4.5 THE FUNDAMENTAL THEOREM OF CALCULUS

This section contains the most important and most used theorem of calculus, THE Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz in the late 1600s, it establishes the connection between derivatives and integrals, provides a way of easily calculating many integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure.

The previous sections emphasized the meaning of the definite integral, defined it, and began to explore some of its applications and properties. In this section, the emphasis is on the Fundamental Theorem of Calculus. You will use this theorem **often** in later sections.

There are two parts of the Fundamental Theorem. They are similar to results in the last section but more general. Part 1 of the Fundamental Theorem of Calculus says that **every** continuous function has an antiderivative and shows how to differentiate a function defined as an integral. Part 2 shows how to evaluate the definite integral of any function if we know an antiderivative of that function.

Part 1: Antiderivatives

Every continuous function has an antiderivative, even those nondifferentiable functions with "corners" such as absolute value.

The Fundamental Theorem of Calculus (Part 1)

If f is continuous and $A(x) = \int_a^x f(t)\,dt$

then $\frac{d}{dx}\left(\int_a^x f(t)\,dt \right) = \frac{d}{dx} A(x) = f(x)$. $A(x)$ is an antiderivative of $f(x)$.

Proof: Assume f is a continuous function and let $A(x) = \int_a^x f(t)\,dt$. By the definition of derivative of A,

$$\frac{d}{dx} A(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left\{ \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right\} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt .$$

By Property 6 of definite integrals (Section 4.3), for $h > 0$

$$\{ \text{ min of f on } [x, x+\text{h}] \ \}\cdot\text{h} \le \int_x^{x+\text{h}} \text{f}(t)\ \text{dt} \ \le \{ \text{ max of f on } [x, x+\text{h}] \ \}\cdot\text{h} \ . \quad \text{(Fig. 1)}$$

Dividing each part of the inequality by h, we have that $\frac{1}{\text{h}} \int_x^{x+\text{h}} \text{f}(t)\ \text{dt}$ is between the minimum and the maximum of f on the interval $[x, x+\text{h}]$. The function f is continuous (by the hypothesis) and the interval $[x, x+\text{h}]$ is shrinking (since h approaches 0), so

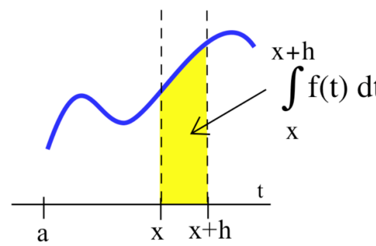

Fig. 1

$$\lim_{h\to 0} \ \{ \text{ min of f on } [x, x+\text{h}] \ \} = \text{f}(x) \quad \text{and}$$

$$\lim_{h\to 0} \ \{ \text{ max of f on } [x, x+\text{h}] \ \} = \text{f}(x).$$ Therefore, $\frac{1}{\text{h}} \int_x^{x+\text{h}} \text{f}(t)\ \text{dt}$ is stuck between two

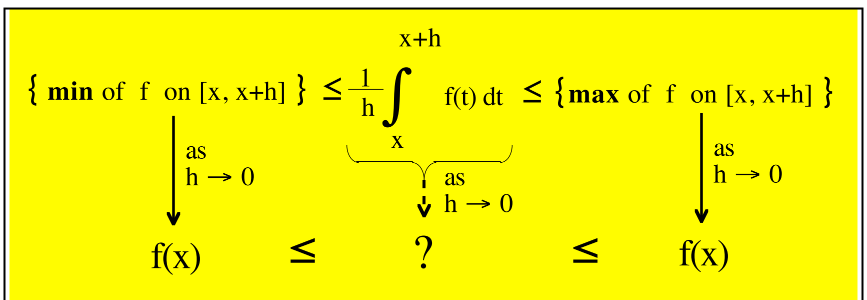

Fig. 2

quantities (Fig. 2) which both approach f(x).

