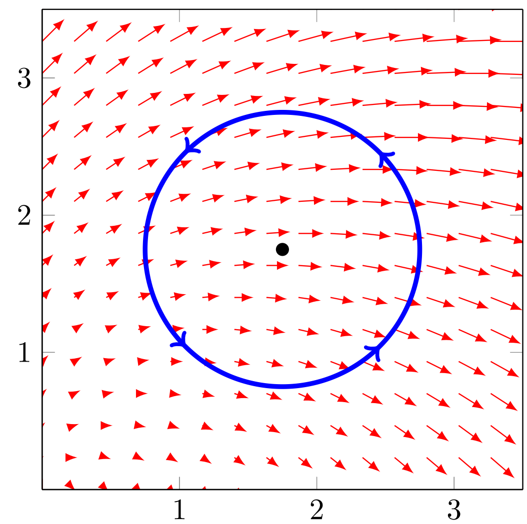


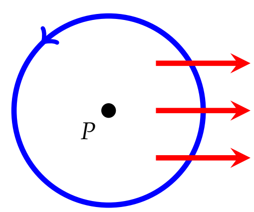
In Problems 25–30, a few vectors of a vector field $\mathbf{F}$ are shown near a point P. In each problem, draw additional vectors so that:

(a) $\text{curl}_{2D}(\mathbf{F})(P) > 0$

(b) $\text{curl}_{2D}(\mathbf{F})(P) < 0$

(c) $\text{curl}_{2D}(\mathbf{F})(P) \approx 0$

25.



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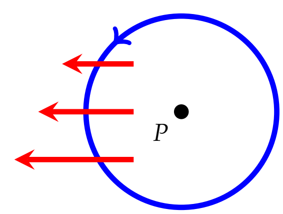

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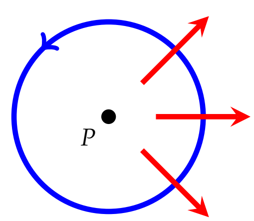

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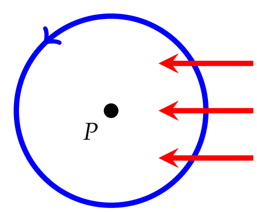

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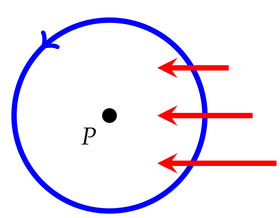

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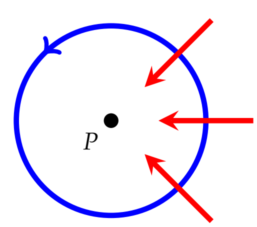


In 31–35, compute the 2D curl of the vector field, assuming f, g, φ and ψ are all differentiable.

31. $\mathbf{F}(x, y) = \langle f(x), g(y) \rangle$

32. $\mathbf{G}(x, y) = \langle \varphi(y), \psi(x) \rangle$

33. $\mathbf{F}(x,y) = \langle f(x) \cdot \varphi(y), g(x) \cdot \psi(y) \rangle$

34. $\mathbf{G}(x,y) = \langle f(x) + \varphi(y), g(x) + \psi(y) \rangle$

35. $\mathbf{G}(x,y) = \langle f(x+y), g(x-y) \rangle$

36. Show that, for constants a, b, c, α, β and γ, and the vector field:

$$\mathbf{F}(x,y) = \langle c + ax + by, \gamma + \alpha x + \beta y \rangle$$

the circulation of $\mathbf{F}$ across any rectangle $\mathcal{R}$ (with sides parallel to the coordinate axes) equals:

$$\text{curl}_{2D}(\mathbf{F}) \cdot (\text{area enclosed by } \mathcal{R})$$

What must be true about the constants a, b, c, α, β and γ if the circulation equals 0?

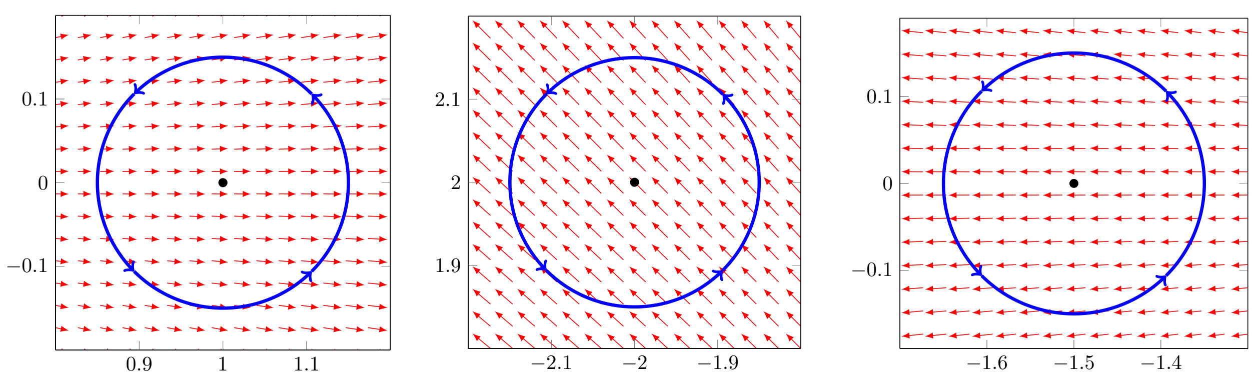
37. Given a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ and a circle $\mathcal{C}$ with center $A = (a,b)$ and small radius, consider what would happen if you imposed a different coordinate system (u,v). Would the value of the 2D curl of the vector field at A change? What about the value of the circulation of the vector field around the circle?

38. Given a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ and a circle $\mathcal{C}$ with center $A = (a,b)$ and small radius, consider what would happen if you measured distances in mm instead of cm. How would the value of 2D curl of the vector field at A change? What about the value of the circulation of the vector field around the circle?

