

2.4 THE CHAIN RULE

The Chain Rule is the most important and most used of the differentiation patterns. It enables us to differentiate **composites** of functions such as $y = \sin(x^2)$. It is a powerful tool for determining the derivatives of some **new functions** such as logarithms and inverse trigonometric functions. And it leads to important **applications** in a variety of fields. You will need the Chain Rule hundreds of times in this course, and practice with it now will save you time and points later. Fortunately, with some practice, the Chain Rule is also easy to use.

We already know how to differentiate the composition of some functions.

Example 1: For $f(x) = 5x - 4$ and $g(x) = 2x + 1$, find $f \circ g(x)$ and $D(f \circ g(x))$.

Solution: $f \circ g(x) = f(g(x)) = 5(2x+1) - 4 = 10x + 1$, so $D(f \circ g(x)) = D(10x + 1) = 10$.

Practice 1: For $f(x) = 5x - 4$ and $g(x) = x^2$, find $f \circ g(x)$, $D(f \circ g(x))$, $g \circ f(x)$, and $D(g \circ f(x))$.

Some compositions, however, are still very difficult to differentiate. We know the derivatives of $g(x) = x^2$ and $h(x) = \sin(x)$, and we know how to differentiate some combinations of these functions such as $x^2 + \sin(x)$, $x^2 \cdot \sin(x)$, and even $\sin^2(x)$, but the derivative of the simple composition $f(x) = h \circ g(x) = \sin(x^2)$ is hard — until we know the Chain Rule. To see just how hard, try using the definition of derivative on it.

Example 2:

- (a) Suppose amplifier Y doubles the strength of the output signal from amplifier U, and U triples the strength of the original signal x. How does the final signal out of Y compare with the signal x?

original signal $x \rightarrow$ amplifier U $\rightarrow$ amplifier Y $\rightarrow$ final signal

- (b) Suppose y changes twice as fast as u, and u changes three times as fast as x. How does the rate of change of y compare with the rate of change of x?

Solution: In each case we are comparing the result of a composition, and the answer to each question is 6, the product of the two amplifications or rates of change.

In (a), we have that $\frac{\text{signal out of Y}}{\text{signal } x} = \frac{\text{signal out of Y}}{\text{signal out of U}} \cdot \frac{\text{signal out of U}}{\text{signal } x} = (2)(3) = 6$.

In (b), $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = (2)(3) = 6$.

These examples are simple cases of the Chain Rule for differentiating a composition of functions.

THE CHAIN RULE

Chain Rule (Leibniz notation form)

If y is a differentiable function of u, and u is a differentiable function of x,

then y is a differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Idea for a proof: $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad (\text{if } \Delta u \neq 0)$

$$= \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \right) \cdot \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right) \quad \text{u is continuous, so } \Delta x \rightarrow 0 \text{ implies } \Delta u \rightarrow 0$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

Although this nice short argument gets to the heart of why the Chain Rule works, it is not quite valid. If $du/dx \neq 0$, then it is possible to show that $\Delta u \neq 0$ for all very small values of Δx , and the "idea for a proof" is a real proof. There are, however, functions for which $\Delta u = 0$ for lots of small values of Δx , and these create problems for the previous argument. A justification which is true for ALL cases is much more complicated.

The symbol $\frac{dy}{du}$ is a single symbol (as is $\frac{du}{dx}$), and we cannot eliminate du from the product $\frac{dy}{du} \cdot \frac{du}{dx}$ in the Chain Rule. It is, however, perfectly fine to use the idea of eliminating du to help you remember the statement of the Chain Rule.

Example 3: $y = \cos(x^2 + 3)$ is $y = \cos(u)$ with $u = x^2 + 3$. Find dy/dx .

Solution: $y = \cos(u)$ so $dy/du = -\sin(u)$. $u = x^2 + 3$ so $du/dx = 2x$. Finally, using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot 2x = -2x \cdot \sin(x^2 + 3).$$

Practice 2: Find dy/dx for $y = \sin(4x + e^x)$.

There is also a composition form of the Chain Rule. The notation is different, but it means precisely the same as the Leibniz form.