Then $\frac{1}{\text{h}} \int_x^{x+\text{h}} \text{f}(t)\ \text{dt}$ must also approach f(x), and $\frac{\text{d}}{\text{dx}}\ \text{A}(x) = \lim_{\text{h}\to 0} \frac{1}{\text{h}} \int_{\text{x}}^{\text{x+h}} \text{f(t) dt} \ = \text{f}(x).$

Example 1: $\text{A}(x) = \int_0^x \text{f}(t)\ \text{dt}$ for f in Fig. 3. Evaluate A(x) and A'(x) for x= 1, 2 , 3 and 4.

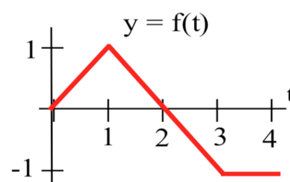

Fig. 3

Solution: $\text{A}(1) = \int_0^1 \text{f}(t)\ \text{dt} \ = 1/2$, $\text{A}(2) = \int_0^2 \text{f}(t)\ \text{dt} \ = 1$, $\text{A}(3) = \int_0^3 \text{f}(t)\ \text{dt} \ = 1/2$,

$\text{A}(4) = \int_0^4 \text{f}(t)\ \text{dt} \ = -1/2$. Since f is continuous, A '(x) = f(x) so

A'(1) = f(1) = 1, A'(2) = f(2) = 0, A'(3) = f(3) = –1, A'(4) = f(4) = –1.

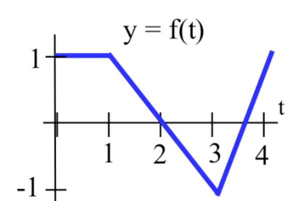

Fig. 4

Practice 1: $A(x) = \int_0^x f(t)\,dt$ for f in Fig. 4. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Example 2: $A(x) = \int_0^x f(t)dt$ for the function f shown in Fig. 5.

For which value of x is $A(x)$ maximum?

For which x is the rate of change of A maximum?

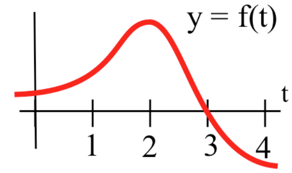

Fig. 5

Solution: Since A is differentiable, the only critical points are where $A'(x) = 0$ or at endpoints. $A'(x) = f(x) = 0$ at $x=3$, and A has a maximum at $x=3$. Notice that the values of $A(x)$ increase as x goes from 0 to 3 and then the A values decrease. The rate of change of $A(x)$ is $A'(x) = f(x)$, and $f(x)$ appears to have a maximum at $x=2$ so the rate of change of $A(x)$ is maximum when $x=2$. Near $x=2$, a slight increase in the value of x yields the maximum increase in the value of $A(x)$.

Part 2: Evaluating Definite Integrals

If we know and can evaluate some antiderivative of a function, then we can evaluate any definite integral of that function.

The Fundamental Theorem of Calculus (Part 2)

If $f(x)$ is continuous and $F(x)$ is **any** antiderivative of f ($F'(x) = f(x)$),

then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

Proof: If F is an antiderivative of f, then $F(x)$ and $A(x) = \int_a^x f(t)\,dt$ are both antiderivatives of f,

$F'(x) = f(x) = A'(x)$, so F and A differ by a constant: $A(x) - F(x) = C$ for all x.

At $x=a$, we have $C = A(a) - F(a) = 0 - F(a) = -F(a)$ so $C = -F(a)$ and the equation $A(x) - F(x) = C$ becomes $A(x) - F(x) = -F(a)$. Then $A(x) = F(x) - F(a)$ for all x so

$A(b) = F(b) - F(a)$ and $\int_a^{\mathbf{b}} f(x)\,dx = A(\mathbf{b}) = F(\mathbf{b}) - F(a)$, the formula we wanted.

The definite integral of a continuous function f can be found by finding an antiderivative of f (any antiderivative of f will work) and then doing some arithmetic with this antiderivative. The theorem does not tell us how to find an antiderivative of f, and it does not tell us how to find the definite integral of a discontinuous function. It is possible to evaluate definite integrals of some discontinuous functions (Section 4.3), but the Fundamental Theorem of Calculus can not be used to do so.

Example 3: Evaluate $\int_0^2 (x^2 - 1)\ dx$.