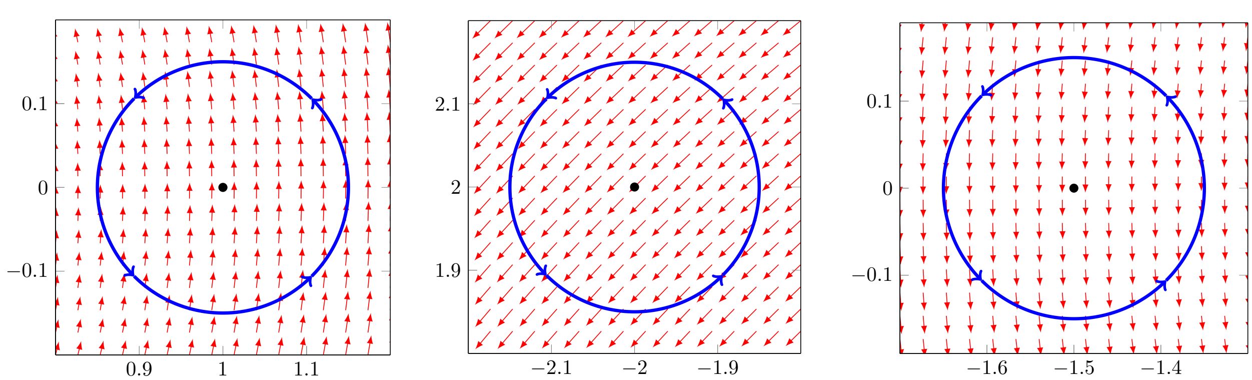
16.7 Practice Answers

1. $\text{curl}_{2D}(\mathbf{G})(x,y) = 0$ everywhere while $\text{curl}_{2D}(\mathbf{H})(x,y) = 2x - 2y$ so therefore $\text{curl}_{2D}(\mathbf{H})(1,1) = 0$ and $\text{curl}_{2D}(\mathbf{H})(1,0) = 2$ while $\text{curl}_{2D}(\mathbf{H})(-1,-1) = 0$

2. $\text{curl}_{2D}(\mathbf{G})(x,y) = 0$ everywhere, corresponding to graphs below, which indicate no rotation in either direction at any point:



$\text{curl}_{2D}(\mathbf{H})(x,y) = 2$ everywhere, corresponding to the graphs below, which indicate positive rotation of a paddlewheel at any point:



16.8 2D Curl Theorem

We have seen that, for a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ (for which P and Q have continuous derivatives near (a,b)) and a circle $\mathcal{C}_h$ of small radius $h > 0$ centered at (a,b):

$$\text{circulation of } \mathbf{F} \text{ along } \mathcal{C}_h \approx \text{curl}_{2D}(\mathbf{F}(a,b)) \cdot (\text{area enclosed by } \mathcal{C}_h)$$

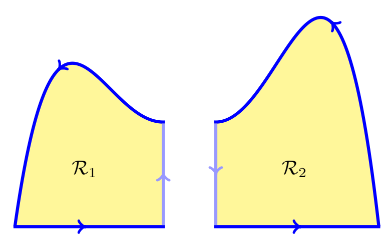
If this holds for more general closed curves $\mathcal{C}$, and if the 2D curl of $\mathbf{F}$ is roughly constant on $\mathcal{R}$, the region enclosed by $\mathcal{C}$, then we can write:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \approx \text{curl}_{2D}(\mathbf{F}(a,b)) \cdot \iint_{\mathcal{R}} 1\, dA$$
$$\Rightarrow \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{r} \approx \iint_{\mathcal{R}} \text{curl}_{2D}(\mathbf{F}(a,b))\, dA$$

Or, in differential form:

$$\int_{\partial\mathcal{R}} [P\,dx \;+\; Q\,dy] = \iint_{\mathcal{R}} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]_{(a,b)} dA$$

This might remind you of the 2D Divergence Theorem. You can, if you wish, verify that this result does in fact hold for reasonably nice vector fields on reasonably nice regions, first checking rectangles, then checking more general simple regions, and finally noting that when you splice together simple regions (as shown in the margin) the line integral across adjacent boundaries cancels. Here, along adjacent sides, the circulation flows one direction ("up") along the right side of $\mathcal{R}_1$ and the opposite direction ("down") along the left side of $\mathcal{R}_2$. These are in fact the same curve and computing the line integral along a curve in the opposite direction multiplies the line integral by -1, so the line integrals along these two adjacent sides cancel. But it turns out that there is a much easier way to prove what we will now call the:



2D Curl Theorem:

If: $\mathcal{R}$ is a finite union of closed, bounded simple regions in the xy-plane with $\partial\mathcal{R}$ a simple, closed, positively oriented curve, and $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ with P and Q both C^1 functions on an open region containing $\mathcal{R}$,

then:

$$\int_{\partial\mathcal{R}} [P\,dx \;+\; Q\,dy] = \iint_{\mathcal{R}} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

or, equivalently:

$$\int_{\partial\mathcal{R}} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{\mathcal{R}} \text{curl}_{2D}(\mathbf{F})\, dA$$

Remember that a C^1 function is differentiable and its derivatives are continuous. Also remember that we are tacitly assuming that our simple, closed boundary curve $\partial\mathcal{R}$ is piecewise smooth.

Some comments about this result are in order.

- The boundary curve $\mathcal{R}$ *must* be closed for this theorem to apply.
- We will often use this theorem to find circulation when the circulation integral is difficult to compute but the corresponding double integral of the 2D curl of the vector field is easier to set up (or to work out).

The theorem applies to the unit circle, $x^2 + y^2 = 1$ (which is closed), but not the top half of that circle, $y = \sqrt{1 - x^2}$.

- In almost every other textbook you will encounter, this result is called the "(circulation-curl form of) Green's Theorem," but "2D Curl Theorem" is a much better name for a variety of reasons.

Proof. Let $M = Q$ and $N = -P$ so that:

$$\int_{\partial\mathcal{R}} [P\,dx \;+\; Q\,dy] = \int_{\partial\mathcal{R}} [-N\,dx \;+\; M\,dy] = \int_{\partial\mathcal{R}} [M\,dy \;-\; N\,dx]$$

This last integral is a flux integral, so we can apply the 2D Divergence Theorem to get:

$$\int_{\partial\mathcal{R}} [M\,dy \;-\; N\,dx] = \iint_{\mathcal{R}} \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right] dA = \iint_{\mathcal{R}} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

which is what we needed to show. □

This proof shows that the 2D Divergence Theorem is, in some sense, equivalent to the 2D Curl Theorem, but note that we are applying each theorem to a *different* (but related) vector field.

Example 1. Compute the line integral of $\mathbf{F}(x,y) = \langle x - y, x\rangle$ along the (positively oriented) circle $x^2 + y^2 = 4$.

Solution. If $\mathbf{F}$ is a gradient field, this is an easy problem, but applying the Mixed Partials Test:

$$\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x - y) = 1 - (-1) = 2 \neq 0$$

so $\mathbf{F}$ is not a gradient field. However, the circle $x^2 + y^2 = 4$ is a closed curve (and is the boundary of $\mathcal{D}$, the closed and bounded disk $x^2 + y^2 \leq 4$, which happens to be a simple region) so we can apply the 2D Curl Theorem:

$$\int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \text{curl}_{2D}(\mathbf{F})\; dA = \iint_{\mathcal{D}} 2\,dA = 2 \cdot \pi \cdot 2^2 = 8\pi$$

You could also evaluate the line integral the "long way" by parameterizing the circle using $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ for $0 \leq t \leq 2\pi$ so that $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle$ and:

$$\begin{aligned}\int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 2\cos(t) - 2\sin(t), 2\cos(t)\rangle \cdot \langle -2\sin(t), 2\cos(t)\rangle\; dt \\ &= \int_0^{2\pi} [-4\sin(t)\cos(t) + 4]\; dt = 8\pi\end{aligned}$$

which agrees with the result of the 2D Curl Theorem. ◀

Practice 1. Compute the circulation of $\mathbf{F}(x,y) = \langle y, -x\rangle$ along $\mathcal{T}$, the (positively oriented) triangle with vertices at $(0,0)$, $(2,0)$ and $(2,2)$.