Chain Rule (composition form)

If g is differentiable at x and f is differentiable at $g(x)$,

then the composite $f \circ g$ is differentiable at x , and $(f \circ g)'(x) = D(f(g(x))) = f'(g(x)) \cdot g'(x)$.

You may find it easier to think of the composition form of the Chain Rule in words:

$(f(g(x)))' =$ "the derivative of the outside function (with respect to the original inside function) times the derivative of the inside function" where f is the outside function and g is the inside function.

Example 4: Differentiate $\sin(x^2)$.

Solution: The function $\sin(x^2)$ is the composition $f \circ g$ of two simple functions: $f(x) = \sin(x)$ and $g(x) = x^2$: $f \circ g(x) = f(g(x)) = f(x^2) = \sin(x^2)$ which is the function we want. Both f and g are differentiable functions with derivatives $f'(x) = \cos(x)$ and $g'(x) = 2x$, so, by the Chain Rule,

$$D(\sin(x^2)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot 2x = \cos(x^2) \cdot 2x = 2x \cos(x^2).$$

If you tried using the definition of derivative to calculate the derivative of this function at the beginning of the section, you can really appreciate the power of the Chain Rule for differentiating compositions.

Example 5: The table gives values for f , f' , g and g' at a number of points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and 0 .

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-1	2	3	1	0		
0	-1	1	3	2		
1	1	0	-1	3		
2	3	-1	0	1		
3	0	2	2	-1		

Solution: $(f \circ g)(-1) = f(g(-1)) = f(3) = 0$ and $(f \circ g)(0) = f(g(0)) = f(1) = 1$.

$$(f \circ g)'(-1) = f'(g(-1)) \cdot g'(-1) = f'(3) \cdot (0) = (2)(0) = 0 \text{ and}$$

$$(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot (2) = (-1)(2) = -2.$$

Practice 3: Fill in the table in Example 5 for $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1, 2$ and 3 .

Neither form of the Chain Rule is inherently superior to the other — use the one you prefer. The Chain Rule will be used hundreds of times in the rest of this book, and it is important that you master its usage. The time you spend now mastering and understanding how to use the Chain Rule will be paid back tenfold in the next several chapters.

Example 6: Determine $D(e^{\cos(x)})$ using each form of the Chain Rule.

Solution: Using the Leibniz notation: $y = e^u$ and $u = \cos(x)$. $dy/du = e^u$ and $du/dx = -\sin(x)$ so

$$dy/dx = (dy/du) \cdot (du/dx) = (e^u) \cdot (-\sin(x)) = -\sin(x) \cdot e^{\cos(x)}.$$

The function $e^{\cos(x)}$ is also the composition of $f(x) = e^x$ with $g(x) = \cos(x)$, so

$$\begin{aligned} D(e^{\cos(x)}) &= f'(g(x)) \cdot g'(x) && \text{by the Chain Rule} \\ &= e^{g(x)} \cdot (-\sin(x)) && \text{since } D(e^x) = e^x \text{ and } D(\cos(x)) = -\sin(x) \\ &= -\sin(x) \cdot e^{\cos(x)}. \end{aligned}$$

Practice 4: Calculate $D(\sin(7x - 1))$, $\frac{d}{dx}(\sin(ax + b))$, and $\frac{d}{dt}(e^{3t})$.

Practice 5: Use the graph of g in Fig. 1 and the Chain Rule to estimate $D(\sin(g(x)))$ and $D(g(\sin(x)))$ at $x = \pi$.

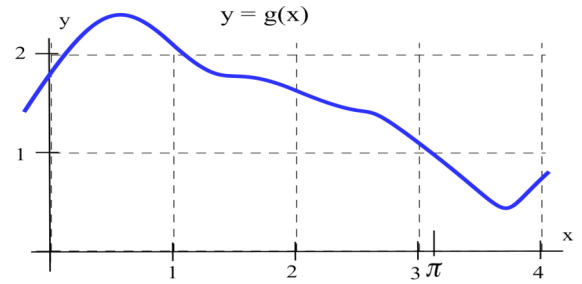


Fig. 1

The Chain Rule is a general differentiation pattern, and it can be used with the other general patterns such as the Product and Quotient Rules.