Solution: $F(x) = \frac{x^3}{3} - x$ is an antiderivative of $f(x) = x^2 - 1$ (check that $D(\frac{x^3}{3} - x) = x^2 - 1$), so

$$\int_0^2 (x^2 - 1)\ dx = \frac{x^3}{3} - x \Big|_0^2 = \left\{ \frac{2^3}{3} - 2 \right\} - \left\{ \frac{0^3}{3} - 0 \right\} = 2/3 - 0 = 2/3.$$

If friends had picked a different antiderivative of $x^2 - 1$, say $F(x) = \frac{x^3}{3} - x \mathbf{+ 4}$, then their calculations would be slightly different but the result would be the same:

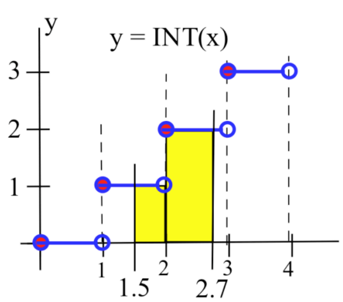

Fig. 6

$$\int_0^2 (x^2 - 1)\ dx = \frac{x^3}{3} - x \mathbf{+ 4} \Big|_0^2 = (\frac{2^3}{3} - 2 \mathbf{+ 4}) - (\frac{0^3}{3} - 0 \mathbf{+ 4}) = 14/3 - 4 = 2/3.$$

Practice 2: Evaluate $\int_1^3 (3x^2 - 1)\ dx$.

Example 4: Evaluate $\int_{1.5}^{2.7} \mathrm{INT}(x)\ dx$. (INT(x) is the largest integer less than or equal to x. Fig. 6)

Solution: $f(x) = \mathrm{INT}(x)$ is not continuous at $x = 2$ in the interval $[1.5, 2.7]$ so the Fundamental Theorem of Calculus can not be used. We can, however, use our understanding of the meaning of an integral to get

$$\int_{1.5}^{2.7} \mathrm{INT}(x)\ dx = \text{(area for } x \text{ between 1.5 and 2)} + \text{(area for } x \text{ between 2 and 2.7)}$$

$$= \text{(base)(height)} + \text{(base)(height)} = (.5)(1) + (.7)(2) = 1.9 .$$

Practice 3: Evaluate $\int_{1.3}^{3.4} \mathrm{INT}(x)\ dx$.

Calculus is the study of derivatives and integrals, their meanings and their applications. The Fundamental Theorem of Calculus shows that differentiation and integration are closely related and that integration is really antidifferentiation, the inverse of differentiation.

Applications — The Future

Calculus is important for many reasons, but students are usually required to study calculus because it is needed for understanding concepts and doing applications in a variety of fields. The Fundamental Theorem of Calculus is very important to both pursuits.

Most applied problems in integral calculus require the following steps to get from the problem to a numerical answer:

$$\text{Applied problem} \xrightarrow{1} \text{Riemann sum (or area)} \xrightarrow{2} \text{definite integral} \xrightarrow{3} \text{number}$$

In some cases, the path from the problem to the answer may be abbreviated, but the three steps are commonly used.

Step 1 is absolutely vital. If we can not translate the ideas of an applied problem into an area or a Riemann sum or a definite integral, then we can not use integral calculus to solve the problem. For a few types of applied problems, we will be able to go directly from the problem to an integral, but usually it will be easier to first break the problem into smaller pieces and to build a Riemann sum. Section 4.8 and all of chapter 5 will focus on translating different types of applied problems into Riemann sums and definite integrals. **Computers and calculators are seldom of any help with Step 1.**

Step 2 is usually easy. If we have a Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ on the interval [a,b], then the limit of the sum is simply the definite integral $\int_a^b f(x)\, dx$.

Step 3 can be handled in several ways.

If the function f is relatively simple, there are several ways to find an antiderivative of f (sections 4.6, parts of chapter 6 and others), and then Part 2 of the Fundamental Theorem of Calculus can be used to get a numerical answer.

If the function f is more complicated, then integral tables (section 4.8) or computers (symbolic manipulators such as Maple or Mathematica) can be used to find an antiderivative of f . Then Part 2 of the Fundamental Theorem of Calculus can be used to get a numerical answer.