Practice 2. Compute the circulation of $\mathbf{F}(x,y) = \langle y, -x\rangle$ along any positively oriented triangle in the xy-plane with base b and height h.

Example 2. If $\mathcal{K}$ is the (positively oriented) rectangle with opposing vertices at $(0,0)$ and $(2,1)$, compute:

$$\int_{\mathcal{K}} \left[-y^2\,dx \;+\; x^2 y\,dy\right]$$

Solution. Applying the 2D Curl Theorem on the rectangular region $\mathcal{D}$ where $\partial\mathcal{D} = \mathcal{K}$, this integral becomes:

$$\iint_{\mathcal{D}} [2xy - (-2y)]\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=1} 2y\,(x+1)\; dy\, dx = 4$$

(Try evaluating the line integral directly to verify this answer.) ◀

Practice 3. If $\mathcal{K}$ is the rectangle from Example 2, compute:

$$\int_{\mathcal{K}} \left[\left(x^2 - y^2\right)\, dx \;+\; \left(x^2 y - 14y^3\right)\, dy\right]$$

Computing Area with the 2D Curl Theorem

We can express the area of any "nice" 2D region $\mathcal{R}$ using the double integral:

$$\iint_{\mathcal{R}} 1\, dA$$

If the 2D curl of a vector field $\mathbf{F} = \langle P, Q\rangle$ happens to equal 1 we could use the 2D Curl Theorem to rewrite this area integral as:

$$\iint_{\mathcal{R}} 1\, dA = \iint_{\mathcal{R}} \text{curl}_{2\text{D}}\,(\mathbf{F})\; dA = \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{r}$$

There are many vector fields for which $\text{curl}_{2\text{D}}\,(\mathbf{F}) = 1$, including:

$$\langle 0, x, \rangle\,, \quad \langle -y, 0\rangle \quad \text{and} \quad \left\langle -\frac{1}{2}y, \frac{1}{2}x\right\rangle$$

The third is the average of the first two.

so if $\mathcal{C}$ is any simple closed curve in the xy-plane, then the area of the region enclosed by $\mathcal{C}$ is equal to:

$$\int_{\mathcal{C}} x\, dy = \int_{\mathcal{C}} -y\, dx = \int_{\mathcal{C}} \left[-\frac{1}{2}y\, dx \;+\; \frac{1}{2}x\, dy\right]$$

In some situations, one or more of these line integrals might be easier to evaluate than the original double integral.

Example 3. Compute the area of $\mathcal{E}$, the elliptical region $\dfrac{x^2}{4} + \dfrac{y^2}{25} \leq 1$.

Solution. We can parameterize the boundary of this region, $\partial\mathcal{E}$, using $\mathbf{r}(t) = \langle 2\cos(t), 5\sin(t)\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -2\sin(t), 5\cos(t)\rangle$ for $0 \leq t \leq 2\pi$. Applying the 2D Curl Theorem:

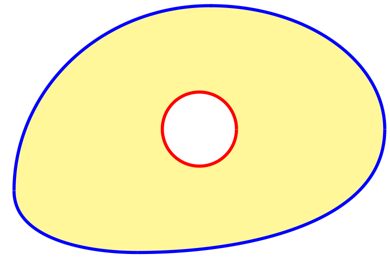
$$\begin{aligned}\iint_{\mathcal{E}} 1\, dA &= \int_{\partial\mathcal{E}} \left[-\frac{1}{2}y\, dx \;+\; \frac{1}{2}x\, dy\right] \\ &= \int_0^{2\pi} \left[-\frac{1}{2} \cdot 5\sin(t)\,(-2\sin(t)) + \frac{1}{2} \cdot 2\cos(t) \cdot 5\cos(t)\right]\; dt \\ &= 5\int_0^{2\pi} \left[\sin^2(t) + \cos^2(t)\right]\; dt = 5\int_0^{2\pi} 1\, dt = 5 \cdot 2\pi = 10\pi\end{aligned}$$

This agrees with the standard formula for the area of an ellipse. ◀

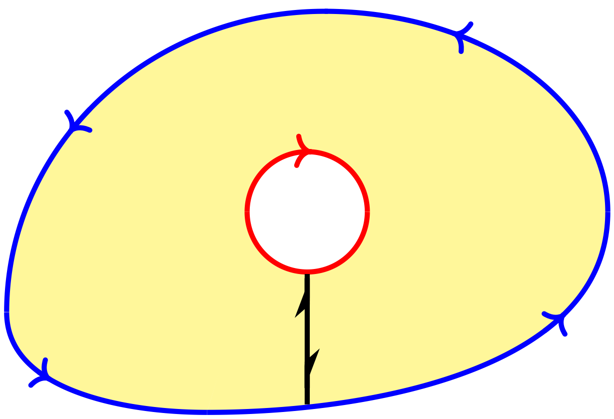
$\pi ab = \pi(2)(5) = 10\pi$

Practice 4. Compute the area of the region enclosed by the curve traced out by $\mathbf{r}(t) = \left\langle \cos^3(t), \sin^3(t)\right\rangle$ for $0 \leq t \leq 2\pi$.

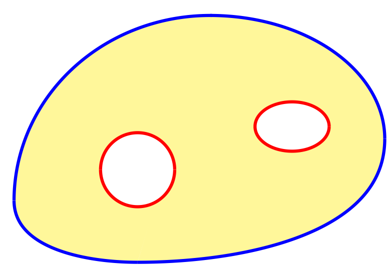
Regions with Holes



So far we have applied the 2D Divergence Theorem and 2D Curl Theorem to regions without "holes." Consider a region $\mathcal{R}$ that consists of points inside one closed curve but outside of one or more other closed curves contained inside the first (such as the region shown in the margin). We can create a single curve that traverses the entire boundary for $\mathcal{R}$ and consistently keeps the rightward-pointing unit normal vector $\mathbf{n}$ pointing "away" from $\mathcal{R}$ as we traverse the curve in the direction of its orientation by adding paths that connect the inner parts of $\partial\mathcal{R}$ to the outer boundary of $\partial\mathcal{R}$, as shown below:



To traverse the entire boundary, start at the bottom of the vertical line segment, go up to the inner circle, around that circle once (in the clockwise direction), then back down the vertical line segment, and finally around the outer boundary curve (in the counterclockwise direction). Note that, by going up the vertical line segment and then back down, the line integral of any vector field along these two vertical paths will cancel. Further note that the orientation of the inner circle is clockwise so that when we move along this circle, the unit normal vector $\mathbf{n}$ will point to the right.