Example 7: Determine $D(e^{3x} \cdot \sin(5x + 7))$ and $\frac{d}{dx}(\cos(x \cdot e^x))$.

Solution: (a) $e^{3x} \sin(5x + 7)$ is a product of two functions so we need the product rule first:

$$\begin{aligned} D(e^{3x} \cdot \sin(5x + 7)) &= e^{3x} \cdot D(\sin(5x + 7)) + \sin(5x + 7) \cdot D(e^{3x}) \\ &= e^{3x} \cos(5x + 7) \cdot 5 + \sin(5x + 7) e^{3x} \cdot 3 = 5e^{3x} \cos(5x + 7) + 3e^{3x} \sin(5x + 7). \end{aligned}$$

(b) $\cos(x \cdot e^x)$ is a composition of cosine with a product so we need the Chain Rule first:

$$\begin{aligned} \frac{d}{dx}(\cos(x \cdot e^x)) &= -\sin(x \cdot e^x) \cdot \frac{d}{dx}(x \cdot e^x) \\ &= -\sin(x \cdot e^x) \cdot \left\{ x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x) \right\} = -\sin(x \cdot e^x) \cdot \{ x e^x + e^x \}. \end{aligned}$$

Sometimes we want to differentiate a composition of more than two functions. We can do so if we proceed in a careful, step-by-step way.

Example 8: Find $D(\sin(\sqrt{x^3 + 1}))$

Solution: The function $\sin(\sqrt{x^3 + 1})$ can be considered as a composition $f \circ g$ of

$f(x) = \sin(x)$ and $g(x) = \sqrt{x^3 + 1}$. Then

$$(\sin(\sqrt{x^3 + 1}))' = f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot g'(x) = \cos(\sqrt{x^3 + 1}) D(\sqrt{x^3 + 1})$$

For the derivative of $\sqrt{x^3 + 1}$, we can use the Chain Rule again or its special case, the Power Rule:

$$D(\sqrt{x^3 + 1}) = D((x^3 + 1)^{1/2}) = \frac{1}{2}(x^3 + 1)^{-1/2} D(x^3 + 1) = \frac{1}{2}(x^3 + 1)^{-1/2} 3x^2.$$

$$\text{Finally, } (\sin(\sqrt{x^3 + 1}))' = \cos(\sqrt{x^3 + 1}) D(\sqrt{x^3 + 1})$$

$$= \cos(\sqrt{x^3 + 1}) \cdot \frac{1}{2}(x^3 + 1)^{-1/2} (3x^2) = \frac{3x^2 \cos(\sqrt{x^3 + 1})}{2\sqrt{x^3 + 1}}.$$

This example was more complicated than the earlier ones, but it is just a matter of applying the Chain Rule twice, to a composition of a composition. If you proceed step-by-step and don't get lost in the problem, these multiple applications of the Chain Rule are relatively straightforward.

We can also use the Leibniz form of the Chain Rule for a composition of more than two functions. If

$y = \sin(\sqrt{x^3 + 1})$, then $y = \sin(u)$ with $u = \sqrt{w}$ and $w = x^3 + 1$. The Leibniz form of the Chain

$$\text{Rule is } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}, \text{ so } \frac{dy}{dx} = \cos(u) \frac{1}{2\sqrt{w}} \cdot 3x^2 = \cos(\sqrt{x^3 + 1}) \cdot \frac{1}{2\sqrt{x^3 + 1}} \cdot 3x^2.$$

Practice 6: (a) Find $D(\sin(\cos(5x)))$. (b) For $y = e^{\cos(3x)}$, find dy/dx .

CHAIN RULE AND TABLES OF DERIVATIVES

With the Chain Rule, the derivatives of all sorts of strange and wonderful functions are available. If we know f' and g' , then we also know the derivatives of their composition: $(f(g(x)))' = f'(g(x))g'(x)$.

Example 9: Given $D(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$, find $D(\arcsin(5x))$ and $\frac{d(\arcsin(e^x))}{dx}$.