If an antiderivative of f cannot be found, approximate numerical answers for the definite integral can be found by various summation methods (section 4.9). These summation methods are typically done on computers, and program listings are included in an Appendix.

Usually the difficulties in solving an applied problem come with the 1st and 3rd steps, and the most time will be spent working with them. There are techniques and details to master and understand, but it is also important to keep in mind where these techniques and details fit into the bigger picture.

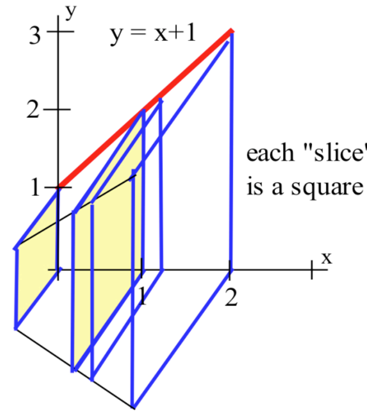

Fig. 7

The next Example illustrates these steps for the problem of finding a volume of a solid. Problems of finding volumes of solids will be examined in more detail in Section 5.1.

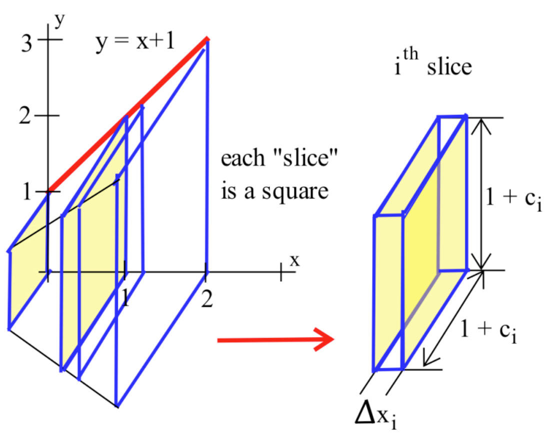

Fig. 8

$\text{volume}_i = (1 + c_i)^2 \Delta x_i$

Example 5: Find the volume of the solid in Fig. 7 for $0 \le x \le 2$. (Each perpendicular "slice" through the solid is a square.)

Solution:

Step 1: Going from the figure to a Riemann sum.

If we break the solid into n "slices" with cuts perpendicular to the x–axis, at $x_1, x_2, x_3, \ldots, x_{n-1}$ (like cutting a loaf of bread), then the volume of the original solid is the sum of the volumes of the "slices" (Fig. 8):

$$\text{Total Volume} = \sum_{i=1}^{n} (\text{volume of the } i^{th} \text{ slice}) .$$

The volume of the i^{th} slice is approximately equal to the volume of a box:

(height of the slice)·(base of the slice)·(thickness)

$\approx (c_i + 1)\cdot(c_i + 1)\cdot\Delta x_i$

where c_i is any value between x_{i-1} and x_i .

Therefore,

$$\text{Total Volume} \approx \sum_{i=1}^{n} (c_i + 1)\cdot(c_i + 1)\cdot\Delta x_i$$

which is a Riemann sum.

Step 2: Going from the Riemann sum to a definite integral.

The Riemann sum approximation of the total volume in Step 1 is improved by taking thinner slices (making all of the Δx_i small), and

$$\text{Total volume} = \lim_{mesh \to 0} \left\{ \sum_{i=1}^{N} (c_i + 1) \cdot (c_i + 1) \cdot \Delta x_i \right\}$$

$$= \int_0^2 (x+1)(x+1)\,dx = \int_0^2 (x^2 + 2x + 1)\,dx .$$

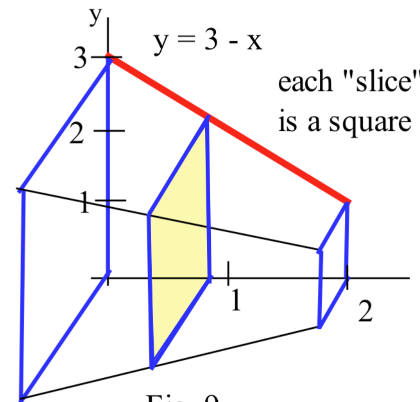

Fig. 9

Step 3: Going from the definite integral to a numerical answer.