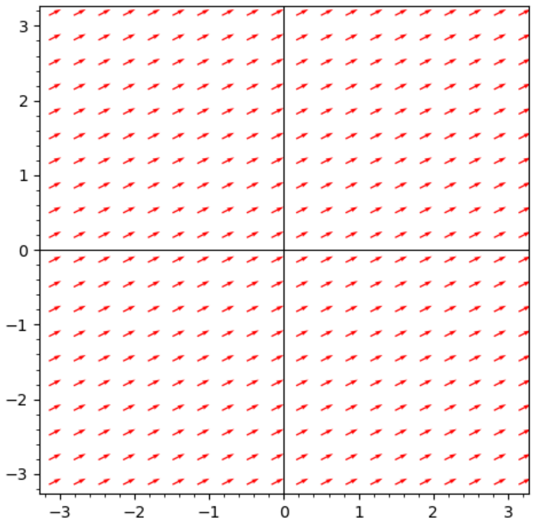
Practice 5. For the region $\mathcal{R}$ shown in the margin, indicate the orientation of all boundary curves so that $\partial\mathcal{R}$ has positive orientation, then add additional oriented curves as necessary to create one "super curve" that traverses all of $\partial\mathcal{R}$ in the proper direction exactly once (and any additional curves exactly twice, but in opposite directions).

A
Answers

Important Note about Precision of Answers: In many of the problems in this book you are required to read information from a graph and to calculate with that information. You should take reasonable care to read the graphs as accurately as you can (a small straightedge is helpful), but even skilled and careful people make slightly different readings of the same graph. That is simply one of the drawbacks of graphical information. When answers are given to graphical problems, the answers should be viewed as the best approximations we could make, and they usually include the word "approximately" or the symbol "≈" meaning "approximately equal to." Your answers should be close to the given answers, but you should not be concerned if they differ a little. (Yes those are vague terms, but it is all we can say when dealing with graphical information.)

Section 16.1

1.



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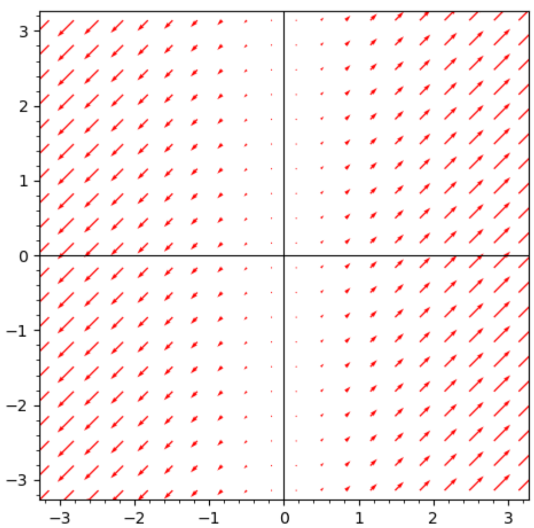

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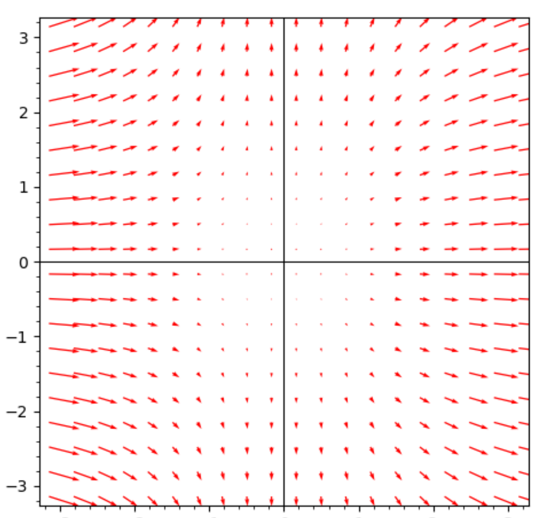

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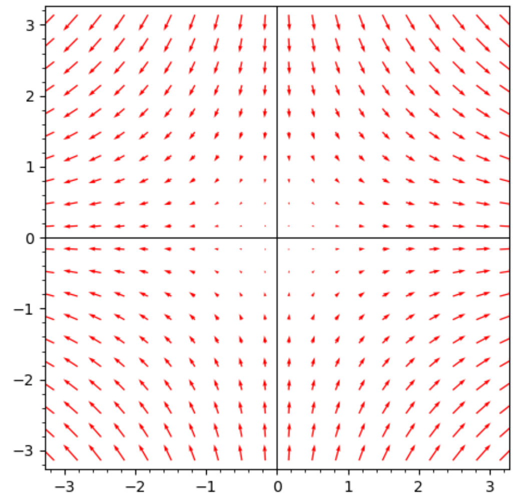
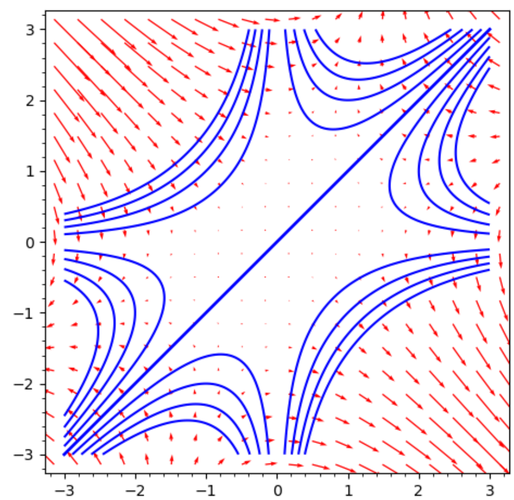


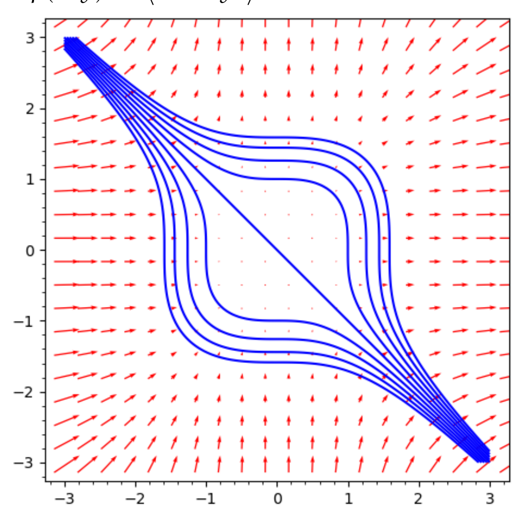
9. One possibility: $\langle 2, 0 \rangle$ 11. $\langle -x, -y \rangle$

13. The only possibility: $\left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$

15. $\nabla\varphi(x,y) = \langle y^2 - 2xy, 2xy - x^2 \rangle$:



17. $\nabla\varphi(x,y) = \langle 3x^2, 3y^2 \rangle$:



19. $\varphi_x = 3x^2+4 \Rightarrow \varphi = x^3+4x+g(y)$ so that $\varphi_y = g'(y) = 6 \Rightarrow g(y) = 6y+C$, hence $\varphi(x,y) = x^3+4x+6y$ works.