Solution: (a) $\arcsin(5x)$ is the composition of $f(x) = \arcsin(x)$ with $g(x) = 5x$. We know $g'(x) = 5$,

$$\text{and } f'(x) = \frac{1}{\sqrt{1-x^2}} \text{ so } f'(g(x)) = \frac{1}{\sqrt{1-(g(x))^2}} = \frac{1}{\sqrt{1-25x^2}}.$$

$$\text{Then } D(\arcsin(5x)) = f'(g(x))g'(x) = \frac{1}{\sqrt{1-25x^2}} \cdot (5) = \frac{5}{\sqrt{1-25x^2}}.$$

(b) $y = \arcsin(e^x)$ is $y = \arcsin(u)$ with $u = e^x$. We know $dy/du = \frac{1}{\sqrt{1-u^2}}$ and $du/dx = e^x$

$$\text{so } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot e^x = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

$$\text{In general, } D(\arcsin(f(x))) = \frac{f'(x)}{\sqrt{1-(f(x))^2}} \text{ and } \frac{d(\arcsin(u))}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}.$$

Practice 7: $D(\arctan(x)) = \frac{1}{1+x^2}$. Find $D(\arctan(x^3))$ and $\frac{d(\arctan(e^x))}{dx}$.

Appendix B in the back of this book shows the derivative patterns for a variety of functions. You may not know much about some of the functions, but with the differentiation patterns given and the Chain Rule you should be able to calculate derivatives of compositions. It is just a matter of following the pattern.

Practice 8: Use the patterns $D(\sinh(x)) = \cosh(x)$ and $D(\ln(x)) = 1/x$ to determine

(a) $D(\sinh(5x - 7))$ (b) $\frac{d}{dx}(\ln(3 + e^{2x}))$ (c) $D(\arcsin(1 + 3x))$.

Example 10: If $D(F(x)) = e^x \cdot \sin(x)$, find $D(F(5x))$ and $\frac{d(F(t^3))}{dt}$

Solution: (a) $D(F(5x)) = D(F(g(x)))$ with $g(x) = 5x$. $F'(x) = e^x \cdot \sin(x)$ so

$$D(F(5x)) = F'(g(x)) \cdot g'(x) = e^{g(x)} \cdot \sin(g(x)) \cdot 5 = e^{5x} \cdot \sin(5x) \cdot 5.$$

(b) $y = F(u)$ with $u = t^3$. $\frac{dy}{du} = e^u \cdot \sin(u)$ and $\frac{du}{dt} = 3t^2$ so

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = e^u \cdot \sin(u) \cdot 3t^2 = e^{(t^3)} \cdot \sin(t^3) \cdot 3t^2$$

Proof of the Power Rule For Functions

We started using the Power Rule For Functions in section 2.3. Now we can easily prove it.

Power Rule For Functions: If $y = f^n(x)$ and f is differentiable, then $\frac{dy}{dx} = n \cdot f^{n-1}(x) \cdot f'(x)$.

Proof: $y = f^n(x)$ is $y = u^n$ with $u = f(x)$. Then $\frac{dy}{du} = n \cdot u^{n-1}$ and $\frac{du}{dx} = f'(x)$ so by the Chain

Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = n \cdot u^{n-1} \cdot f'(x) = n \cdot f^{n-1}(x) \cdot f'(x)$.

PROBLEMS

In problems 1 – 6, find two functions f and g so that the given function is the composition of f and g .

1. $y = (x^3 - 7x)^5$ 2. $y = \sin^4(3x - 8)$ 3. $y = \sqrt{(2 + \sin(x))^5}$

4. $y = \frac{1}{\sqrt{x^2 + 9}}$ 5. $y = |x^2 - 4|$ 6. $y = \tan(\sqrt{x})$

7. For each function in problems 1 – 6, write y as a function of u for some u which is a function of x .

Problems 8 and 9 refer to the values given in this table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-2	2	-1	1	1		
-1	1	2	0	2		
0	-2	1	2	-1		
1	0	-2	-1	2		
2	1	0	1	-1		

8. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1$ and 2 .

9. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -2, -1$ and 0 .