We can use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$F(x) = \frac{1}{3} x^3 + x^2 + x$ is an antiderivative of $x^2 + 2x + 1$

(check by differentiating F(x)), so

$$\int_0^2 (x^2 + 2x + 1)\,dx = F(2) - F(0) = \left\{ \tfrac{1}{3}\, \mathbf{2}^3 + \mathbf{2}^2 + \mathbf{2} \right\} - \left\{ \tfrac{1}{3}\, \mathbf{0}^3 + \mathbf{0}^2 + \mathbf{0} \right\}$$

$$= \left\{ \frac{26}{3} \right\} - \{ 0 \} = \frac{26}{3} = 8\frac{2}{3} .$$

The volume of the solid shape in Fig. 7 is exactly $8\frac{2}{3}$ cubic inches.

Practice 4: Find the volume of the solid shape in Fig. 9 for $0 \le x \le 2$. (Each "slice" through the solid perpendicular to the x–axis is a square.)

Leibniz' Rule For Differentiating Integrals

If the endpoint of an integral is a function of x rather than simply x,
then we need to use the Chain Rule together with part 1 of the Fundamental Theorem of Calculus to calculate the derivative of the integral. According to the Chain Rule, if

$\frac{d}{dx} A(x) = f(x)$, then $\frac{d}{dx} A(x^2) = f(x^2) \cdot \mathbf{2}x$ and, applying the Chain Rule to the derivative of the integral,

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt \right) = \frac{d}{dx} A(\, g(x)\,) = f(\, g(x)\,) \cdot \mathbf{g'}(x).$$

If f is a continuous function and $A(x) = \int_a^x f(t)dt$

then $\frac{d}{dx}\left(\int_a^x f(t)dt \right) = \frac{d}{dx} A(x) = f(x)$ **(Fundamental Theorem, Part I)**

and, if g is differentiable, $\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt \right) = \frac{d}{dx} A(\, g(x)\,) = f(\, g(x)\,)\cdot g'(x)$ **(Leibniz' Rule)**

Example 6: Calculate $\frac{d}{dx}\left(\int_a^{\mathbf{5x}} t^2 \, dt \right)$, $\frac{d}{dx}\left(\int_a^{\mathbf{x^2}} \cos(u)\, du \right)$, $\frac{d}{dw}\left(\int_a^{\mathbf{\sin(w)}} z^3 \, dz \right)$.

Solution: $\frac{d}{dx}\left(\int_a^{\mathbf{5x}} t^2 \, dt \right) = (5x)^2 \cdot\mathbf{5} = 125x^2$. $\frac{d}{dx}\left(\int_a^{\mathbf{x^2}} \cos(u)\, du \right) = \cos(\mathbf{x^2})\cdot\mathbf{2x} = 2x\cdot\cos(x^2)$

$\frac{d}{dw}\left(\int_a^{\mathbf{\sin(w)}} z^3 \, dz \right) = (\mathbf{\sin(w)})^3 \cdot\mathbf{\cos(w)} = \sin^3(w)\cos(w)$.

Practice 5: Find $\frac{d}{dx}\left(\int_0^{x^3} \sin(t)\, dt \right)$.

PROBLEMS:

1. $A(x) = \int_0^x 3t^2 \, dt$ (a) Use part 2 of the Fundamental Theorem to find a formula for $A(x)$ and then differentiate $A(x)$ to obtain a formula for $A'(x)$. Evaluate $A'(x)$ at $x = 1, 2,$ and 3.

   (b) Use part 1 of the Fundamental Theorem to evaluate $A'(x)$ at $x = 1, 2,$ and 3.

2. $A(x) = \int_1^x (\, 1 + 2t\,)\, dt$ (a) Use part 2 of the Fundamental Theorem to find a formula for $A(x)$ and then differentiate $A(x)$ to obtain a formula for $A'(x)$. Evaluate $A'(x)$ at $x = 1, 2,$ and 3.

   (b) Use part 1 of the Fundamental Theorem to evaluate $A'(x)$ at $x = 1, 2,$ and 3.

In problems 3 – 8 , evaluate $A'(x)$ at $x = 1, 2,$ and 3.