21. $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2x^2 \neq 0$ so $\mathbf{H}$ is not a gradient field.

23. $\varphi(x,y) = x \cdot \sin(y)$

25. $x(t) = 3t + A$, $y(t) = -2t + B$

27. $x(t) = \frac{1}{2}t^2 + At + B$, $y(t) = t + A$

29. $x(t) = \sin(t)$, $y(t) = \cos(t)$

Section 16.2

1. $P\Delta y - Q\Delta x = 5\cdot 5 - 12\cdot 0 = 25$

3. $P\Delta y - Q\Delta x = 5\cdot 0 - 12\cdot(-5) = 60$

5. With $P=-7$, $Q=10$ and $y = 2+x+x^2 \Rightarrow dy = (1+2x)\,dx$:

$$\int_C [P\,dy \;-\; Q\,dx] = \int_0^2 [-7\,(1+2x) - 10]\;dx = \int_0^2 [-17-14x]\;dx = \left[-17x-7x^2\right]_0^2 = -62$$

7. With $P=2x$, $Q=3y$ and $x = 2+y+y^2 \Rightarrow dx = (1+2y)\,dy$:

$$\int_C [P\,dy \;-\; Q\,dx] = \int_{-2}^1 \left[2\left(2+y+y^2\right) - 3y\,(1+2y)\right]\;dy = \int_{-2}^1 \left[4-y-4y^2\right]\;dy = \frac{3}{2}$$

9. With $P=-y$, $Q=x$, $x=\cos(t) \Rightarrow dx = -\sin(t)\,dt$ and $y = \sin(t) \Rightarrow dy = \cos(t)\,dt$:

$$\int_C [P\,dy \;-\; Q\,dx] = \int_0^{\frac{\pi}{2}} [-\sin(t)\cdot\cos(t) - \cos(t)\,(-\sin(t))]\;dt = \int_0^{\frac{\pi}{2}} 0\,dt = 0$$

11. With $P = 2xy^2$, $Q = 2x^2y$, $x=t^2 \Rightarrow dx = 2t\,dt$ and $y=t^3 \Rightarrow dy = 3t^2\,dt$:

$$\int_C [P\,dy \;-\; Q\,dx] = \int_1^3 \left[2t^2\left(t^3\right)^2\left(t^2\right) - 2\left(3t^2\right)^2\cdot t^3\,(2t)\right]\;dt = \int_1^3 \left[6t^{10}-4t^8\right]\;dt = 87877.\overline{53}$$

13. $\varphi = x^3y + 4xy^2 \Rightarrow P = 3x^2y+4y^2$, $Q = x^3+8xy$, so with $y=2x \Rightarrow dy = 2\,dx$:

$$\int_C [P\,dy \;-\; Q\,dx] = \int_0^2 \left[\left(3x^2\cdot 2x + 4\,(2x)^2\right)\cdot 2 - \left(x^3+8x\cdot 2x\right)\right]\;dx = \int_0^2 \left[11x^3+16x^2\right]\;dx = \frac{260}{3}$$

15. With $y=x \Rightarrow dy = dx$ for $-2\le x\le 3$: $\displaystyle\int_C [5\,dy \;-\; 3\,dx] = \int_{-2}^3 2\,dx = 10$

17. With $y=x \Rightarrow dy = dx$ for $-2\le x\le 3$: $\displaystyle\int_C [7\,dy \;+\; 5\,dx] = \int_{-2}^3 12\,dx = 60$

19. With $y = \dfrac{7}{4}x \Rightarrow dy = \dfrac{7}{4}\,dx$ for $0\le x\le 4$: $\displaystyle\int_C [x\,dy \;-\; y\,dx] = \int_0^4 \left[x\cdot\frac{7}{4} - \frac{7}{4}x\right]\;dx = 0$

21. $y = x^2+7 \Rightarrow dy = 2x\,dx$ for $-1\le x\le 1$: $\displaystyle\int_C \left[x^2\,dy \;-\; y^2\,dx\right] = \int_{-1}^1 \left[x^2\cdot 2x - \left(x^2+7\right)^2\right]\;dx = -\frac{1616}{15}$

23. (a) $\displaystyle\int_0^1 [2\cdot 6 - 5\cdot 3]\;dt = -3$ (b) $\displaystyle\int_0^1 [2\cdot 12t - 5\cdot 6t]\;dt = \int_0^1 -6t\,dt = -3$

Section 16.3

1. $\operatorname{div}(\mathbf{F})(x,y) = \partial_x[x] + \partial_y[y] = 1 + 1 = 2$
3. $\operatorname{div}(\mathbf{F})(x,y) = \partial_x[-y] + \partial_y[x] = 0 + 0 = 0$
5. $\partial_x[-1+3x] + \partial_y[7-4y] = 3 - 4 = -1$
7. $\partial_x[-1+3y] + \partial_y[7-4x] = 0 + 0 = 0$
9. $\partial_x\left[2-x^3\right] + \partial_y\left[pi^4 + y^5\right] = -3x^2 + 5y^4$
11. $\operatorname{div}(\mathbf{F})(x,y) = 3x^2y^2 + 3x^2y^2 = 6x^2y^2$
13. $\operatorname{div}(\mathbf{F})(x,y) = 2x + 2$ so that $\operatorname{div}(\mathbf{F})(1,1) = 4$, $\operatorname{div}(\mathbf{F})(2,-1) = 6$ and $\operatorname{div}(\mathbf{F})(1,3) = 4$
15. $\operatorname{div}(\mathbf{F})(x,y) = 5 + 2 = 7$ so $\operatorname{div}(\mathbf{F})(1,1) = 7$, $\operatorname{div}(\mathbf{F})(2,-1) = 7$ and $\operatorname{div}(\mathbf{F})(1,3) = 7$
17. $\operatorname{div}(\mathbf{F})(x,y) = 0 + x = x$ so $\operatorname{div}(\mathbf{F})(3,2) = 3$, $\operatorname{div}(\mathbf{F})(0,3) = 0$ and $\operatorname{div}(\mathbf{F})(1,4) = 1$