10. Use Fig. 2 to estimate the values of $g(x)$, $g'(x)$, $(f \circ g)(x)$, $f'(g(x))$, and $(f \circ g)'(x)$ at $x = 1$.
11. Use Fig. 2 to estimate the values of $g(x)$, $g'(x)$, $(f \circ g)(x)$, $f'(g(x))$, and $(f \circ g)'(x)$ for $x = 2$.

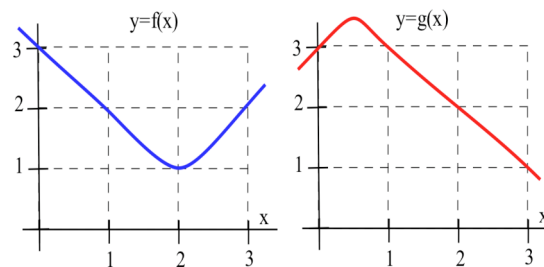


Fig. 2

In problems 12 - 20, differentiate each function.

12. $D((x^2 + 2x + 3)^{87})$ 13. $D((1 - \frac{3}{x})^4)$ 14. $\frac{d}{dx}(x + \frac{1}{x})^5$
15. $D(\frac{5}{\sqrt{2 + \sin(x)}})$ 16. $\frac{d}{dt} t \sin(3t + 2)$ 17. $D(x^2 \sin(x^2 + 3))$
18. $\frac{d}{dx} \sin(2x) \cos(5x + 1)$ 19. $D(\frac{7}{\cos(x^3 - x)})$ 20. $\frac{d}{dt} \frac{5}{3 + e^t}$
21. $D(e^x + e^{-x})$ 22. $\frac{d}{dx}(e^x - e^{-x})$

23. An object attached to a spring is at a height of $h(t) = 3 - \cos(2t)$ feet above the floor t seconds after it is released.
- At what height was it released?
 - Determine its height, velocity and acceleration at any time t .
 - If the object has mass m , determine its kinetic energy $K = \frac{1}{2}mv^2$ and dK/dt at any time t .
24. A manufacturer has determined that an employee with d days of production experience will be able to produce approximately $P(d) = 3 + 15(1 - e^{-0.2d})$ items per day. Graph $P(d)$.
- Approximately how many items will a beginning employee be able to produce each day?
 - How many items will an experienced employee be able to produce each day?
 - What is the marginal production rate of an employee with 5 days of experience? (What are the units of your answer, and what does this answer mean?)
25. The air pressure $P(h)$, in pounds per square inch, at an altitude of h feet above sea level is approximately $P(h) = 14.7e^{-0.0000385h}$.
- What is the air pressure at sea level? What is the air pressure at an altitude of 30,000 feet?
 - At what altitude is the air pressure 10 pounds per square inch?
 - If you are in a balloon which is 2000 feet above the Pacific Ocean and is rising at 500 feet per minute, how fast is the air pressure on the balloon changing?
 - If the temperature of the gas in the balloon remained constant during this ascent, what would happen to the volume of the balloon?

Find the derivatives in problems 26 – 33 .

$$26. \quad D\left(\frac{(2x+3)^2}{(5x-7)^3}\right)$$

$$27. \quad \frac{d}{dz}\sqrt{1+\cos^2(z)}$$

$$28. \quad D(\sin(3x+5))$$

$$29. \quad \frac{d}{dx}\tan(3x+5)$$

$$30. \quad \frac{d}{dt}\cos(7t^2)$$

$$31. \quad D(\sin(\sqrt{x+1}))$$

$$32. \quad D(\sec(\sqrt{x+1}))$$

$$33. \quad \frac{d}{dx}(e^{\sin(x)})$$

In problems 34 – 37 , calculate $\frac{df(x)}{dx}$ and $\frac{dx(t)}{dt}$ when $t = 3$ and use these values to determine the value of $\frac{d f(x(t))}{dt}$ when $t = 3$.

$$34. \quad f(x) = \cos(x), \quad x = t^2 - t + 5$$

$$35. \quad f(x) = \sqrt{x}, \quad x = 2 + \frac{21}{t}$$

$$36. \quad f(x) = e^x, \quad x = \sin(t)$$

$$37. \quad f(x) = \tan^3(x), \quad x = 8$$

In problems 38 – 43, find a function which has the given function as its derivative. (You are given $f'(x)$ in each problem and are asked to find a function $f(x)$.)