3. $A(x) = \int_0^x 2t \; dt$ 4. $A(x) = \int_1^x 2t \; dt$ 5. $A(x) = \int_{-3}^x 2t \; dt$

6. $A(x) = \int_0^x (3 - t^2) \; dt$ 7. $A(x) = \int_0^x \sin(t) \, dt$ 8. $A(x) = \int_1^x | t - 2 | \, dt$

In problems 9 – 13, $A(x) = \int_0^x f(t) \; dt$ for the functions in Figures 10 – 14. Evaluate $A'(1), A'(2), A'(3)$.

9. f in Fig. 10 10. f in Fig. 11 11. f in Fig. 12 12. f in Fig. 13

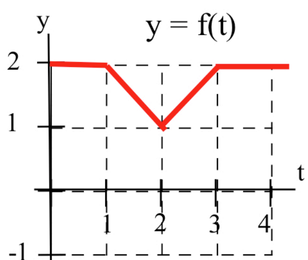

Fig. 10

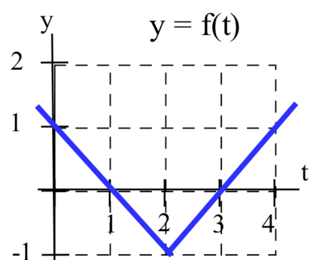

Fig. 11

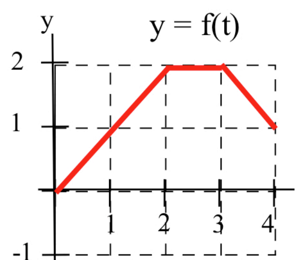

Fig. 12

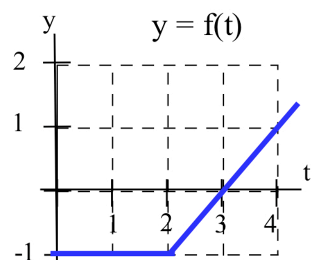

Fig. 13

In problems 13 – 33, verify that $F(x)$ is an antiderivative of the integrand $f(x)$ and use Part 2 of the Fundamental Theorem to evaluate the definite integrals.

13. $\int_0^1 2x \, dx$, $F(x) = x^2 + 5$ 14. $\int_1^4 3x^2 \, dx$, $F(x) = x^3 + 2$ 15. $\int_1^3 x^2 \, dx$, $F(x) = \frac{1}{3} x^3$

16. $\int_0^3 (x^2 + 4x - 3) \, dx$, $F(x) = \frac{1}{3} x^3 + 2x^2 - 3x$ 17. $\int_1^5 \frac{1}{x} dx$, $F(x) = \ln(x)$

18. $\int_2^5 \frac{1}{x} dx$, $F(x) = \ln(x) + 4$ 19. $\int_{1/2}^3 \frac{1}{x} dx$, $F(x) = \ln(x)$ 20. $\int_1^3 \frac{1}{x} dx$, $F(x) = \ln(x) + 2$

21. $\int_0^{\pi/2} \cos(x) \, dx$, $F(x) = \sin(x)$ 22. $\int_0^{\pi} \sin(x) \, dx$, $F(x) = -\cos(x)$ 23. $\int_0^1 \sqrt{x} \, dx$, $F(x) = \frac{2}{3} x^{3/2}$

24. $\int_1^4 \sqrt{x} \, dx$, $F(x) = \frac{2}{3} x^{3/2}$ 25. $\int_1^7 \sqrt{x} \, dx$, $F(x) = \frac{2}{3} x^{3/2}$ 26. $\int_1^4 \frac{1}{2\sqrt{x}} dx$, $F(x) = \sqrt{x}$

27. $\int_1^9 \frac{1}{2\sqrt{x}}\, dx$, $F(x) = \sqrt{x}$

28. $\int_2^5 \frac{1}{x^2}\, dx$, $F(x) = -\frac{1}{x}$

29. $\int_{-2}^3 e^x\, dx$, $F(x) = e^x$

30. $\int_0^3 \frac{2x}{1+x^2}\, dx$, $F(x) = \ln(1+x^2)$

31. $\int_0^{\pi/4} \sec^2(x)\, dx$, $F(x) = \tan(x)$

32. $\int_1^e \ln(x)\, dx$, $F(x) = x\cdot\ln(x) - x$

33. $\int_0^3 2x\sqrt{1+x^2}\, dx$, $F(x) = \frac{2}{3}(1+x^2)^{3/2}$

For problems 34 – 48, find an antiderivative of the integrand and use Part 2 of the Fundamental Theorem to evaluate the definite integral.