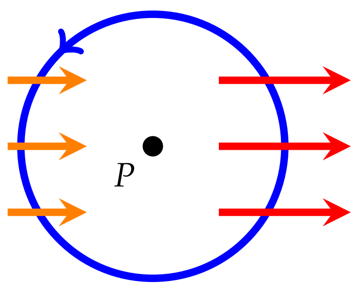
19. $\operatorname{div}(\mathbf{F}) > 0$ 21. $\operatorname{div}(\mathbf{F}) \approx 0$ 23. $\operatorname{div}(\mathbf{F}) < 0$

25. On your own.

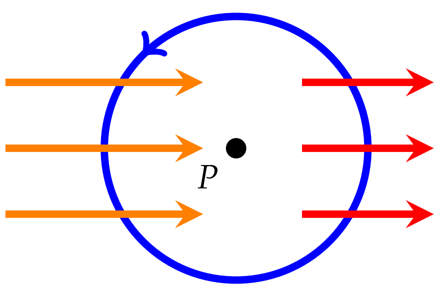
27. 0 29. $f'(x) + \psi'(y)$

31. Some possibilities:

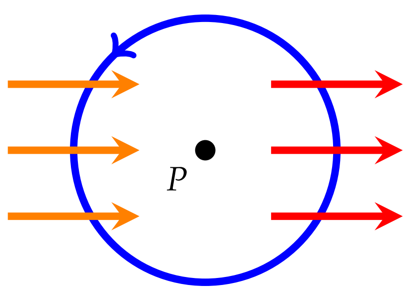
(a)



(b)

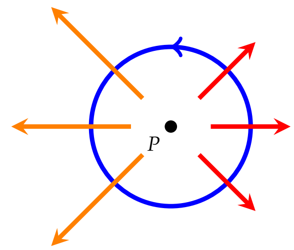


(c)

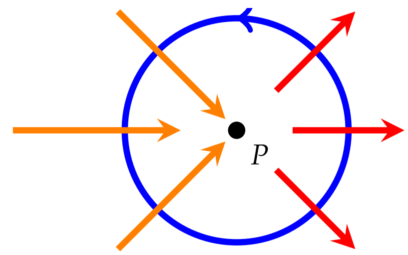


33. Some possibilities:

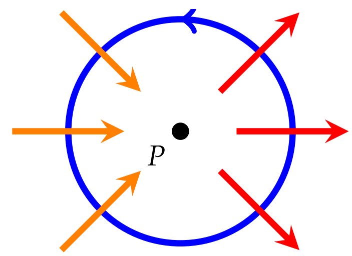
(a)



(b)

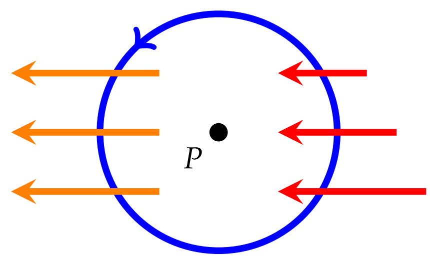


(c)

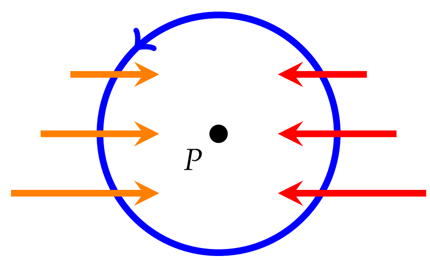


35. Some possibilities:

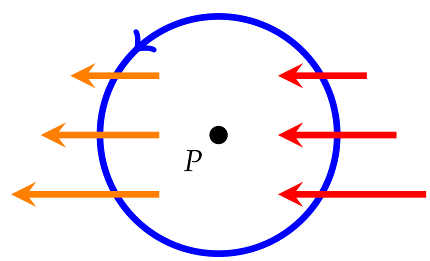
(a)



(b)



(c)



37. Both would remain the same.

Section 16.4

1. If $\mathcal{D}$ is the interior of the circle, applying the 2D Divergence Theorem yields:

$$\int_{\mathcal{C}} \left[y^2\,dy \;-\; x^2\,dx\right] = \iint_{\mathcal{D}} \operatorname{div}\left(\left\langle y^2, x^2\right\rangle\right)\,dA = \iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}\left(y^2\right) + \frac{\partial}{\partial y}\left(x^2\right)\right]\,dA = \iint_{\mathcal{D}} 0\,dA = 0$$

3. If $\mathcal{D}$ is the interior of the unit circle, applying the 2D Divergence Theorem yields:

$$\int_{\mathcal{C}} [5x\,dy \;-\; 8y\,dx] = \iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(8y)\right]\,dA = \iint_{\mathcal{D}} 13\,dA = 13\cdot\pi\cdot 1^2 = 13\pi$$

5. If $\mathcal{R}$ is the rectangular region with $\partial\mathcal{R} = \mathcal{C}$, applying the 2D Divergence Theorem yields:

$$\int_{\mathcal{C}} [(17x - 5y)\,dy \;-\; (8x + 11y)\,dx] = \iint_{\mathcal{D}} [17 + 11]\,dA = 28(13)(9) = 3276$$

7. If $\mathcal{D}$ is the interior of the circle, applying the 2D Divergence Theorem yields:

$$\int_{\mathcal{C}} \left[x^3\,dy \;-\; y^3\,dx\right] = \iint_{\mathcal{D}} \operatorname{div}\left(\left\langle x^3, y^3\right\rangle\right)\,dA = \iint_{\mathcal{D}} \left[3x^2 + 3y^2\right]\,dA = 0$$

Using polar coordinates, this integral becomes:

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4} 3r^2\cdot r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left[\frac{3}{4}r^4\right]_{r=0}^{r=4}\,d\theta = \int_0^{2\pi} 192\,d\theta = 384\pi$$

9. Applying the 2D Divergence Theorem:

$$\int_{\partial\mathcal{R}} \left[2x^3\,dy \;-\; 4y^2\,dx\right] = \iint_{\mathcal{R}} \left[6x^2 + 8y\right]\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \left[6x^2 + 8y\right]\,dy\,dx = \frac{11}{6}$$

11. Applying the 2D Divergence Theorem: $\displaystyle\int_{\partial\mathcal{R}} \left[x^2y\,dy \;-\; xy^2\,dx\right] = \int_{x=-3}^{x=3}\int_{y=0}^{y=9-x^2} 4xy\,dA = 0$

13. Applying the 2D Divergence Theorem:

$$\int_{\partial\mathcal{R}} \left[x^2y\,dy \;-\; y\,dx\right] = \iint_{\mathcal{R}} [2xy + 1]\,dA = \int_{x=-3}^{x=3}\int_{y=0}^{y=9-x^2} [2xy + 1]\,dy\,dx = 36$$

15. If $\mathcal{C}$ is the unit circle and $\mathcal{D}$ its interior, applying the 2D Divergence Theorem:

$$\int_{\mathcal{C}} [x\,dy \;-\; y\,dx] = \iint_{\mathcal{D}} [2]\,dA = 2\cdot\pi\cdot 1^2 = 2\pi$$

17. If $\mathcal{C}$ is the unit circle and $\mathcal{D}$ its interior, applying the 2D Divergence Theorem:

$$\int_{\mathcal{C}} \left[(x^3 - 7y)\,dy \;-\; (8x + y^3)\,dx\right] = \iint_{\mathcal{D}} \left[3x^2 + 3y^2\right]\,dA = \int_0^{2\pi}\int_0^1 3r^2\cdot r\,dr\,d\theta = \frac{3\pi}{2}$$

19. The divergence of the vector field is $-5 + 14 = 9$ so the flux is $9\cdot 100\pi^2 = 900\pi^2$.

21. $64\cdot\pi\left(\sqrt{10}\right)^2 = 640\pi$.