$$38. \quad f'(x) = (3x+1)^4$$

$$39. \quad f'(x) = (7x-13)^{10}$$

$$40. \quad f'(x) = \sqrt{3x-4}$$

$$41. \quad f'(x) = \sin(2x-3)$$

$$42. \quad f'(x) = 6e^{3x}$$

$$43. \quad f'(x) = \cos(x) \cdot e^{\sin(x)}$$

If two functions are equal, then their derivatives are also equal. In problems 44 – 47 , differentiate each side of the trigonometric identity to find a new identity.

$$44. \quad \sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$$

$$45. \quad \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$46. \quad \sin(2x) = 2 \sin(x) \cos(x)$$

$$47. \quad \sin(3x) = 3 \sin(x) - 4 \sin^3(x)$$

Derivatives of Families of Functions

So far we have emphasized derivatives of particular functions, but sometimes we want to look at the derivatives of a whole family of functions. In problems 48 – 71, the letters A–D represent constants and the given formulas describe families of functions.

For problems 48 – 65, calculate $y' = \frac{dy}{dx}$.

$$48. \quad y = Ax^3 - B$$

$$49. \quad y = Ax^3 + Bx^2 + C$$

$$50. \quad y = \sin(Ax + B)$$

$$51. \quad y = \sin(Ax^2 + B)$$

$$52. \quad y = Ax^3 + \cos(Bx)$$

$$53. \quad y = \sqrt{A + Bx^2}$$

$$54. \quad y = \sqrt{A - Bx^2}$$

$$55. \quad y = A - \cos(Bx)$$

$$56. \quad y = \cos(Ax + B)$$

57. $y = \cos(Ax^2 + B)$

58. $y = A \cdot e^{Bx}$

59. $y = x \cdot e^{Bx}$

60. $y = e^{Ax} + e^{-Ax}$

61. $y = e^{Ax} - e^{-Ax}$

62. $y = \frac{\sin(Ax)}{x}$

63. $y = \frac{Ax}{\sin(Bx)}$

64. $y = \frac{1}{Ax + B}$

65. $y = \frac{Ax + B}{Cx + D}$

In problems 66 – 71, (a) find y' , (b) find the value(s) of x so that $y' = 0$, and (c) find y'' .

Typically your answer in part (b) will contain As , Bs and (sometimes) Cs .

66. For $y = Ax^2 + Bx + C$, (a) find y' , (b) find the value(s) of x so that $y' = 0$, and (c) find y'' .

(You should recognize the part (b) answer from intermediate algebra. What is it?)

67. $y = Ax(B - x) = ABx - Ax^2$.

68. $y = Ax(B - x^2) = ABx - Ax^3$.

69. $y = Ax^2(B - x) = ABx^2 - Ax^3$.

70. $y = Ax^2 + \frac{B}{x}$.

71. $y = Ax^3 + Bx^2 + C$.

Use the given differentiation patterns to differentiate the composite functions in problems 72 – 83. We have not derived the derivatives for these functions (yet), but if you are handed the derivative pattern for a function then you should be able to take derivatives of a composition involving that function.

Given: $D(\arctan(x)) = \frac{1}{1+x^2}$, $D(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$, $D(\ln(x)) = \frac{1}{x}$.

72. $D(\arctan(7x))$

73. $\frac{d}{dx}(\arctan(x^2))$

74. $\frac{d}{dt}(\arctan(\ln(t)))$

75. $D(\arctan(e^x))$

76. $\frac{d}{dw}(\arcsin(4w))$

77. $D(\arcsin(x^3))$

78. $D(\arcsin(\ln(x)))$

79. $\frac{d}{dt}(\arcsin(e^t))$

80. $D(\ln(3x + 1))$

81. $\frac{d}{dx}(\ln(\sin(x)))$

82. $D(\ln(\arctan(x)))$

83. $\frac{d}{ds}(\ln(e^s))$

Section 2.4

PRACTICE Answers

Practice 1: $f(x) = 5x - 4$ and $g(x) = x^2$ so $f'(x) = 5$ and $g'(x) = 2x$. $f \circ g(x) = f(g(x)) = f(x^2) = 5x^2 - 4$.