34. $\int_2^5 3x^2\, dx$

35. $\int_{-1}^2 x^2\, dx$

36. $\int_1^3 (x^2 + 4x - 3)\, dx$

37. $\int_1^e \frac{1}{x}\, dx$

38. $\int_{\pi/4}^{\pi/2} \sin(x)\, dx$

39. $\int_{25}^{100} \sqrt{x}\, dx$

40. $\int_3^5 \sqrt{x}\, dx$

41. $\int_1^{10} \frac{1}{x^2}\, dx$

42. $\int_1^{1000} \frac{1}{x^2}\, dx$

43. $\int_0^1 e^x dx$

44. $\int_{-2}^2 \frac{2x}{1+x^2}\, dx$

45. $\int_{\pi/6}^{\pi/4} \sec^2(x)\, dx$

46. $\int_0^1 e^{2x}\, dx$

47. $\int_3^3 \sin(x)\cdot\ln(x)\, dx$

48. $\int_2^4 (x-2)^3\, dx$

In problems 49 – 54 , find the area of each shaded region.

49. Region in Fig. 14.

50. Region in Fig. 15.

51. Region in Fig. 16.

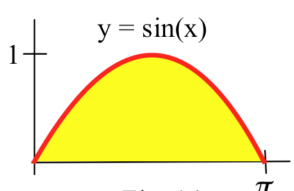

Fig. 14

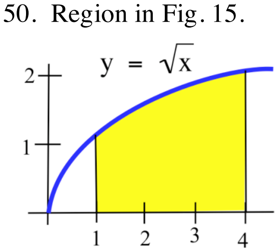

Fig. 15

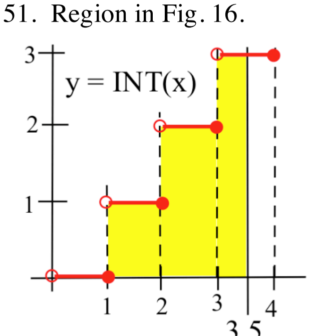

Fig. 16

52. Region in Fig. 17.

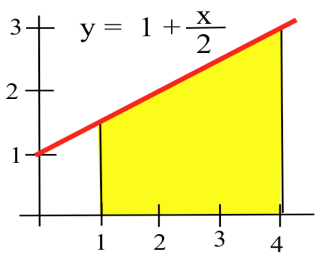

Fig. 17

53. Region in Fig. 18.

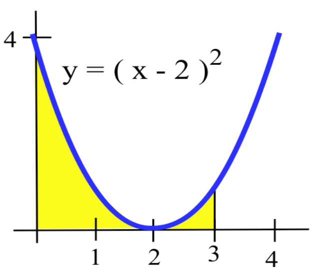

Fig. 18

54. Region in Fig. 19.

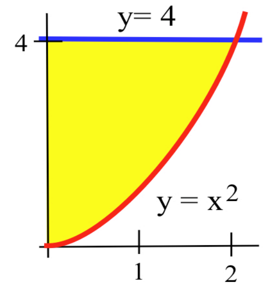

Fig. 19

Leibniz' Rule

55. If **D**(A(x)) = tan(x) , then find **D**(A($3x$)) , **D**(A(x^2)) , and **D**(A(sin(x))) .

56. If **D**(B(x)) = sec(x) , then find **D**(B($3x$)) , **D**(B(x^2)) , and **D**(B(sin(x))) .

57. $\frac{d}{dx}\left(\int_1^{5x} \sqrt{1+t}\ dt \right)$

58. $\frac{d}{dx}\left(\int_2^{x^2} \sqrt{1+t}\ dt \right)$

59. $\frac{d}{dx}\left(\int_0^{\sin(x)} \sqrt{1+t}\ dt \right)$

60. $\frac{d}{dx}\left(\int_1^{2+3x} t^2 + 5\ dt \right)$

61. $\frac{d}{dx}\left(\int_0^{1-2x} 3t^2 + 2\ dt \right)$

62. $\frac{d}{dx}\left(\int_x^{9} 3t^2 + 2\ dt \right)$

63. $\frac{d}{dx}\left(\int_x^{\pi} \cos(3t)\ dt \right)$

64. $\frac{d}{dx}\left(\int_{7x}^{\pi} \cos(2t)\ dt \right)$

65. $\frac{d}{dx}\left(\int_x^{x^2} \tan(t)\ dt \right)$

66. $\frac{d}{dx}\left(\int_0^{\pi} \cos(3t)\ dt \right)$