23. By symmetry, the integral of y over a disk centered at the origin is 0.

Section 16.5

1. $\mathbf{F} \cdot \mathbf{T} > 0$ everwhere along curve $\mathcal{A}$ so the work should be positive.
3. $\mathbf{F} \cdot \mathbf{T} > 0$ everwhere along curve $\mathcal{C}$ so the work should be positive.
5. $\mathbf{F} \cdot \mathbf{T} < 0$ everwhere along curve $\mathcal{A}$ so the work should be negative.
7. $\mathbf{F} \cdot \mathbf{T} > 0$ everwhere along curve $\mathcal{C}$ so the work should be positive.
9. $\int_0^3 \langle 2+3t, 2+7t\rangle \cdot \langle 3,4\rangle \, dt = \int_0^3 [14+37t] \, dt = 208.5$
11. With $\mathbf{r}(t) = \langle 1+t, 3+4t\rangle$ for $0 \le t \le 1$: $\int_0^1 \langle 1+t, 1+t\rangle \cdot \langle 1,4\rangle \, dt = \int_0^1 [5+5t] \, dt = 7.5$
13. With $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$: $\int_0^{\frac{\pi}{2}} \langle \cos(t), \sin(t)\rangle \cdot \langle -3\sin(t), 3\cos(t)\rangle \, dt = 0$
15. With $\mathbf{r}(t) = \langle 2-t, 2-t\rangle$ for $0 \le t \le 1$:

$$\int_0^1 \left\langle \frac{2-t}{\sqrt{(2-t)^2+(2-t)^2}}, \frac{2-t}{\sqrt{(2-t)^2+(2-t)^2}} \right\rangle \cdot \langle -1,-1\rangle \, dt = \int_0^1 -\sqrt{2}\, dt = -\sqrt{2}$$

17. With $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$: $\int_0^{\frac{\pi}{2}} \langle -\sin(t), \cos(t)\rangle \cdot \langle -3\sin(t), 3\cos(t)\rangle \, dt = \frac{3\pi}{2}$
19. With $\mathbf{r}(t) = \langle 2-t, 2-t\rangle$ for $0 \le t \le 1$:

$$\int_0^1 \left\langle \frac{-(2-t)}{\sqrt{(2-t)^2+(2-t)^2}}, \frac{2-t}{\sqrt{(2-t)^2+(2-t)^2}} \right\rangle \cdot \langle -1,-1\rangle \, dt = \int_0^1 0\, dt = 0$$

21. With $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$ for $0 \le t \le 2\pi$: $\int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -4sin(t), 4\cos(t)\rangle \, dt = 8\pi$
23. Along the bottom side of the square $y = 0 \Rightarrow dy = 0$, so the line integral evaluates to 0. Along the right side, $x = 1 \Rightarrow dx = 0$ so here the integral becomes:

$$\int_{y=0}^{y=1} [0 + 3y\, dy] = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$$

Along the top side of the square $y = 1 \Rightarrow dy = 0$ so here the integral becomes:

$$\int_{x=1}^{x=0} [5 \cdot 1\, dx + 0] = -5$$

And along the left side of the square $x = 0 \Rightarrow dx = 0$ so here the integral becomes:

$$\int_{y=1}^{y=0} [0 + 3y\, dy] = -\frac{3}{2}$$

Summing these values yields $0 + \frac{3}{2} - 5 - \frac{3}{2} = -5$.

25. Along the parabola portion of $\mathcal{C}$, $y = 1 - x^2 \Rightarrow dy = -2x\, dx$, so the line integral becomes:

$$\int_{x=1}^{x=-1} \left[5(1-x^2) + 3x(-2x)\right] \, dx = \int_1^{-1} \left[5 - 11x^2\right] \, dx \left[5x - \frac{11}{3}x^3\right]_1^{-1} = -\frac{8}{3}$$

And along the line-segment portion, $y = 0 \Rightarrow dy = 0$, so the line integral becomes:

$$\int_{x=-1}^{x=1} [0\, dx + 3x \cdot 0] \, dx = 0$$

hence the value of the integral around the closed curve is $-\frac{8}{3}$.

27. With $\mathbf{r}(t) = \langle 3\cos(t), 2\sin(t)\rangle$ for $0 \le t \le 2\pi$: $\displaystyle\int_0^{2\pi} \left[8\sin^3(t)\left(-3\sin(t)\right) + 27\cos^3(t)\cdot 2\cos(t)\right]\, dt = \frac{45\pi}{2}$

29. $\displaystyle\int_0^{10} 5x\,dx + \int_0^{12} (110+8y)\, dy + \int_{10}^0 (5x - 17\cdot 12)\, dx + \int_{12}^0 8y\,dy = 3360$

31. $\displaystyle\int_0^L \alpha x\,dx + \int_0^H (\gamma L + \delta y)\, dy + \int_L^0 (\alpha x \beta H)\, dx + \int_H^0 \delta y\,dy = (\gamma - \beta)\, LH$

33. With $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$ for $0 \le t \le 2\pi$: $\displaystyle\int_0^{2\pi} \left[64\cos^3(t)\left(-4\sin(t)\right) + 64\sin^3(t)\cdot 4\cos(t)\right]\, dt = 0$

35. With $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$ for $0 \le t \le 2\pi$: $\displaystyle\int_0^{2\pi} \left[64\sin^3(t)\left(-4\sin(t)\right) + 64\cos^3(t)\cdot 4\cos(t)\right]\, dt = 0$

37. $\displaystyle\int_0^1 x^3\,dx + \int_0^1 \left(1+y^3\right)\, dy + \int_1^0 \left(2x^3 + x\right)\, dx = \frac{1}{2}$

39. With $\mathbf{r}(t) = \left\langle t, t^2\right\rangle$ for $1 \le t \le 2$: $\displaystyle\int_1^2 (1+t)\sqrt{1+4t^2}\,dt \approx 8.0772$