$$D(f \circ g(x)) = f'(g(x)) \cdot g'(x) = 5 \cdot 2x = 10x \text{ or } D(f \circ g(x)) = D(5x^2 - 4) = 10x.$$

$$g \circ f(x) = g(f(x)) = g(5x - 4) = (5x - 4)^2 = 25x^2 - 40x + 16.$$

$$D(g \circ f(x)) = g'(f(x)) \cdot f'(x) = 2(5x - 4) \cdot 5 = 50x - 40 \text{ or}$$

$$D(g \circ f(x)) = D(25x^2 - 40x + 16) = 50x - 40.$$

Practice 2: $\frac{d}{dx}(\sin(4x + e^x)) = \cos(4x + e^x) \cdot D(4x + e^x) = \cos(4x + e^x) \cdot (4 + e^x)$

Practice 3: Fill in the table in Example 6 for $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1, 2$ and 3 .

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
-1	2	3	1	0	0	0
0	-1	1	3	2	1	-2
1	1	0	-1	3	-1	$f'(g(1)) \cdot g'(1) = f'(0) \cdot (3) = (3)(3) = 9$
2	3	-1	0	1	2	$f'(g(2)) \cdot g'(2) = f'(-1) \cdot (1) = (1)(1) = 1$
3	0	2	2	-1	3	$f'(g(3)) \cdot g'(3) = f'(2) \cdot (-1) = (0)(-1) = 0$

Practice 4: $D(\sin(7x - 1)) = \cos(7x - 1)D(7x - 1) = 7 \cdot \cos(7x - 1).$

$$\frac{d}{dx} \sin(ax + b) = \cos(ax + b) D(ax + b) = a \cdot \cos(ax + b) \quad \frac{d}{dt}(e^{3t}) = e^{3t} \frac{d}{dt}(3t) = 3 \cdot e^{3t}$$

Practice 5: $D(\sin(g(x))) = \cos(g(x)) \cdot g'(x)$. At $x = \pi$, $\cos(g(\pi)) \cdot g'(\pi) \approx \cos(0.86) \cdot (-1) \approx -0.65$.

$$D(g(\sin(x))) = g'(\sin(x)) \cdot \cos(x). \text{ At } x = \pi, g'(\sin(\pi)) \cdot \cos(\pi) = g'(0) \cdot (-1) \approx -2$$

Practice 6:

$$D(\sin(\cos(5x))) = \cos(\cos(5x)) \cdot D(\cos(5x)) = \cos(\cos(5x)) \cdot (-\sin(5x)) \cdot D(5x) = -5 \cdot \sin(5x) \cdot \cos(\cos(5x))$$

$$\frac{d}{dx} e^{\cos(3x)} = e^{\cos(3x)} D(\cos(3x)) = e^{\cos(3x)} (-\sin(3x)) D(3x) = -3 \cdot \sin(3x) \cdot e^{\cos(3x)}.$$

$$\text{Practice 7: } D(\arctan(x^3)) = \frac{1}{1 + (x^3)^2} D(x^3) = \frac{3x^2}{1 + x^6}$$

$$\frac{d}{dx}(\arctan(e^x)) = \frac{1}{1 + (e^x)^2} D(e^x) = \frac{e^x}{1 + e^{2x}}$$

$$\text{Practice 8: } D(\sinh(5x - 7)) = \cosh(5x - 7) D(5x - 7) = 5 \cdot \cosh(5x - 7)$$

$$\frac{d}{dx} \ln(3 + e^{2x}) = \frac{1}{3 + e^{2x}} D(3 + e^{2x}) = \frac{2e^{2x}}{3 + e^{2x}}$$

$$D(\arcsin(1 + 3x)) = \frac{1}{\sqrt{1 - (1 + 3x)^2}} D(1 + 3x) = \frac{3}{\sqrt{1 - (1 + 3x)^2}}$$