67. $\frac{d}{dx}\left(\int_2^{\ln(x)} 5t \cdot \cos(3t)\ dt) \right)$

68. $\frac{d}{dx}\left(\int_0^{\pi} \tan(7t)\ dt \right)$

Very Optional Problems

a. $\int_0^{ice} 3x^2\ dx$ What a calculus student puts in a drink.

b. $\int_1^{cabin} \frac{1}{x}\ dx$ Where Abe Lincoln was born.

c. $\int_0^{cerely} \cos(x)\ dx$ How the calculus student ended a letter.

d. $\int_1^{jam} \frac{1}{x}\ dx$ What a forester puts on toast.

Section 4.5 **PRACTICE Answers**

Practice 1: A(1) = 1, A(2) = 1.5, A(3) = 1, A(4) = 0.5

A '(x) = f(x) so A '(1) = f(1) = 1, A '(2) = f(2) = 0, A '(3) = –1, A '(4) = 0.

Practice 2: $F(x) = x^3 - x$ is one antiderivative of $f(x) = 3x^2 - 1$ (F ' = f) so

$$\int_1^3 3x^2 - 1\,dx = x^3 - x \Big|_1^3 = (3^3 - 3) - (1^3 - 1) = 24.$$

$F(x) = x^3 - x \mathbf{+ 7}$ is another antiderivative of $f(x) = 3x^2 - 1$ so

$$\int_1^3 3x^2 - 1\,dx = x^3 - x \mathbf{+ 7} \Big|_1^3 = (3^3 - 3 \mathbf{+ 7}) - (1^3 - 1 \mathbf{+ 7}) = 24.$$

No matter which antiderivative of $f(x) = 3x^2 - 1$ you use, the value of the definite integral $\int_1^3 3x^2 - 1\,dx$ is **24.**

Practice 3: $\int_{1.3}^{3.4} \text{INT}(x)\,dx = \mathbf{3.9}$. Since $f(x) = \text{INT}(x)$ is not continuous on the interval [1.3, 3.4] so we can not use the Fundamental Theorem of Calculus. Instead, we can think of the definite integral as an area (Fig. 20).

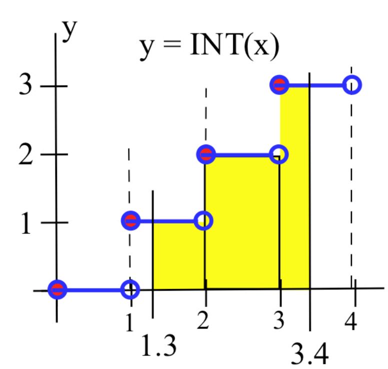

Fig. 20

Practice 4: Total volume $= \lim_{mesh \to 0} \left\{ \sum_{i=1}^{N} (3 - c_i) \cdot (3 - c_i) \cdot \Delta x_i \right\}$

$$= \int_0^2 (3 - x)(3 - x)\,dx$$

$$= \int_0^2 (9 - 6x + x^2)\,dx$$

$$= 9x - 3x^2 + \frac{x^3}{3} \Big|_0^2 = (18 - 12 + \frac{8}{3}) - (0–0+0) = \mathbf{8\tfrac{2}{3}}.$$

Practice 5: $\frac{d}{dx}\left(\int_0^{x^3} \sin(t)\,dt \right) = \sin(x^3)\,\frac{d\,x^3}{dx} = \mathbf{3x^2 \cdot \sin(x^3)}.$