41. With $\mathbf{r}(t) = \left\langle t, t^3\right\rangle$ for $0 \le t \le 2$: $\displaystyle\int_0^2 \left(1+t^3\right)\sqrt{1+9t^4}\,dt \approx 40.9457$

Section 16.6

1. $\varphi(7,6) - \varphi(1,2) = 144$
3. A potential function is $\varphi(x,y) = \frac{1}{2}x^4 + 5y$ so $\varphi(3,27) - \varphi(1,3) = 160$
5. The vector field is not conservative. With $\mathbf{r}(t) = \langle 10+10t, 10+20t\rangle$ for $0 \le t \le 1$:

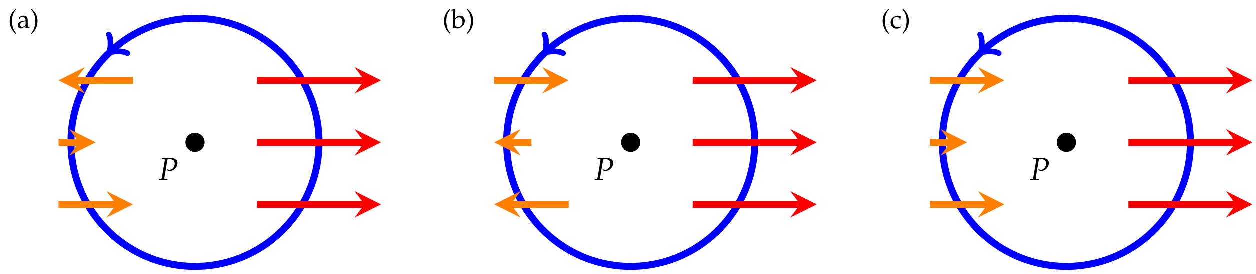
$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 \langle 7, 4(10+10t)\rangle \cdot \langle 10, 20\rangle\, dt = 1270$$

7. A potential function is $\varphi(x,y) = x^2 + y^2$ so the work done is $\varphi(5,1) - \varphi(1,2) = 21$
9. The vector field is not conservative. Along the line segment from A to B the work is 10.5; along the path along the line semgment from A to $(3,2)$ followed by the line segment from $(3,2)$ to B the work is 16.5.
11. A potential function is $\varphi(x,y) = x^3 y$ so the work done is $\varphi(5,1) - \varphi(1,2) = 123$
13. The vector field is the gradient of $x^4 + y^5$ so the line integral around the closed curve evaluates to 0.
15. The vector field is the gradient of $x^2\cos(y)$ so the line integral evaluates to $\left[x^2\cos(y)\right]_{(1,\pi)}^{(9,\pi)} = -80.$
17. The vector field is the gradient of $2x^3y^2$ so the line integral evaluates to $2\cdot 3^3\cdot 5^2 - 2\cdot 1^3\cdot 2^2 = 1342.$
19. The vector field is the gradient of $2x^2 + 5xy + y^4$ so the line integral evaluates to 20.

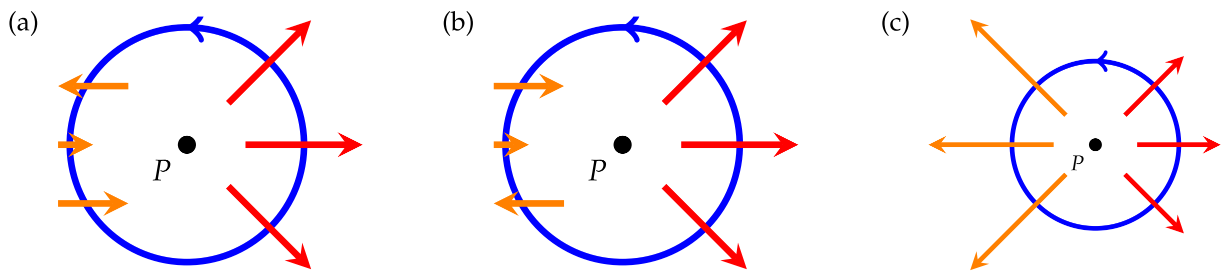
Section 16.7

1. $\partial_x(y) - \partial_y(x) = 0 - 0 = 0$
3. $\partial_x(-2x) - \partial_y(y) = -2 - 1 = -3$
5. $\partial_x(7-4y) - \partial_y(-1+3x) = 0 - 0 = 0$
7. $\partial_x(7-4x) - \partial_y(-1+3y) = -4 - 3 = -7$
9. $\partial_x\left(\pi^4 + x^5\right) - \partial_y\left(2 - y^3\right) = 5x^4 + 3y^2$
11. $2xy^3 - 2x^3y$
13. $\text{curl}_{2D}(\mathbf{F})(x,y) = 1 - 3 = -2$ everywhere
15. $\text{curl}_{2D}(\mathbf{F})(x,y) = 1 - (-3) = 4$ everywhere
17. $\text{curl}_{2D}(\mathbf{F})(x,y) = y - (-3) = y + 3$ so that $\text{curl}_{2D}(\mathbf{F})(1,2) = 5$, $\text{curl}_{2D}(\mathbf{F})(0,3) = 6$ and $\text{curl}_{2D}(\mathbf{F})(1,4) = 7$
19. positive
21. (approximately) 0
23. negative

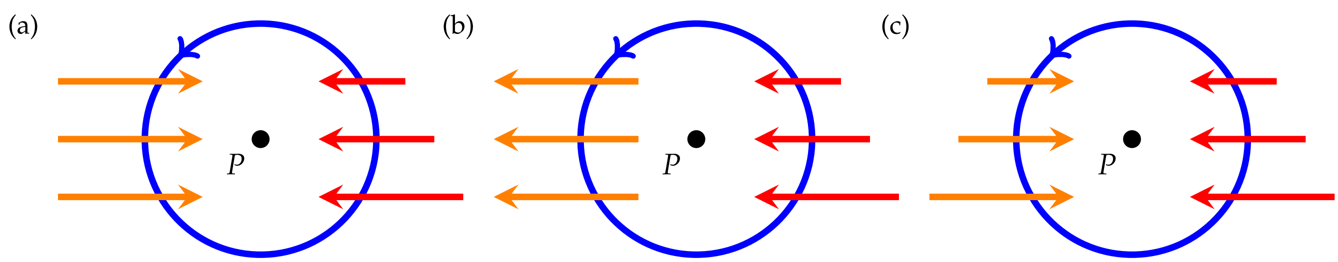
25. Some possibilities:



27. Some possibilities:



29. Some possibilities:



31. 0

33. $g'(x) \cdot \psi(y) - f(x) \cdot \varphi'(y)$

35. $g'(x-y) - f'(x+y)$

37. Both would remain the same.